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Positive solution of non-square fully Fuzzy linear system of equation in general form using least square method

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Abstract. In this paper, we propose the least-squares method for computing the positive solution of a $m \times n$ fully fuzzy linear system (FFLS) of equations, where $m > n$, based on Kaffman's arithmetic operations on fuzzy numbers that introduced in [18]. First, we consider all elements of coefficient matrix are non-negative or non-positive. Also, we obtain 1-cut of the fuzzy number vector solution of the non-square FFLS of equations by using pseudoinverse. If 1-cuts vector is non-negative, we solve constrained least squares problem for computing left and right spreads. Then, in the special case, we consider 0 is belong to the support of some elements of coefficient matrix and solve three overdetermined linear systems and if the solutions of these systems held in non-negative fuzzy solutions then we compute the solution of the non-square FFLS of equations. Else, we solve constrained least squares problem for obtaining an approximated non-negative fuzzy solution. Finally, we illustrate the efficiency of the proposed method by solving some numerical examples.

Keywords: Fuzzy linear system; Fuzzy number; Ranking Function; Fuzzy number vector solution.

1. Introduction

In different areas of science and engineering, there are several linear equation systems. Because of complexity of real world systems, imprecision are often unsolved. In describing uncertain data fuzzy systems are a natural ways. Fuzzy concept was first introduced by Zadeh [24, 25]. So we must solve linear systems with parameters which all or some of them are fuzzy. We have several numerical methods for solving fuzzy linear systems including Jacobi, Gauss-Seidel, Adomian decomposition method and Successive Over Relaxation (SOR) iterative method [1–4]. Cheng's ranking function introduced in [7].

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Then Kumar in [19] obtained exact solution of fully fuzzy linear system by solving a linear programming. In [15], approximated solutions are obtained using Cheng's ranking function and constrained least squares problem. Recently, Ezzati et. al. in [26] obtained nonnegative solution of FFLS of equations in general form by using ranking function.
In this paper, first, we obtain equivalent system including three determined crisp linear systems to given non-square FFLS. Then, by an theorem, we present sufficient conditions for the existence and uniqueness of nonnegative least-squares solution of the given FFLS. Also, we obtain positive fuzzy solution of FFLS using pseudoinverse and constrained least square problem.
This paper is organized as follows:
In Section 2, the basic concepts are brought which are used throughout the paper. In Section 3, the main Section of the paper, a new approach based constrained least square problem for solving FFLS, is suggested. The proposed idea is illustrated by solving some numerical examples in the Section 4. Finally conclusion is drawn in Section 5.

2. Preliminaries

In this section, we give some basic definitions of fuzzy numbers.

Definition 2.1 A fuzzy number is a fuzzy set $\mu_{\tilde{A}} : R \longrightarrow I = [0,1]$ which satisfies:

1. $\mu_{\tilde{A}}$ is upper semi continuous.
2. $\mu_{\tilde{A}}(x) = 0$outside some interval $[c,d]$.
3. There are real numbers $a, b : c \leq a \leq b \leq d$ for which
   a. $\mu_{\tilde{A}}(x)$ is monotonic increasing on $[c,a]$,
   b. $\mu_{\tilde{A}}(x)$ is monotonic decreasing on $[b,d]$,
   c. $\mu_{\tilde{A}}(x) = 1, \quad a \leqslant x \leqslant b$.

The set of all fuzzy numbers (as given by Definition (2.1)) is denoted by $F(\Re)^1$. An alternative definition or parametric form of a fuzzy number which yields the same $F(\Re)^1$ is given by Kaleva [16].

Definition 2.2 A fuzzy number $\tilde{A}$ is $LR-$type if there exit L (for left) and R (for right) and scalars $\alpha > 0$, $\beta > 0$ with

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), \; x \leqslant \alpha, \\ R(\frac{x-a}{\beta}), \; x \geqslant a \end{cases} \tag{1}$$

where L and R are strictly decreasing functions defined on $[0,1]$ and satisfy the conditions:

$$\begin{aligned} &L(0) = R(0) = 1, \\ &L(1) = R(1) = 0, \\ &0 < L(x) < 1, \qquad 0 < R(x) < 1, \, x \neq 0, 1. \end{aligned} \tag{2}$$

The mean value of $\tilde{A}$, m, is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Definition 2.3 $\tilde{M} = (m, \alpha, \beta)_{LR}$ is a triangular fuzzy number if $L = R = max(0, 1-x)$.

