On the superstability of a special derivation

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Abstract. The aim of this paper is to show that under some mild conditions a functional equation of multiplicative \((\alpha, \beta)\)-derivation is superstable on standard operator algebras. Furthermore, we prove that this generalized derivation can be a continuous and an inner \((\alpha, \beta)\)-derivation.

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1. Introduction

Questions concerning the stability of functional equations seems to be originated with S. M. Ulam [15]. In fact if \(X\) and \(Y\) are two Banach spaces and if \(f : X \to Y\) is an approximately additive mapping, he wanted the functional equation for additive functions to be stable.

The case of approximately additive mapping between Banach spaces was solved by D. H. Hyers [9]. In 1968 S. M. Ulam proposed a more general problems: "When is it true that by changing the hypothesis of Hyers theorem a little one can still assert that the thesis of the theorem remains true of approximately true!"

Th. M. Rassias [12] proved a substantial generalization of the result of Hyers. Taking it into account, the additive functional equation is said to have the "Hyers- Ulam- Rassias" stability. And many authors answered the Ulam’s equation for several cases. In [4] the

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author proved that every mapping \( f \) of a Banach algebra \( A \) onto a Banach algebra \( B \) which is approximately multiplicative is a ring homomorphism of \( A \) onto \( B \).

J. A. Baker [2] showed that every approximately multiplicative unbounded complex-valued function defined on a semigroup \( S \) is actually a multiplicative function. We can find further references on problems concerning stability and superstability in survey papers.

In 1994 Peter Semrl [14] proved that the question of multiplicative derivation is superstable on standard operator algebras. In this paper we first prove that the functional equation of linear \((\alpha, \beta)\)-derivation is superstable on whole of \( B(X) \) and furthermore we prove that this \((\alpha, \beta)\)-derivation can be a continuous and inner \((\alpha, \beta)\)-derivation and then we extend superstability to functional equation of multiplicative \((\alpha, \beta)\)-derivation on standard operator algebras. (for further results see [3],[9],[16]).

2. Preliminaries

Let \( R \) and \( S \) be two arbitrary associative rings (not necessarily with identity element). A mapping \( \sigma : R \to S \) such that \( \sigma(x + y) = \sigma(x) + \sigma(y) \quad (x, y \in R) \) is called an additive mapping of \( R \) into \( S \) and is called a multiplicative mapping of \( R \) into \( S \) if \( \sigma(xy) = \sigma(x)\sigma(y) \quad (x, y \in R) \) and a ring homomorphism from \( R \) into \( S \) is a mapping that is additive and also multiplicative. Furthermore a one to one and onto ring homomorphism is called a ring isomorphism and a ring isomorphism from \( R \) into \( R \) is called a ring automorphism of \( R \).

If \( \alpha \) and \( \beta \) are mappings on \( R \), by multiplicative derivation from \( R \) into itself we call a mapping \( D : R \to R \) such that

\[
D(xy) = D(x)y + xD(y) \quad (x, y \in R).
\]

And by a multiplicative \((\alpha, \beta)\)-derivation from \( R \) into itself we call a mapping \( D : R \to R \) such that

\[
D(xy) = D(x)\alpha(y) + \beta(x)D(y) \quad (x, y \in R).
\]

In addition, if there exists \( x_0 \in R \) such that \( d(x) = \beta(x)x_0 - x_0\alpha(x) \) holds for each \( x \in R \), then \( d \) is called an inner \((\alpha, \beta)\)-derivation.

Note that if \( R \subseteq S \) similarly the derivation and \((\alpha, \beta)\)-derivation \( D : R \to S \) can be defined (for further results see [5],[6],[7]).

**Definition 2.1** A mapping \( \sigma \) from a ring \( R \) into a normed linear space \( S \) is approximately additive if there is \( \delta > 0 \) such that

\[
||\sigma(x + y) - \sigma(x) - \sigma(y)|| \leq \delta \quad (x, y \in R).
\]

And is approximately multiplicative if there is \( \varepsilon > 0 \) such that

\[
||\sigma(xy) - \sigma(x)\sigma(y)|| \leq \varepsilon \quad (x, y \in R).
\]

**Definition 2.2** A mapping \( D \) from a normed linear space \( R \) into \( R \) is an approximate multiplicative derivation if there is \( \delta > 0 \) such that

\[
||D(xy) - D(x)y - xD(y)|| \leq \delta \quad (x, y \in R).
\]
And is an approximate multiplicative \((\alpha, \beta)\)-derivation if there is \(\varepsilon > 0\) such that
\[
\|D(xy) - D(x)\alpha(y) - \beta(x)D(y)\| \leq \varepsilon \quad (x, y \in R).
\]

**Definition 2.3** Let \(R\) be a ring and \(S\) be a normed space. If for given \(\varepsilon > 0\) and for an approximate additive mapping \(f : R \to S\) there exists a unique additive mapping \(g : R \to S\) such that \(\|f(x) - g(x)\| \leq \varepsilon\) then we say the functional equation for additive functions is stable. In a situation where an approximate additive mapping must be a true additive mapping we say that the equation of the additive mapping is superstability.

