On the Finite Groupoid $G(n)$

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Abstract. In this paper we study the existence of commuting regular elements, verifying the notion left (right) commuting regular elements and its properties in the groupoid $G(n)$ . Also we show that $G(n)$ contains commuting regular subsemigroup and give a necessary and sufficient condition for the groupoid $G(n)$ to be commuting regular.

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1. Introduction

We use $S$ and $G$ to denote a semigroup and a groupoid, respectively. An element $x$ of a semigroup $S$ is called regular if there exists $y$ in $S$ such that, $x = xyx$ [3]. Two elements $x$ and $y$ of a semigroup $S$ are commuting regular if for some $z$ in $S$, $xy = yzx$ [2]. A semigroup $S$ is called commuting regular if and only if for each $x, y \in S$ there exists an element $z$ of $S$ such that $xy = yzx$ [1]. In [2] Pourfaraj showed that the existence of commuting regular elements for the loop ring $Z_t[L_n(m)]$ when $t$ is an even perfect number or $t$ is the form of $2^ip$ or $3^ip$, where $p$ is an odd prime or in general, when $t = p_1p_2$ ( $p_1$ and $p_2$ are distinct odd primes ). Define a binary operation $*$ on $G = Z_n \cup \{e\}$ as follows,

1) $a * a = a$ for all $a \in G$.
2) $a * e = e * a = a$ for all $a \in G$.
3) $a * b = ta + ub \pmod{n}$ , where $t, u \in Z_n$ are fixed elements and $a, b \in G$ ($a \neq b$), $Z_n = \{0, 1, 2, ..., n - 1\}$, $n \geq 3$ and $e \notin Z_n$.

The properties of these groupoids denote by $G(n)$ has been studied in [5].

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2. Commuting Regular Elements

Definition 2.1 Two elements \(a\) and \(b\) of a groupoid \(G\) are called left commuting regular if for some \(c_1 \in G\), \(ab = ((ba)c_1)(ba)\). Similarly, they are called right commuting regular if for some \(c_2 \in G\), \(ab = (ba)(c_2ba)\). Finally, two elements \(x\) and \(y\) are commuting regular if they are both left and right commuting regular. [see 4]

Definition 2.2 A groupoid \(G\) is called left commuting regular groupoid if for each \(a, b \in G\) there exists \(c_1 \in G\) such that \(ab = (ba)c_1(ba)\). Similarly, right commuting regular groupoid is defined. A groupoid \(G\) is called commuting regular groupoid if \(G\) is both a left and right commuting regular groupoid.[see 4]

Example 2.3 The groupoid \(G(3)\) where \(t = 1\) and \(u = 2\) is given by the following table

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<td>2</td>
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<td>1</td>
<td>e</td>
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</table>

We have:

\[
(2 * 1) * (0 * (2 * 1)) = 1 * (0 * 1) = 1 * 2.
\]

So, 1 and 2 are right commuting regular. On the other hand,

\[
\begin{align*}
1 * 2 & \neq ((2 * 1) * 0) * (2 * 1) \\
1 * 2 & \neq ((2 * 1) * 1) * (2 * 1) \\
1 * 2 & \neq ((2 * 1) * 2) * (2 * 1) \\
1 * 2 & \neq ((2 * 1) * e) * (2 * 1).
\end{align*}
\]

Thus, 1 and 2 aren’t left commuting regular. 2 and 2 are commuting regular,

\[
2 * 2 = (2 * 2) * e * (2 * 2).
\]

Proposition 2.4 Let the \(G(n)\) be a groupoid, where \(n = tu - 1\). Suppose that \(a, b \in G(n)\) and pair of elements \(\{b * a, c_1, (b * a) * c_1\}\) and \(\{b * a, c_2, (b * a) * c_2\}\) are distinct. Then \(a\) and \(b\) are commuting regular elements, where \(b \equiv au \pmod{n}\), \(c_1 \equiv -bt^3 - b \pmod{n}\) and \(c_2 \equiv -au^3 - a \pmod{n}\).

Proof We consider two follows case:

Case1) If \(a * b = b * a\) then:

\[
a * b = (b * a) * (a * b) * (b * a)
\]

Case2) If \(a * b \neq b * a\) then:
\[(b * a) \ast c_1 \ast (b * a) = \]
\[= ((bt + au) \ast c_1) \ast (bt + au) \]
\[= ((bt + au) \ast t + c_1 u) \ast (bt + au) \]
\[= bt^3 + au^2 + c_1 tu + btu + au^2 \]
\[= bt^3 + at + bt^3 - b + b + bu \ (\text{since} \ tu \equiv 1 \ (\text{mod} \ n) \ \text{and} \ b \equiv au \ (\text{mod} \ n)) \]
\[= at + bu \]
\[= a \ast b \]

Similarly,

\[a \ast b = (b \ast a) \ast (c_2 \ast (b \ast a)).\]

**Proposition 2.5** Let the \(G(n)\) be a groupoid, where \(n \equiv tu + 1\). Suppose that \(a, b \in G(n)\) and pair of elements in \(\{b \ast a, c_1, (b \ast a) \ast c_1\}\) and \(\{b \ast a, c_2, (b \ast a) \ast c_2\}\) are distinct. Then \(a\) and \(b\) are commuting regular elements, where, \(b \equiv au \ (\text{mod} \ n), c_1 \equiv -2at + bt^3 - b \ (\text{mod} \ n)\) and \(c_2 \equiv -2at - 2bu + au^3 - a \ (\text{mod} n)\).