We denote the set of triangular fuzzy numbers by $F(\Re)^1_T$.

In this paper, we use ranking function $R(\widetilde{x}) = \sqrt{x_0^2 + y_0^2}$ introduced by Cheng [7] which is based on centroid point where for any triangular fuzzy number $\widetilde{x} = (m, \alpha, \beta)$, x_0 and y_0 are as follows:

$$x_0 = m + \tfrac{1}{3}(\beta - \alpha),$$
$$y_0 = \tfrac{1}{3}\left(\tfrac{6m+(\beta-\alpha)}{4m+(\beta-\alpha)}\right).$$

Remark 2.1 According to Eq. (1) and Definition (2.3), throughout the paper, we assume that all the fuzzy numbers are triangular in the form $(m - \alpha, m, m + \beta)$.

Arithmetic operations between two triangular fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are reviewed [18].

Theorem 2.4 [17], Let $\tilde{M} = (a, b, c)$ and $\tilde{N} = (x, y, z)$ are two arbitrary triangular fuzzy numbers and $\lambda > 0$ is a real number. Then

(1) $\tilde{M} \oplus \tilde{N} = (a + x, b + y, c + z)$,
(2) $-\tilde{M} = (-c, -b, -a)$,
(3) $\tilde{M} \ominus \tilde{N} = (a - z, b - y, c - x)$,
(4) Let $\tilde{M} = (a, b, c)$ be any triangular fuzzy number and $\tilde{N} = (x, y, z)$ be a non-negative triangular fuzzy number, then

$$\tilde{M} \otimes \tilde{N} = \begin{cases} (ax, by, cz),\ a \geqslant 0, \\ (az, by, cz),\ a < 0, \quad c \geqslant 0, \\ (az, by, cx),\ c < 0. \end{cases}$$

Definition 2.5 A matrix $\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^n$ is called a fuzzy matrix if for all i and j, $\tilde{a}_{ij} \in F(\Re)_T^1$. $\widetilde{A}$ will be positive (negative) and denoted by $\widetilde{A} > 0$ ($\widetilde{A} < 0$) if for all i and j, $\widetilde{a}_{ij} > 0$ ($\widetilde{a}_{ij} < 0$). Clearly, $\widetilde{N} = (a, b, c)$ is positive (negative), if and only if $a > 0$ ($c < 0$). Non-negative and non-positive fuzzy matrices will be defined similarly.

Definition 2.6 A vector $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$, denoted by $\tilde{X} \in F(\Re)_T^n$, is called a fuzzy numbers vector, where $\tilde{x}_i \in F(\Re)_T^1$, $i = 1, \ldots, n$.

Definition 2.7 [22], Consider an $m \times n$ system of linear equations given by

$$Ax = b \qquad (A \in M_{mn}, b \in R^M) \tag{3}$$

If $m > n$, the system is called overdetermined. An overdetermined system typically has no solution, i.e, typically, there is no $x \in R^n$ such that

$$Ax = b, \quad or \quad b - Ax = 0.$$

when there is no exact solution, we can form the residual

$$r(x) = b - Ax, \qquad (x \in R^n)$$

and seek a vector $x \in R^n$ for which

$$\parallel r(x) \parallel = \parallel b - Ax \parallel$$

is a minimum.

Definition 2.8 [22], A vector x is minimizes $\parallel r(x) \parallel_2$ is called a least-squares solution to the system (3). The least-squares solution x which has the minimum 2-norm is called the minimum norm least-squares solution, i.e, if z is any other least-squares solution to the system $Ax = b$, then we must have

$$\parallel x \parallel_2 \leqslant \parallel z \parallel_2 .$$

It can be shown that the minimum norm least-squares solution to the system (3) is given by

$$x = A^{\dagger} b$$

where $A^{\dagger}$ is the pseudoinverse of A.

Definition 2.9 [22], Let A is any real $m \times n$ matrix. The pseudoinverse of A is an $n \times m$ matrix X satisfying the following More-Penrose conditions:
$AXA = A$
$XAX = X$
$(AX)^T = AX$
$(XA)^T = XA$

Theorem 2.10 [22], Let A be any real $m \times n$ matrix. Then:
The pseudoinverse of A is unique.
The pseudoinverse of the pseoduinverse of A is A, i.e, $(A^{\dagger})^{\dagger} = A$.
The pseudoinverse of A^T is the transpose of the pseudoinverse of A ,i.e, $(A^T)^{\dagger} = (A^{\dagger})^T$.