In a similar fashion we can define stability and superstability of the functional equations of multiplicative functions and multiplicative derivations and multiplicative \((\alpha, \beta)\)-derivations.

**Definition 2.4** A ring \(R\) is called a prime ring if \(xRy = 0\) for \(x, y \in R\), implies that \(x = 0\) or \(y = 0\).

Let \(X\) be a Banach space over \(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}\). We denote by \(B(X)\), the algebra of all bounded linear operators of \(X\) and \(\mathcal{F}(X)\) the subalgebra of all bounded linear finite rank operators and \(\mathcal{F}_1(X)\) the subalgebra of all bounded linear rank one operators. We shall call a subalgebra \(A\) of \(B(X)\) standard provided \(A\) contains \(\mathcal{F}(X)\).

**Definition 2.5** Let \(A\) be a standard operator algebra on a Banach space \(X\).

A mapping \(D : A \to B(X)\) is called a linear derivation if
\[
\begin{align*}
(i) & \quad D(\lambda A) = \lambda D(A), \\
(ii) & \quad D(A + B) = D(A) + D(B), \\
(iii) & \quad D(AB) = AD(B) + D(A)B.
\end{align*}
\]

For each \(A, B \in A\) and \(\lambda \in \mathbb{F}\), a mapping satisfying \((ii)\) and \((iii)\) is called a ring derivation. Multiplicative derivations are mapping satisfying only \((iii)\). Linear \((\alpha, \beta)\)-derivation and ring \((\alpha, \beta)\)-derivation are defined similarly.

Given a Banach algebra \(A\) it is also to consider \(n \times n\) matrix algebra \(M_n(A)\) with the following standard operations,
\[
(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}), \quad \lambda (a_{ij}) = (\lambda a_{ij}), \quad (a_{ij})(b_{ij}) = (\sum_{k=1}^{n} a_{ik} b_{kj}) \quad (i, j = 1, 2, ..., n)
\]

And the norm \(\|(a_{ij})\| = \sup_{1 \leq i \leq n} (\|a_{i1}\| + ... + \|a_{in}\|)\).

Note that if \(X\) is finite dimensional vector space then all norms defined on \(X\) are equivalent. So if \(A\) is finite dimensional then the above norm on \(M_n(A)\) is equivalent to the operator norm.

Let \(X\) be a Banach space and \(X^*\) the dual space of \(X\). If \(x \in X\) and \(f \in X^*\), then \(x \otimes f\) denotes the operator defined by \((x \otimes f)(z) = f(z)x \quad (z \in X)\).

In particular if \(H\) is a Hilbert space and \(x, y \in H\), then \(x \otimes y\) denotes the operator defined by \((x \otimes y)(z) = (z, y)x \quad (z \in H)\).

Clearly if \(A \in B(X)\), then \(A(x \otimes f) = A(x) \otimes f\).

**Definition 2.6** A Banach space \(X\) is called simple if \(B(X)\) has a unique nontrivial norm-closed two-sided ideal. For example, \(l^p\; (1 \leq p < \infty), c_0\) (The Banach space of all sequences which converges to zero, with \(\ell^\infty\) norm) and a separable infinite dimensional Hilbert space \(H\) are simple. In this case, the norm closure of all the finite rank operators is the ideal of compact operators, which is dense in \(B(X)\) with weak operator topology and is the unique nontrivial norm-closed two-sided ideal of \(B(X)\). (for further results see [1],[11],[13]).
3. Main Results

**Lemma 3.1** [10] Let $R$ be a ring containing a family $\{e_\alpha : \alpha \in A\}$ of idempotent which satisfies:

1. $xR = 0$ implies $x = 0$.
2. If $e_\alpha Rx = 0$ for each $\alpha \in A$, then $x = 0$ (and hence $Rx = 0$ implies $x = 0$).
3. For each $\alpha \in A$, $e_\alpha xe_\alpha R(1 - e_\alpha) = 0$ implies $e_\alpha xe_\alpha = 0$.

