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Some algebraic properties of Lambert Multipliers on L^2 spaces

A. Zohri[a,*] and S. Khalil Sarbaz[a]

[a] *Faculty of Mathematical Sciences, Payame Noor University,*
P. O. BOX 19395-3697, Tehran, I. R. Iran.

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Abstract. In this paper, we determine the structure of the space of multipliers of the range of a composition operator C_φ that induces by the conditional expectation between two $L^p(\Sigma)$ spaces.

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1. Introduction and Preliminaries

Let $L(X, \Sigma, \mu)$ be a σ-finite measure space. For any complete σ-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leqslant p \leqslant \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|\mathcal{A})$ is abberivated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. We understand $L^p(\mathcal{A})$ as a Banach sub-space of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

To examine the weighted composition operators efficiently, Lambert in [5] associated with each transformation T, the so-called conditional expectation operator $E(\cdot|\mathcal{A}) = E(\cdot)$ is defined for each non-negative measurable function f or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions:
(i) $E(f)$ is $\mathcal{A}$-measurable and

*Corresponding author.
E-mail address: zohri_a@pnu.ac.ir (A. Zohri).

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(ii) If A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ converges we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

This operator will play a major role in our work, and we list here some of its useful properties:
- If g is $\mathcal{A}$-measurable then $E(fg) = E(f)g$.
- $|E(f)|^p \leqslant E(|f|^p)$.
- $\|E(f)\|_p \leqslant \|f\|_p$.
- If $f \geqslant 0$ then $E(f) \geqslant 0$; if $f > 0$ then $E(f) > 0$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^p(\mathcal{A})$.

As an operator on $L^p(\Sigma)$, $E(\cdot)$ is the contractive idempotent and $E(L^p(\Sigma)) = L^p(\mathcal{A})$. The real-valued Σ-measurable function f is said to be conditionable with respect to $\mathcal{A}$ if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}) = 0$. In this case $E(f) := E(f^+) - E(f^-)$. If f is complex-valued, then f is conditionable if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. In this case, $E(f) := E(Ref) + iE(Imf)$ (see [3]). We denote the linear space of all conditionable Σ-measurable functions on X by $L^0(\Sigma)$. For f and g in $L^0(\Sigma)$, we define $f \star g = fE(g) + gE(f) - E(f)E(g)$. Let $1 \leqslant p,\ q \leqslant \infty$. A measurable function $u \in L^0(\Sigma)$ for which $u \star f \in L^q(\Sigma)$ for each $f \in L^p(\Sigma)$, is called Lambert multiplier (see [6]). In other words, $u \in L^0(\Sigma)$ is Lambert multiplier if and only if the corresponding $\star$-multiplication operator $T_u : L^p(\Sigma) \to L^q(\Sigma)$ defined as $T_u f = u \star f$ is bounded.

In the next section, Lambert multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator. In section 3, Fredholmness of corresponding $\star$-multiplication operators will be investigated.

2. New results of Lambert multipliers on L^2 spaces

In this paper we will assume $\mu(X) < \infty$.

Definition 2.1 For f and g in $L^0(\Sigma)$, we define

$$f \star g = fE(g) + gE(f) - E(f)E(g).$$

Definition 2.2 A measurable function $u \in L^0(\Sigma)$ for which $T_u(f) = u \star f$ for each $f \in L^2(\Sigma)$, is called Lambert operator.

Definition 2.3 A measurable function $u \in L^0(\Sigma)$ for which $u \star f \in L^2(\Sigma)$ for each $f \in L^2(\Sigma)$, is called Lambert multiplier.

In other words, $u \in L^0(\Sigma)$ is Lambert multiplier if and only if the corresponding $\star$-multiplication operator $T_u : L^2(\Sigma) \to L^2(\Sigma)$ defined as $T_u f = u \star f$ is bounded.