Theorem 2.11 [22], Consider the linear system

$$Ax = b \tag{4}$$

where A is a real $m \times n$ matrix, where $m \geqslant n$, and $b \in \mathbb{R}^n$. Then
The linear system (4) has infinitely many least-squares solutions if and only if A is rank-deficient.
The linear system (4) has a unique least-squares solutions x if and only if A has full rank.
When A has full rank, the unique least-squares solution to the system (4) is given by

$$x = A^{\dagger} b = (A^T A)^{-1} A^T b. \tag{5}$$

3. Solutions of non-square, $m \times n$, FFLS

Definition 3.1 The $m \times n$ linear system

$$\begin{cases} (\widetilde{a}_{11} \otimes \widetilde{x}_1) \oplus (\widetilde{a}_{12} \otimes \widetilde{x}_2) \oplus \cdots \oplus (\widetilde{a}_{1n} \otimes \widetilde{x}_n) = \widetilde{b}_1 \\ (\widetilde{a}_{21} \otimes \widetilde{x}_1) \oplus (\widetilde{a}_{22} \otimes \widetilde{x}_2) \oplus \cdots \oplus (\widetilde{a}_{2n} \otimes \widetilde{x}_n) = \widetilde{b}_2 \\ \vdots \\ (\widetilde{a}_{m1} \otimes \widetilde{x}_1) \oplus (\widetilde{a}_{m2} \otimes \widetilde{x}_2) \oplus \cdots \oplus (\widetilde{a}_{mn} \otimes \widetilde{x}_n) = \widetilde{b}_m, \end{cases} \tag{6}$$

or in its matrix form,

$$\widetilde{A} \otimes \widetilde{x} = \widetilde{b}, \tag{7}$$

where $m > n$ is called a FFLS of equations, where the coefficient matrix $\tilde{A} = [\tilde{a}_{ij}]$ is a $m \times n$ fuzzy matrix and $\tilde{b} = [\tilde{b}_1, \ldots, \tilde{b}_m]^T$ is a fuzzy number vector and the fuzzy number vector $\widetilde{x}$ is the unknown to be found.

Definition 3.2 We call $\tilde{x} \in F(\Re)^n_T$ a solution of $\tilde{A} \otimes \tilde{x} = \tilde{b}$ with respect to the ranking function R if and only if we have,

$$\begin{cases} By = b_2, \\ R(\tilde{a}_i^T \tilde{x}) = R(\tilde{b}_i), \quad i = 1, \ldots, n \end{cases}$$

where the crisp linear system $By = b_2$ is the $1-$cut or mean value of system $\tilde{A} \otimes \tilde{x} = \tilde{b}$ and $\tilde{a}_i^T$ is the $i-$th row of A.

Notation 3.3 Set

$$\widetilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}), \quad \widetilde{b}_i = (d_{i1}, d_{i2}, d_{i3}), \quad \widetilde{x}_i = (x_i, y_i, z_i),$$

$$A = \left(a_{ij}\right)_{m\times n}, \quad B = \left(b_{ij}\right)_{m\times n}, \quad C = \left(c_{ij}\right)_{m\times n},$$

$$d_1 = \left(d_{i1}\right)_{m\times 1}, \quad d_2 = \left(d_{i2}\right)_{m\times 1}, \quad d_3 = \left(d_{i3}\right)_{m\times 1}.$$

Notation 3.4 We break up the matrix A into two $m\times n$ matrices such that their additions is A. Let $A^{+} = \left(a^{+}_{ij}\right)_{m\times n}$ and $A^{-} = \left(a^{-}_{ij}\right)_{m\times n}$, where

$$a^{+}_{ij} = \begin{cases} a_{ij} & a_{ij} \geqslant 0 \\ 0 & a_{ij} < 0 \end{cases}, \qquad a^{-}_{ij} = \begin{cases} 0 & a_{ij} \geqslant 0 \\ a_{ij} & a_{ij} < 0 \end{cases}.$$

Then $A^{+} + A^{-} = A$. We also break up the matrix C into two $m \times n$ matrices, similarly.