**Example 2.6** Let \(G(20)\) where \(t = 3\) and \(u = 7\), then \(a = 11\) and \(b = 17\) are commuting regular elements:

\[((17 \ast 11) \ast 4) \ast (17 \ast 11) = (17 \ast 11) \ast (16 \ast (17 \ast 11)) = 11 \ast 17.\]

Note that 17 \(\equiv 11 \times 7 \ (\text{mod} \ 20)\).

**Proposition 2.7** Let \(G(n)\) be a groupoid, where \(t \equiv -u \ (\text{mod} \ n)\), then \(a, b \in G(n)\) are commuting regular elements, where \(at \equiv bt \ (\text{mod} \ n)\).

**Proof** Since \(at \equiv bt \ (\text{mod} \ n)\) and \(t \equiv -u \ (\text{mod} \ n)\):

\[-au \equiv -bu \ (\text{mod} \ n).\]

So in \(G(n)\),

\[a \ast b = at + bu = bt + au = b \ast a.\]

And therefore:

\[a \ast b = (b \ast a) \ast (a \ast b) \ast (b \ast a).\]

So \(a\) and \(b\) are commuting regular.

**Proposition 2.8** Let \(G(n)\) be a groupoid, where \(n = (t - u)k, \ k \in \mathbb{Z}\), if for some \(a, b \in G(n)\), \(a - b \equiv k \ (\text{mod} \ n)\), then \(a\) and \(b\) are commuting regular elements.

**Proof** We have \(a - b \equiv \frac{n}{t - u} \ (\text{mod} \ n)\), so

\[(a - b)(t - u) \equiv 0 \ (\text{mod} \ n)\]

Therefore, in \(G(n)\):

\[at - au - bt + bu = 0\]

\[at + bu = bt + au\]
Proposition 2.9 Let $G(n)$ be a groupoid, then $a, b \in G(n)$ are commuting regular elements where $at \equiv au \pmod{n}$ and $bt \equiv bu \pmod{n}$.

Proof We have $a \ast b = at + bu = bt + au = b \ast a$ So

$$a \ast b = (b \ast a) \ast (a \ast b) \ast (b \ast a)$$

Thus $a$ and $b$ are commuting regular elements.

Proposition 2.10 Let $G(n)$ be a groupoid, where $t + u = n$. Suppose that $a \in G(n)$ and $k \in \mathbb{Z}$. Then $a$ and $ka$ are commuting regular elements, where $au \equiv -au \pmod{n}$.

Proof Since $t \equiv u \pmod{n}$, for all $a \in G(n)$ we have $at \equiv -au \pmod{n}$ and by $au \equiv -au \pmod{n}$, $at \equiv au \pmod{n}$. So $kat \equiv kau \pmod{n}$. Now by the proposition 2.9, $a$ and $ka$ are commuting regular elements.

3. Commuting Regular Groupoids

Proposition 3.1 The groupoid $G(n)$ for all $a \in G(n)$ contains the commuting regular subgroupoid $\{e, a\}$.

Proof The subgroupoid $\{e, a\}$ given by the following table,

$$
\begin{array}{c|cc}
\ast & e & a \\
\hline
  e  & e & a \\
  a  & a & e \\
\end{array}
$$

$e \ast a = (a \ast e) \ast a \ast (a \ast e)$

$a \ast a = (a \ast a) \ast e \ast (a \ast a)$

$e \ast e = (e \ast e) \ast e \ast (e \ast e)$

Proposition 3.2 Let $G(n)$ be a groupoid, where $n = 2u$, $u^2 \equiv u \pmod{n}$ and $t = 1$. Then for every $a$ in $G(n)$, $\{e, a, a + u\}$ is a commuting regular groupoid.

Proof Let $b = a + u$. If, we have:

$$x \ast x = e, \ x \ast e = e \ast x = x$$

Also,

$$au = \begin{cases} 
0 & \text{if } a \text{ is even } (mod \ n), \\
 u & \text{if } a \text{ is odd } (mod \ n), 
\end{cases}$$

$$a \ast b = b \ast a \equiv a + u + au \equiv \begin{cases} 
b & \text{if } a \text{ is even } (mod \ n) \\
a & \text{if } a \text{ is odd } (mod \ n) 
\end{cases}$$
So \( \{e, a, b\} \) is groupoid.
For all \( x, y \in \{e, a, b\} \) we have \( x \ast y = y \ast x \). So
\[
x \ast y = (y \ast x) \ast (x \ast y) \ast (y \ast x)
\]
Thus \( \{e, a, b\} \) is a commuting regular groupoid.