Theorem 2.4 Suppose $u \in L^0(\Sigma)$. Then $u \in K_2^\star$ if and only if $E(|u|^2) \in L^\infty(\mathcal{A})$.

Proof. See [4]. ∎

Definition 2.5 Define $K_2^\star$, the set of all Lambert multiplier from $L^2(\Sigma)$ into $L^2(\Sigma)$, as follows:

$$K_2^{\star} = \left\{u \in L^0(\Sigma) : u \star L^2(\Sigma) \subseteq L^2(\Sigma)\right\}.$$

Note $K_2^{\star}$ is a vector subspace of $L^0(\Sigma)$.

Definition 2.6 Suppose $u \in L^0(\Sigma)$. Then we define

$$\mathcal{R} = \{\lambda I + T_u : u \in K_2^{\star} \textit{ and } \lambda \in \mathbb{C}\}.$$

Hereafter we shall denote $\lambda I + T_u$ (I is identity operator) simply as $\lambda + T_u$.

Proposition 2.7 Show $\mathcal{R}$ is closed under addition and $T_{u+v} = T_u + T_v$ for every $u, v \in K_2^{\star}$.

Proof. Let $\lambda + T_u$ and $\gamma + T_v$ be in $\mathcal{R}$. Then, for any $f \in L^2(\Sigma)$,

$$\begin{aligned}
[(\lambda + T_u) + (\gamma + T_v)]f &= (\lambda + T_u)f + (\gamma + T_v)f \\
&= (\lambda f + T_u f) + (\gamma f + T_v f) \\
&= \lambda f + \Big(uE(f) + fE(u) - E(u)E(f)\Big) \\
&\quad + \gamma f + \Big(vE(f) + fE(v) - E(v)E(f)\Big) \\
&= (\lambda + \gamma)f + (u + v)E(f) + fE(u + v) - E(u + v)E(f) \\
&= (\lambda + \gamma)f + T_{u+v}f.
\end{aligned}$$

Then $(\lambda + \gamma)f + T_{u+v}f = (\delta + T_y)f$ where $\delta = \lambda + \gamma$ and $y = u + v$. Since $|u + v|^2 \leqslant 2\Big(|u|^2 + |v|^2\Big)$ and E is a linear operator, thus $E\Big(|u + v|^2\Big) \leqslant 2\Big(E(|u|^2) + E(|v|^2)\Big)$. It follows that $y \in K_2^{\star}$. Therefore, $\mathcal{R}$ is closed under addition and $T_{u+v} = T_u + T_v$. ■

Proposition 2.8 a. Show $\mathcal{R}$ is closed under multipliers and $T_{\lambda u} = \lambda T_u$ for any $f \in L^2(\Sigma)$ and $\lambda \in \mathbb{C}$.

b. $\mathcal{R}$ is commutative under composition operators.

Proof. a. Suppose $f \in L^2(\Sigma)$ and $\lambda \in \mathbb{C}$, thus we have

$$\begin{aligned}
(\lambda T_u)f &= \lambda T_u f \\
&= \lambda u E(u) + fE(u) - E(u)E(f) \\
&= (\lambda u)E(f) + fE(\lambda u) - E(\lambda u)E(f) \\
&= T_{\lambda u} f,
\end{aligned}$$

it follows $\lambda T_u = T_{\lambda u}$. Otherwise since

$$E\Big(|\lambda u|^2\Big) = E\Big(|\lambda|^2 |u|^2\Big) = |\lambda|^2 E(|u|^2),$$

this equality shows that if $E(|u|^2) \in L^\infty(\mathcal{A})$, then $E\Big(|\lambda u|^2\Big)$. Thus if $T_u \in \mathcal{R}$, then $T_{\lambda u} = \lambda T_u$ for any $\lambda \in \mathbb{C}$.