Let $C^{+}=\left(c_{ij}^{+}\right)$, and $C^{-}=\left(c_{ij}^{-}\right)$ be $m\times n$ matrices, where

$$c_{ij}^{+}=\begin{cases} c_{ij} & c_{ij}\geqslant 0 \\ 0 & c_{ij}<0 \end{cases},\qquad c_{ij}^{-}=\begin{cases} 0 & c_{ij}\geqslant 0 \\ c_{ij} & c_{ij}<0 \end{cases}.$$

Theorem 3.5 The system (7) with the multiplication defined in Theorem 2.4 is equivalent to

$$\begin{cases} \begin{pmatrix} A^{+} & A^{-} \\ C^{-} & C^{+} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}=\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}, \\ \\ By=d_2, \end{cases} \tag{8}$$

where $\widetilde{x}_j=(x_j,y_j,z_j)$, $j=1,\ldots,n$, are nonnegative and $x=(x_1,\ldots,x_n)^T$, $y=(y_1,\ldots,y_n)^T$, $z=(z_1,\ldots,z_n)^T$.

Proof. Using the multiplication defined in Theorem 2.4, we have

$$\begin{aligned}
(\widetilde{A}\otimes\widetilde{x})_i &= \textstyle\sum_{j=1}^{n}(\widetilde{a}_{ij}\otimes\widetilde{x}_j) \\
&= \textstyle\sum_{j=1}^{n}(a_{ij},b_{ij},c_{ij})\otimes(x_j,y_j,z_j) \\
&= \textstyle\sum_{j,a_{ij}\geqslant 0}(a_{ij}x_j,b_{ij}y_j,c_{ij}z_j)+\sum_{j,a_{ij}<0,c_{ij}\geqslant 0}(a_{ij}z_j,b_{ij}y_j,c_{ij}z_j) \\
&\quad +\textstyle\sum_{j,c_{ij}<0}(a_{ij}z_j,b_{ij}y_j,c_{ij}x_j) \\
&= \left(\textstyle\sum_{j,a_{ij}\geqslant 0}a_{ij}x_j+\sum_{j,a_{ij}<0}a_{ij}z_j,\sum_{j=1}^{n}b_{ij}y_j,\sum_{j,c_{ij}\geqslant 0}c_{ij}z_j+\sum_{j,c_{ij}<0}c_{ij}x_j\right)
\end{aligned}$$

■

Proposition 3.6 Suppose that the matrices $M=\begin{pmatrix} A^{+} & A^{-} \\ C^{-} & C^{+} \end{pmatrix}$ and B have full rank and $x=(x_1,x_2,...,x_n)^T$, $y=(y_1,y_2,...,y_n)^T$ and $z=(z_1,z_2,...,z_n)^T$ be the solution of (3.8). Then $(\widetilde{x}_1,\ldots,\widetilde{x}_n)^T$ given by $\widetilde{x}_j=(x_j,y_j,z_j)$, $j=1,\ldots,n$, is a nonnegative unique fuzzy least-squares solution of (7) if it satisfies $0\leqslant x_i\leqslant y_i\leqslant z_i$, $i=1,\ldots,n$.

Proof. Using Theorem 3.5, the proof is clear. ■

According to Theorem 3.5, we know that the system (7), $\widetilde{A}\otimes\widetilde{x}=\widetilde{b}$, is equivalent to

$$(A^{+}x+A^{-}z,By,C^{-}x+C^{+}z)=\widetilde{b}=(d_1,d_2,d_3). \tag{9}$$

Clearly, we have:

$$x=(A^{+})^{\dagger}(d_1-A^{-}z). \tag{10}$$

By substituting above equation in $C^{-}x+C^{+}z=d_3$, we conclude that

$$(C^{+}-C^{-}(A^{+})^{\dagger}A^{-})z=d_3-C^{-}(A^{+})^{\dagger}d_1\Longrightarrow z=(C^{+}-C^{-}(A^{+})^{\dagger}A^{-})^{\dagger}(d_3-C^{-}(A^{+})^{\dagger}d_1). \tag{11}$$

So, we proved the following theorem:

Theorem 3.7 The system (7) with the multiplication defined in Theorem 2.4 is equivalent to

$$\begin{cases} A^+x = d_1 - A^-(C^+ - C^-(A^+)^\dagger A^-)^\dagger(d_3 - C^-(A^+)^\dagger d_1), \\ (C^+ - C^-(A^+)^\dagger A^-)z = d_3 - C^-(A^+)^\dagger d_1 \\ By = d_2, \end{cases} \tag{12}$$

where $\widetilde{x}_j = (x_j, y_j, z_j)$, $j = 1, \ldots, n$, are nonnegative and $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$, $z = (z_1, \ldots, z_n)^T$. □