Then every multiplicative isomorphism $\sigma$ of $R$ onto an arbitrary ring is additive.

As a special case of Lemma 3.1, we conclude the following theorem:

**Theorem 3.2** Suppose that $R$ is a ring containing a family $\{e_\alpha\}_{\alpha \in A}$ of idempotents, such that for each $\alpha \in A$ and $x \in R$ satisfies the following conditions:

(i) $xR = 0$ implies $x = 0$;
(ii) $e_\alpha Rx = 0$ implies $x = 0$;
(iii) If $e_\alpha xe_\alpha R(1 - e_\alpha) = 0$ then $e_\alpha xe_\alpha = 0$.

If $\alpha$ and $\beta$ are ring homomorphisms on $R$ and at least one of $\alpha$ and $\beta$ is one to one then every multiplicative $(\alpha, \beta) - derivation$ of $R$ is additive.

**Proof.** Let $d : R \to R$ be a multiplicative $(\alpha, \beta) - derivation$, and let

$$S = \left\{ \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix} \middle| x \in R \right\}.$$  

Obviously $S$ is a ring. Define $\sigma : R \to S$ by

$$\sigma(x) = \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix},$$

for each $x \in R$. Then $\sigma$ is onto and one to one, since one of $\alpha$ and $\beta$ is one to one.

For every $x, y \in R$, we have

$$\sigma(xy) = \begin{pmatrix} \beta(xy) & d(xy) \\ 0 & \alpha(xy) \end{pmatrix}$$

$$= \begin{pmatrix} \beta(x)\beta(y) & d(x)\alpha(y) + \beta(x)d(y) \\ 0 & \alpha(x)\alpha(y) \end{pmatrix}$$

$$= \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix}$$

$$= \begin{pmatrix} \beta(y) & d(y) \\ 0 & \alpha(y) \end{pmatrix}$$

$$= \sigma(x)\sigma(y).$$

Then $\sigma$ is multiplicative. Hence it is an isomorphism and by Lemma 3.1, it is additive.

$$\sigma(x + y) = \begin{pmatrix} \beta(x + b) & d(x + y) \\ 0 & \alpha(x + y) \end{pmatrix}$$

$$= \sigma(x) + \sigma(y)$$

$$= \begin{pmatrix} \beta(x) + \beta(y) & d(x) + d(y) \\ 0 & \alpha(x) + \alpha(y) \end{pmatrix}$$

Hence $d$ is additive.

**Lemma 3.3** [8] Let $X$ be a complex Banach space, $\alpha$ and $\beta$ be mappings from $B(X)$
into itself. Let \( D : \mathcal{B}(X) \to \mathcal{B}(X) \) be a linear \((\alpha, \beta)\)-derivation. Then \( D \) is continuous if one of the following conditions holds:

(i) \( \alpha \) is an automorphism, \( \beta \) is continuous at 0 and the set \( \{ \beta(T) : T \in \mathcal{F}_1(X) \} \) separates the points of \( X \) in the sense that, for each pair \( \xi, \eta \in X \) with \( \xi \neq \eta \), there is a rank one operator \( T \) such that \( \beta(T)\xi \neq \beta(T)\eta \), equivalently, the set \( \{ \beta(T) : T \in \mathcal{F}_1(X) \} \) has no nonzero right annihilators in \( \mathcal{B}(X) \).

(ii) \( \beta \) is an automorphism, \( \alpha \) is continuous at 0 and the set \( \{ \alpha(T) : T \in \mathcal{F}_1(X) \} \) has no nonzero right annihilators in \( \mathcal{B}(X) \).

(iii) \( \alpha \) and \( \beta \) are continuous at 0, \( \text{span}\{ \alpha(T)\xi : T \in \mathcal{F}_1(X), \xi \in X \} \) is dense in \( X \) and there is a rank one \( S \) such that \( \beta(S) \) is injective.

(iv) \( X \) is simple and \( \alpha, \beta \) are surjective and continuous at zero.

(v) \( \alpha, \beta \) are surjective and multiplicative and there are rank one operators \( T_0 \) and \( S_0 \) such that \( (T_0) \neq 0 \) and \( (S_0) \neq 0 \).

Moreover, if either (i), or \( X \) is reflexive and (ii), holds, \( D \) is \((\alpha, \beta)\)-inner.

Let \( \mathbb{R}_+ \) the set of all nonnegative real numbers. We prove that the equation of a special multiplicative \((\alpha, \beta)\) - derivation is superstble on standard operator algebras.