b. We show $\mathcal{R}$ is commutative. Suppose T_u and T_v be in $\mathcal{R}$. Thus

$$\begin{aligned} T_uT_vf &= T_u\Big(vE(f) + fE(v) - E(v)E(f)\Big) \\ &= vE(u)E(f) + fE\Big(vE(u)\Big) - E\Big(vE(u)\Big)E(f) + uE(v)E(f) - E\Big(uE(v)\Big)E(f). \end{aligned}$$

similarly we have

$$\begin{aligned} T_vT_uf &= T_v\Big(uE(f) + fE(u) - E(u)E(f)\Big) \\ &= vE(u)E(f) + fE\Big(vE(u)\Big) - E\Big(vE(u)\Big)E(f) + uE(v)E(f) - E\Big(uE(v)\Big)E(f). \end{aligned}$$

The above equality shows that $T_uT_v = T_vT_u$. Therefor, $\mathcal{R}$ is commutative.

Now, we show $T_uT_v \in \mathcal{R}$. Note that

$$T_uT_vf = T_{(uEv+vEu)}f - T_{EvEu}f = T_{(uEv+vEu)}f - T_{E(uEv)}f.$$

also

$$\begin{aligned} E\left(|uEv + vEu|^2\right) &\leqslant 2\left(E|uEv|^2 + E|vEu|^2\right) \\ &\leqslant 2\left(E|u|^2E|v|^2 + E|v|^2E|u|^2\right) \\ &= 4E|v|^2E|u|^2, \end{aligned}$$

and

$$E|EvEu|^2 = E\left(|Ev|^2|Eu|^2\right) \leqslant E\left(E|v|^2E|u|^2\right) = E|v|^2E|u|^2,$$

which implies T_vT_u be in $\mathcal{R}$, since by $E|u|^2$ and $E|v|^2$ are bounded operators. In general, for every $\gamma, \lambda \in \mathbb{C}$ we have

$$(\lambda + T_v)(\gamma + T_u) = \lambda\gamma + \lambda T_v + \gamma T_u + T_vT_u.$$

Thus by the above equalities (b) hold.

■

Theorem 2.9 $\mathcal{R}$ is a commutative operator algebra.

Proof. By Propositions [2.7] and [2.8], it is trivial. ■

If $\mathfrak{U}$ is an algebra of bounded operators, then its commutant $\mathfrak{U}''$ is the set of all bounded operators that commute with every element in $\mathfrak{U}$ (see [1, 2]). In symbols and in the context of Lambert multipliers:

$$\mathcal{R}'' = \left\{A \in B\left(L^2(\Sigma)\right) : AT = TA \; for \; any \; T \in \mathcal{R}\right\}.$$

Theorem 2.10 Suppose M_h is multiplication operator. Set

$$\mathcal{L}^\infty(\mathcal{A}) := \Big\{M_h : h \in L^\infty(\mathcal{A})\Big\},$$

then $\mathcal{L}^\infty(\mathcal{A}) \subseteq \mathcal{R}''$.

Proof. Let $f \in L^2(\Sigma)$. Then for any $h \in L^\infty(\mathcal{A})$, we have

$$\begin{aligned} T_u M_h f = T_u(hf) &= uE(hf) + hfE(u) - E(u)E(hf) \\ &= h\Big(uEf + fEu - uEf\Big) = hT_u f = M_h T_u f. \end{aligned}$$

Therefor $\mathcal{L}^\infty(\mathcal{A}) \subseteq \mathcal{R}''$. ■

3. Some results of Lambert multipliers on L^2 spaces

In this section we bring some facts and definitions, which will be used later.

Definition 3.1 Let $T_u : L^2(\Sigma) \longrightarrow L^2(\Sigma)$. Define

$$W := \left\{u \in L^0(\Sigma) : T_u \; is \; bounded \; on \; L^2(\Sigma)\right\}.$$

We already know one important property of function in W, namely, $E(|u|^2)$ is bounded. However, Since

$$|E(u)|^2 \leqslant E|u|^2,$$

we see that $u \in W$ implies that $E(u)$ is bounded. Therefore, if a function is both $\mathcal{A}-$measurable and in W, then it must be bounded. Our next Lemma states that the converse also holds.