Now, we present sufficient conditions for the existence and uniqueness of nonnegative least-squares solution of (7) as follows:

Theorem 3.8 Suppose that the matrices A^+, $C^+ - C^-(A^+)^\dagger A^-$ and B have full rank and $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ and $z = (z_1, z_2, ..., z_n)^T$ be the solution of (3.12) or (3.8). Then $(\widetilde{x}_1, \ldots, \widetilde{x}_n)^T$ given by $\widetilde{x}_j = (x_j, y_j, z_j)$, $j = 1, \ldots, n$, is a nonnegative unique fuzzy least-squares solution of (7) if it satisfies $0 \leqslant x_i \leqslant y_i \leqslant z_i$, $i = 1, 2, ..., n$, where

$$y = (B^+)^\dagger d_2,$$

$$z = (C^+ - C^-(A^+)^\dagger A^-)^\dagger(d_3 - C^-(A^+)^\dagger d_1),$$

$$x = (A^+)^\dagger(d_1 - A^-z).$$

Proof. Using Theorems 2.11 and 3.7, the proof is clear. ■

To solve (7), we first solve system $By = d_2$. Clearly, we obtain $y = B^\dagger d_2$. Now, if the solution vector y is negative, the system hasn't the positive fuzzy solution, else, (y is non negative) for obtaining left and right spreads, we solve the following constrained least-squares problem:

$$\begin{aligned} &min \textstyle\sum_{i=1}^n (R((A^+x + A^-z)_i \ , \sum_{j=1}^n b_{ij}y_j \ , (C^-x + C^+z)_i) - R(\tilde{b}_i))^2 \\ &s.t. \quad x_i \geqslant 0, && i = 1, \ldots, n, \\ &\qquad\; y_i - x_i \geqslant 0, && i = 1, \ldots, n, \\ &\qquad\; z_i - y_i \geqslant 0, && i = 1, \ldots, n. \end{aligned} \tag{13}$$

or

$$\begin{aligned} &min \textstyle\sum_{i=1}^n \{R((A^+x)_i \ , \sum_{j=1}^n b_{ij}y_j \ , ((C^+ - C^-(A^+)^\dagger A^-)z)_i) - R(\Gamma_i)\}^2 \\ &s.t. \quad x_i \geqslant 0, && i = 1, \ldots, n, \\ &\qquad\; y_i - x_i \geqslant 0, && i = 1, \ldots, n, \\ &\qquad\; z_i - y_i \geqslant 0, && i = 1, \ldots, n, \end{aligned} \tag{14}$$

where $\Gamma_i = (((d_1 - A^-(C^+ - C^-(A^+)^\dagger A^-)^\dagger(d_3 - C^-(A^+)^\dagger d_1))_i, (d_2)_i, (d_3 - C^-(A^+)^\dagger d_1))_i)$

Remark 3.1 Let for all i and j, $0 \in supp(\tilde{a}_{ij})$, i.e., for all i and j, $a_{ij} < 0$ and $c_{ij} \geqslant 0$. Let A, B and C have full rank. Using Theorem (2.4), the system (7) is transformed to the following form:

$$\left(\sum_{j=1}^{n} a_{ij} z_j, \sum_{j=1}^{n} b_{ij} y_j, \sum_{j=1}^{n} c_{ij} z_j\right) = (d_{i1}, d_{i2}, d_{i3}), \quad i = 1, \ldots, n. \tag{15}$$

Also, we can rewrite (15) in the matrix form as:

$$(Az, By, Cz) = (d_1, d_2, d_3). \tag{16}$$

$$(z, y, z) = (A^{\dagger} d_1, B^{\dagger} d_2, C^{\dagger} d_3). \tag{17}$$

It is clear that, the least-squares solution ,y, is unique, but there are two vectors for z, i.e., $z_1 = A^{\dagger} d_1$ and $z_2 = C^{\dagger} d_3$. In order to find the fuzzy number vector solution, we have the following cases:

(I) Let one of the two vectors z_1 or z_2 satisfy in the conditions $0 \leqslant y \leqslant z_1$ or $0 \leqslant y \leqslant z_2$, respectively. For example, without the loss generality, suppose z_1 satisfies in the condition $0 \leqslant y \leqslant z_1$, so we set $x = y$, and therefore we obtain (y, y, z_1) as a positive fuzzy number vector solution of (7).