**Theorem 3.4** Let \( X \) be a complex Banach space with \( \dim X > 1 \) and suppose that \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a mapping such that \( \lim_{t \to \infty} \frac{\phi(t)}{t} = 0 \) and \( D : \mathcal{B}(X) \to \mathcal{B}(X) \) is a mapping satisfying \( D(\lambda A) = \lambda D(A) \) \((\lambda \in \mathbb{C}, A \in \mathcal{B}(X)) \) and

\[
||D(AB) - \beta(A)D(B) - D(A)\alpha(B)|| < \phi(||A|| ||B||) \quad (A, B \in \mathcal{B}(X))
\]

If \( \alpha, \beta \) are ring homomorphisms on \( \mathcal{B}(X) \) and at least one of \( \alpha \) and \( \beta \) is one to one in which \( \alpha \) is a scalar multiplicative preserving map then the following statements holds:

(1) \( D \) is \((\alpha, \beta)\)-inner if either condition (i) or when \( X \) is reflexive, condition (ii) of lemma 3.3 holds.

(2) \( D \) is continuous if one of conditions (i),(ii), (iii), (iv) and (v) of Lemma 3.3 holds.

**Proof.** At first we show that if \( A, B \in \mathcal{B}(X) \), then \( AB(X)B = 0 \) implies \( A = 0 \) or \( B = 0 \). In fact if \( B \neq 0 \), then there exists \( z \in A \) such that \( B(z) \neq 0 \) and then from Hahn Banach theorem there exists \( f \in X^* \) such that \( f(B(z)) \neq 0 \). Now for every arbitrary \( x \in X \) we have:

\[
AB(X)B = 0
\]

\[
A(x \otimes f)B(z) = 0
\]

\[
(Ax \otimes f)(B(z)) = 0
\]

\[
f(B(z))A(x) = 0
\]

\[
A(x) = 0 \quad (x \in X)
\]

\[
A = 0.
\]
Furthermore if \( \{x_i\} \) is a base for \( X \) and \( j \) be considered fixed we can define \( T : X \to X \) by \( T(\sum a_i x_i) = a_j x_j \), then clearly \( T \) is a nontrivial idempotent of rank one. Therefore \( B(X) \) is a prime ring with nontrivial idempotent and so satisfies the conditions of Theorem 3.2 and so every multiplicative \((\alpha, \beta)\)-derivation on \( B(X) \) is a ring \((\alpha, \beta)\)-derivation.

Replacing \( B \) by \( tB \) in which \( t \) is a positive real number in \( \| D(AB) - \beta(A)D(B) - D(A)\alpha(B) \| < \phi(\| A \||\| B \|) \), and then by dividing the above inequality by \( t \), we obtain \( \| D(AB) - \beta(A)D(B) - D(A)\alpha(B) \| < \frac{\phi(\| A \||\| B \|)}{t} \). By taking a limit when \( t \to \infty \) we see that \( D \) is multiplicative \((\alpha, \beta)\)-derivation and hence the above observations and assumption imply that \( D \) is a linear \((\alpha, \beta)\)-derivation. Now, the result follows from Lemma 3.3. \( \blacksquare \)

Now we want extend Theorem 3.4 for the case \( D \) is not necessarily scalar multiplicative preserving maps.

**Theorem 3.5** Let \( X \) be a Banach space with \( \text{dim}X > 1 \) and \( A \) be a standard operator algebra on \( X \). Assume that \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function satisfying \( \lim_{t \to \infty} \frac{\phi(t)}{t} = 0 \).

Suppose \( \alpha : A \to A \) is a ring automorphism and \( \beta : A \to A \) be a ring automorphism and suppose that \( D : A \to B(X) \) is a mapping such that

\[
\| D(AB) - \beta(A)D(B) - D(A)\alpha(B) \| < \phi(\| A \||\| B \|) \quad (A, B \in A).
\]

Then \( D \) is multiplicative \((\alpha, \beta)\)-derivation.