**Example 3.3** Let \( G(n) \) be a groupoid, where \( n = 6 \), \( u = 3 \) and \( t = 1 \) is given by the following table,
\[
\begin{array}{c|ccccc}
\ast & e & 0 & 1 & 2 & 3 & 4 & 5 \\
e & e & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & e & 3 & 0 & 3 & 0 & 3 \\
1 & 1 & 1 & e & 1 & 4 & 1 & 4 \\
2 & 2 & 2 & 5 & e & 5 & 2 & 5 \\
3 & 3 & 3 & 0 & 3 & e & 3 & 0 \\
4 & 4 & 4 & 1 & 4 & 1 & e & 1 \\
5 & 2 & 5 & 2 & 5 & 2 & 5 & e \\
\end{array}
\]
\( \{e, 0, 3\} \), \( \{e, 1, 4\} \) and \( \{e, 2, 5\} \) are commuting regular groupoids.

**Proposition 3.4** Let \( G(n) \) be a groupoid, where \( t = 0 \), \( n = 2u \) and \( u \) is an odd element. Therefore groupoid \( G(n) \) contains commuting regular and commutative groupoids \( G_1 = \{e, 1, 3, \ldots, n - 1\} \) and \( G(2) = \{e, 0, 2, \ldots, n - 2\} \). In particular, if \( u^2 \equiv u \pmod{n} \), then \( G_1 \) and \( G_2 \) are commuting regular and commutative semigroup.

**Proof** For all \( a, b \in G_1 - \{e\} \), if \( a \neq b \) we have \( a \ast b = b \ast a = u \). So, we have:
\[
a \ast b = (b \ast a) \ast (a \ast b) \ast (b \ast a)
\]
In particular, if \( u^2 = u \pmod{n} \) for all \( a, b, c \in G_1 \) we have:
\[
(a \ast b) \ast c = bu \ast c = cu
\]
\[
a \ast (b \ast c) = a \ast cu = cu^2
\]
Therefore \( G_1 \) is a semigroup. The proof for \( G_2 \) is the same as above.

**Corollary 3.5** Let \( G(n) \) be a groupoid, where \( u = 0, n = 2t \) and \( t \) is odd element. Then groupoid \( G(n) \) contains commuting regular and commutative groupoids \( G_1 = \{e, 1, 3, \ldots, n - 1\} \) and \( G(2) = \{e, 0, 2, \ldots, n - 2\} \). In particular, if \( t^2 \equiv t \pmod{n} \) then \( G_1 \) and \( G_2 \) are commuting regular and commutative semigroup.

**Proposition 3.6** Let \( G(n) \) be a groupoid, where \( t = 0, n = 3u \) and \( u = 3k + 1 \) for some \( k \in \mathbb{Z} \). Then groupoid \( G(n) \) contains commuting regular and commutative groupoids \( G_1 = \{e, 2, 5, \ldots, n - 1\} \), \( G(2) = \{e, 1, 4, \ldots, n - 2\} \) and \( G_3 = \{e, 0, 3, \ldots, n - 3\} \). In particular, if \( u^2 \equiv u \pmod{n} \), then \( G_1, G_2 \) and \( G_3 \) are commuting regular and commutative semigroups.

**Theorem 3.7** Let \( G(n) \) be a groupoid, where \( t = 0, n = mu \) and \( u = mk + 1 \), for some \( m, k \in \mathbb{Z} \). Then groupoid \( G(n) \) contains commuting regular and commutative groupoids. In particular, if \( u^2 \equiv u \pmod{n} \) then \( G(n) \) contains commuting regular and commutative semigroups.

**Example 3.8** Let \( G(n) \) be a groupoid, where \( t = 0, u = 5 \) and \( n = 10 \) is given in the following table,
<table>
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<tr>
<th>*</th>
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Clearly, the semigroups \{e, 0, 2, 4, 6, 8\}, \{e, 1, 3, 5, 7, 9\} are commuting regular and commutative.

**Theorem 3.9** Let \(G(n)\) be a groupoid, where \(t = u \mod n\) then \(G(n)\) is a commuting regular and commutative semigroup.

**Proof** Let \(a, b \in G(n) - \{e\}\),

1) If \(a \neq b\), then \(a\) and \(b\) are commuting regular elements [4, Theorem 3.8].

2) If \(a = b\) then \(a * b = b * a = e\), so \(a * b = (b * a) * e * (b * a)\),

3) If \(b = e\) then \(a * e = e * a = a\), so \(a * e = (e * a) * a * (e * a)\).

On the other hand,

\[
a * (b * c) = a * (bt + ct) = at + bt^2 + ct^2
\]

\[
(a * b) * c = (at + bt) * c = at^2 + bt^2 + ct.
\]

So, the groupoid \(G(n)\) is a semigroup.

**References**