(II) Let both of the two vector z_1 and z_2 satisfy in the conditions $0 \leqslant y \leqslant z_1$ and $0 \leqslant y \leqslant z_2$, respectively. In this case, we set $x = y$. So, two positive fuzzy number vectors $\widetilde{S}_1 = (y, y, z_1)$ and $\widetilde{S}_2 = (y, y, z_2)$ are solutions of (7). To choose a solution for (7) from $\{\widetilde{S}_1, \widetilde{S}_2\}$, we will use a distance function to measure the closeness of the vectors $\widetilde{A}\widetilde{S}_1$ and $\widetilde{A}\widetilde{S}_2$ to $\widetilde{b}$. To this end, we use the Ming et. al. [19] metric proposed for triangular fuzzy numbers. If $\tilde{x} = (x, \alpha_x, \beta_x)$ and $\tilde{y} = (y, \alpha_y, \beta_y)$ are two triangular fuzzy numbers, then Ming et al. [19] introduced the distance function,

$$D_2^2(\tilde{x}, \tilde{y}) = \frac{1}{2}\left(4(x-y)^2 + (\alpha_y - \alpha_x)^2 + (\beta_y - \beta_x)^2\right) + (x-y)(\beta_y + \alpha_y - \beta_x - \alpha_x), \tag{18}$$

and for two LR fuzzy vectors $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ and $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ defined the distance between $\tilde{x}$ and $\tilde{y}$ to be,

$$D_n^2(\tilde{x}, \tilde{y}) = \sum_{i=1}^{n} D_2^2(\tilde{x}_i, \tilde{y}_i). \tag{19}$$

Using (19), we obtain $D_n(\widetilde{A}\widetilde{S}_1, \widetilde{b})$ and $D_n(\widetilde{A}\widetilde{S}_2, \widetilde{b})$ and choose the nearest solution to $\widetilde{b}$.

(III) Let vectors y or z_1 and z_2 don't hold in condition $y \geqslant 0$ or $y \leqslant z_1$ and $y \leqslant z_2$, respectively. In this case, we solve the constrained least squares problem (13) to obtain the approximation of positive fuzzy number vector solution of (7).

4. Numerical examples

***Example* 4.1** Consider the following system:

$$\begin{cases} (-2,1,3)\otimes(x_1,y_1,z_1)\oplus(2,3,4)\otimes(x_2,y_2,z_2) & \oplus(1,1,2)\otimes(x_3,y_3,z_3) \\ & =(7,8,9) \\ (3,4,5)\otimes(x_1,y_1,z_1)\oplus(-1,0,1)\otimes(x_2,y_2,z_2) & \oplus(1,1,1)\otimes(x_3,y_3,z_3) \\ & =(13,14,15) \\ (3,4,7)\otimes(x_1,y_1,z_1)\oplus(-3,-2,-1)\otimes(x_2,y_2,z_2) & \oplus(-4,0,1)\otimes(x_3,y_3,z_3) \\ & =(9,10,11) \\ (1,2,5)\otimes(x_1,y_1,z_1)\oplus(3,4,6)\otimes(x_2,y_2,z_2) & \oplus(-1,1,2)\otimes(x_3,y_3,z_3) \\ & =(5,12,12) \end{cases} \tag{20}$$

Using constrained least-squares problem (13), we have

$$\begin{bmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} (2.96,3,4.23) \\ (1,1,2) \\ (1.88,2,2) \end{bmatrix}$$

as a positive fuzzy number vector solution of (20).