Lemma 3.2 $W \bigcap L^0(\mathcal{A}) = L^\infty(\mathcal{A})$.

Proof. Let $s \in L^\infty(\mathcal{A})$. Since s is $\mathcal{A}-$measurable, then $T_s f = sf$ for $f \in L^2(\Sigma)$. Also, we know $L^2(\mathcal{A}) \subset L^2(\Sigma)$, thus we get

$$\|T_s f\|_2^2 = \int_X |T_s f|^2 d\mu = \int_X |sf|^2 d\mu \leqslant \|s\|_\infty^2 \int_X |f|^2 d\mu = \|s\|_\infty^2 \|f\|_2^\infty$$

so that $T_s f \in L^2(\sigma)$ for all f. Hence, $W \bigcap L^0(\mathcal{A})$. The converse we proved in the remarks leading up to the Lemma. ■

Theorem 3.3 T_u is normal if and only if $u \in L^\infty(\mathcal{A})$.

Proof. Assume T_u is normal. Then, for any $f \in L^2(\Sigma)$,

$$T_u T_u^* f = T_u^* T_u f.$$

Now, we have

$$T_u^* T_u f = E(u)E(\bar{u}f) \tag{1}$$

and

$$T_u^* T_u f = E(f)E(|u|^2) + E(u)E(\bar{u}f) - E(\bar{u})E(u)E(f). \tag{2}$$

Therefore we conclude from [1] and [2]

$$E(u)E(|u|^2) = E(f)|E(u)|^2.$$

The last equation holds for every L^2 function, so in particular it must hold for any strictly positive $\mathcal{A}-$ measurable L^2 function s:

$$E(s)E(|u|^2) = E(s)|E(u)|^2,$$

then by letting $E(s) = s$ we have

$$sE(|u|^2) = s|E(u)|^2.$$

Since $s > 0$, this gives

$$|E(u)|^2 = E(|u|^2).$$

However, we saw that this is equivalent to u being $\mathcal{A}-$ measurable. By [3.2], $u \in L^\infty(\mathcal{A})$.

Conversely, now suppose, that $u \in L^\infty(\mathcal{A})$. Then, for $f \in L^2(\Sigma)$, $T_u f = uf$ and $T_u^* f = \bar{u}f$. Therefore,

$$T_u T_u^* f = T_u(\bar{u}f) = u(\bar{u}f) = |u|^2 f$$

and

$$T_u^* T_u f = T_u^*(uf) = \bar{u}(uf) = |u|^2 f.$$

Hence, $T_u T_u^* = T_u^* T_u$. Thus, T_u is normal. ■

Theorem 3.4 T_u is self-adjoint if and only if $u \in L^\infty(\mathcal{A})$ is real-valued.

Proof. Assume T_u is self-adjoint. Then, T_u is normal and by Theorem [3.3] $u \in L^\infty(\mathcal{A})$. Therefore, we must only show that u is real-valued. Let $f \in L^2(\Sigma)$. Then, $T_u^* f = T_u f$ can be written as

$$E(\bar{u}f) + E(\bar{u})\,(f - E(f)) = uE(f) + fE(u) - E(u)E(f).$$

Since u is $\mathcal{A}-$ measurable, $\bar{u}f = uf$ which implies $(u - \bar{u})f = 0$. This last equality holds for any L^2 function. In particular, it holds for strictly positive $s \in L^2(\mathcal{A})$. Therefore, $u = \bar{u}$.

Conversely, suppose $u \in L^\infty(\mathcal{A})$ is real-valued. For $f \in L^2(\Sigma)$,

$$T_u^* f = uf = T_u f.$$

Hence, T_u is self-adjoint. ■

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