**Proof.** Let us define a mapping \( \phi : A \to B(X \oplus X) \) by

\[
\phi(A) = \begin{pmatrix} \beta(A) & D(A) \\ 0 & \alpha(A) \end{pmatrix}
\]

We have

\[
\phi(AB) - \phi(A)\phi(B)
\]

\[
= \begin{pmatrix} \beta(AB) & D(AB) \\ 0 & \alpha(AB) \end{pmatrix} - \begin{pmatrix} \beta(A) & D(A) \\ 0 & \alpha(A) \end{pmatrix} \begin{pmatrix} \beta(B) & D(B) \\ 0 & \alpha(B) \end{pmatrix}
\]

\[
= \begin{pmatrix} \beta(AB) & D(AB) \\ 0 & \alpha(AB) \end{pmatrix} - \begin{pmatrix} \beta(A)\beta(B) & \beta(A)D(B) + D(A)\alpha(B) \\ 0 & \alpha(A)\alpha(B) \end{pmatrix}
\]

Consequently

\[
\| \phi(AB) - \phi(A)\phi(B) \| = \left\| \begin{pmatrix} 0 & D(AB) - \beta(A)D(B) - D(A)\alpha(B) \\ 0 & 0 \end{pmatrix} \right\|
\]
\[= \sup(||0|| + ||D(AB) - \beta(A)D(B) - D(A)\alpha(B)||, ||0|| + ||0||)\]

\[= ||D(AB) - \beta(A)D(B) - D(A)\alpha(B)|| < \phi(||A|| ||B||) \quad (A, B \in \mathcal{A}).\]

So we have
\[
(\phi(AB) - \phi(A)\phi(B)) \phi(C)
\]

\[
= \begin{pmatrix} 0 & D(AB) - \beta(A)D(B) - D(A)\alpha(B) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta(C) \\ 0 \end{pmatrix} D(C)
\]

\[= \begin{pmatrix} 0 & (D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C) \\ 0 & 0 \end{pmatrix} .\]

It follows that for arbitrary \(A, B, C \in \mathcal{A}\), we have

\[
||D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C)||
\]

\[
= \left| \begin{pmatrix} 0 & (D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C) \\ 0 & 0 \end{pmatrix} \right|
\]

\[= ||(\phi(AB) - \phi(A)\phi(B))\phi(C)||
\]

\[= ||\phi(AB)\phi(C) - \phi(A)\phi(BC) + \phi(A)\phi(BC) - \phi(A)\phi(B)\phi(C)||
\]

\[\leq ||\phi(AB)\phi(C) - \phi(A)\phi(BC)|| + ||\phi(A)\phi(BC) - \phi(A)\phi(B)\phi(C)||
\]

\[= ||\phi(AB)\phi(C) - \phi(ABC) + \phi(ABC) - \phi(A)\phi(BC)|| + ||\phi(A)\phi(BC) - \phi(A)\phi(B)\phi(C)||
\]

\[\leq ||\phi(AB)\phi(C) - \phi(ABC)|| + ||\phi(ABC) - \phi(A)\phi(BC)|| + ||\phi(A)|| ||\phi(BC) - \phi(B)\phi(C)||
\]

\[< \phi(||AB|| ||C||) + \phi(||A|| ||BC||) + ||\phi(A)|| ||\phi(BC) - \phi(B)\phi(C)||.
\]

Replacing \(C\) by \(tC\) in which \(t\) is a positive real number;

\[||D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(tC)||
\]

\[< \phi(t||AB|| ||C||) + \phi(t||A|| ||BC||) + ||\phi(A)|| ||\phi(BC) - \phi(B)\phi(C)||.
\]

Dividing the above inequality by \(t\), we get

\[0 \leq \frac{||D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C)||}{\phi(A)} < \frac{\phi(t||AB|| ||C||)}{t} + \frac{\phi(t||A|| ||BC||)}{t} + \frac{||\phi(A)||}{t}.
\]

Taking \(t\) to infinity we have

\[||D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C)|| = 0.
\]

\[(D(AB) - \beta(A)D(B) - D(A)\alpha(B))\alpha(C) = 0.
\]

For any arbitrary \(E \in \mathcal{A}\), there exists \(C \in \mathcal{A}\), such that \(\alpha(C) = E\), since \(\alpha\) is onto. Hence
for any arbitrary $E \in A \subseteq F(X)$, we have $(D(AB) - D(A)\alpha(B) - \beta(A)D(B))E = 0$. Therefore $(D(AB) - D(A)\alpha(B) - \beta(A)D(B))F(X) = 0$.

So with the similar argument in Theorem 3.4 with application Hahn Banach theorem we have:

$$D(AB) - D(A)\alpha(B) - \beta(A)D(B) = 0.$$  

$$D(AB) = D(A)\alpha(B) + \beta(A)D(B).$$

as desired.

\textbf{Corollary 3.6} Let $X$ and $A$ and $\phi$ be the same as in Theorem 3.5 and $D : A \rightarrow B(X)$ be a mapping satisfies

$$\|D(AB) - AD(B) - D(A)B\| < \phi(\|A\|, \|B\|)$$  

$(A, B \in A)$

then $D$ is a multiplicative derivation.

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\textbf{References}