***Example* 4.2** Consider the following system:

$$\begin{cases} (2,3,6)\otimes(x_1,y_1,z_1)\oplus(-1,0,1)\otimes(x_2,y_2,z_2) & \oplus(-10,-5,-1)\otimes(x_3,y_3,z_3) \\ & =(10,13,15) \\ (5,8,10)\otimes(x_1,y_1,z_1)\oplus(1,3,11)\otimes(x_2,y_2,z_2) & \oplus(6,8,12)\otimes(x_3,y_3,z_3) \\ & =(60,65,76) \\ (-2,1,3)\otimes(x_1,y_1,z_1)\oplus(-4,-3,-1)\otimes(x_2,y_2,z_2) & \oplus(7,9,10)\otimes(x_3,y_3,z_3) \\ & =(0,6,10) \\ (6,8,10)\otimes(x_1,y_1,z_1)\oplus(5,9,15)\otimes(x_2,y_2,z_2) & \oplus(-3,0,3)\otimes(x_3,y_3,z_3) \\ & =(65,75,77) \end{cases} \tag{21}$$

Using constrained least-squares problem (13), we have

$$\begin{bmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} (0,6,6) \\ (3,3,4/39) \\ (1,1,1/64) \end{bmatrix}$$

as a positive fuzzy number vector solution of (21).

***Example* 4.3** Consider the following system:

$$\begin{cases} (3,4,6)\otimes(x_1,y_1,z_1)\oplus(7,9,10)\otimes(x_2,y_2,z_2) & \oplus(-1,3,6)\otimes(x_3,y_3,z_3) \\ & =(43,101,178) \\ (-3,0,3)\otimes(x_1,y_1,z_1)\oplus(1,2,3)\otimes(x_2,y_2,z_2) & \oplus(6,9,10)\otimes(x_3,y_3,z_3) \\ & =(2,43,111) \\ (-10,0,10)\otimes(x_1,y_1,z_1)\oplus(1,3,6)\otimes(x_2,y_2,z_2) & \oplus(-10,-7,-1)\otimes(x_3,y_3,z_3) \\ & =(-35,3,124) \\ (3,6,9)\otimes(x_1,y_1,z_1)\oplus(8,9,11)\otimes(x_2,y_2,z_2) & \oplus(5,6,8)\otimes(x_3,y_3,z_3) \\ & =(54,120,221) \end{cases} \tag{22}$$

Using (9), we have

$$\begin{bmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} (3,5,7) \\ (5,8,10) \\ (1,3,6) \end{bmatrix}$$

as a positive fuzzy number vector solution of (22).

***Example* 4.4** Consider the following system:

$$\begin{cases} (1,2,4)\otimes(x_1,y_1,z_1)\oplus(3,4,7)\otimes(x_2,y_2,z_2) & \oplus(7,9,10)\otimes(x_3,y_3,z_3) \\ & =(24,49,95) \\ (2,5,6)\otimes(x_1,y_1,z_1)\oplus(0,1,3)\otimes(x_2,y_2,z_2) & \oplus(3,4,6)\otimes(x_3,y_3,z_3) \\ & =(8,31,69) \\ (4,5,7)\otimes(x_1,y_1,z_1)\oplus(-1,0,2)\otimes(x_2,y_2,z_2) & \oplus(2,4,8)\otimes(x_3,y_3,z_3) \\ & =(5,27,77) \\ (-2,-1,0)\otimes(x_1,y_1,z_1)\oplus(1,3,4)\otimes(x_2,y_2,z_2) & \oplus(5,6,9)\otimes(x_3,y_3,z_3) \\ & =(11,27,56) \end{cases} \tag{23}$$

Using (9), we have

$$\begin{bmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \end{bmatrix} = \begin{bmatrix} (1,3,5) \\ (3,4,5) \\ (2,3,4) \end{bmatrix}$$

as a positive fuzzy number vector solution of (23).

5. Conclusion

In this paper, we found the positive fuzzy number vector solution for the non-square fully fuzzy linear system of equations by considering multiplication of fuzzy numbers that introduced by Kaffman in [18]. First, we considered all elements of coefficient matrix are non-negative or non-positive. We solved a non-square system to obtain 1-cut of the fuzzy number vector solution of the non-square fully fuzzy linear system of equations and if obtained solutions are non-negative then by solving constrained least squares problem left and right spreads are computed. Otherwise, i.e. if obtained solutions are non-positive, there is not any positive fuzzy solution for the non-square fully fuzzy linear system of equations. In the second case, we considered 0 is belong to the support of some elements of coefficient matrix. According to multiplication of fuzzy numbers that introduced by Kaffman in [18], the system (15) is obtained that lead to create three determined systems and if the solutions of these systems hold in non-negative fuzzy solutions so, the solution of the non-square FFLS of equations is computed. Else, by solving constrained least squares problem, we obtained an approximated non-negative fuzzy solution. We showed the capability and efficiency of our proposed method by solving some numerical examples.

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