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Generalized f-clean rings

S. Jamshidvand[a*], H. Haj Seyyed Javadi[a] and N. Vahedian Javaheri[a]

[a] *Department of Mathematics, Shahed University, Tehran, Iran.*

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Abstract. In this paper, we introduce the new notion of n-f-clean rings as a generalization of f-clean rings. Next, we investigate some properties of such rings. We prove that $M_n(R)$ is n-f-clean for any n-f-clean ring R. We also, get a condition under which the definitions of n-cleanness and n-f-cleanness are equivalent.

Keywords: Full element, clean ring, n-clean ring, n-f-clean ring.

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1. Introduction

Throughout this papar, all rings are associative rings with identity. We denote the set of all invetrible elements in R by $U(R)$, $Id(R)$ the set of idempotents, $K(R)$ the set of all full elements and n be a positive integer. Nicholson (1977) introduced the notion of clean ring [7]. A ring R is called clean ring if every element of R can be written as the sum of a unit and an idempotent in R. Such rings constitute a subclass of exchange rings in the theory of noncommutative rings. Following Nicholson [7], R is an exchage ring if and only if for any element a in R there exist an idempotent $e \in R$ such that $e \in Ra$ and $(1-e) \in R(1-a)$. In [8], a ring R is called n-clean ring if for any $x \in R$, $x = e + u_1 + ... + u_n$ where $e \in Id(R)$ and $u_i \in U(R)(1 \leqslant i \leqslant n)$. Clean rings are 1-clean ring. In [6], they extended clean rings and introduced the concept of f-clean rings. Recall that, an element in R is said to be full element if there exist $s, t \in R$ such that $sxt = 1$. A ring R is called a f-clean ring if each element in R can be written as the sum of an idempotent and a full element [6]. Now in this paper, we generalize f-

*Corresponding author.
E-mail address: jamshidvand1367@gmail.com (S. Jamshidvand).

clean rings and denote it by n-f-clean. Example of n-f-clean rings is given. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$. We study various properties of n-f-clean rings. We prove that $M_n(R)$ is n-f-clean for any n-f-clean ring R and we also show that the notion of n-clean and n-f-clean are equvalent when R is a left quasi-dou ring.
A Morita Context (A, B, V, W, ψ, ϕ) consists two rings A, B, two bimodules ${}_AV_B$, ${}_BW_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \to A$, $\phi : W \otimes_A V \to B$, such that $\psi(v \otimes w)v' = v\phi(w \otimes v')$, $\phi(w \otimes v)w' = w\psi(v \otimes w')$. We can form

$$C = \{ \begin{bmatrix} a & v \\ w & b \end{bmatrix} | a \in A, b \in B, v \in V, w \in W\}$$

and define a multiplication on C as follows:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' + \psi(v \otimes w') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{bmatrix}.$$

A routine check shows that, with multipilication, C becomes an associative ring. We call C a Morita Context ring. Obviously, the class of the rings of Morita Contexts includes all 2×2 matrix rings [4], [5]. We get the relationship of n-f-cleaness between Morita Context ring C and A, B.
Troughout the paper, $T_n(R)$ denote the triangular matrix ring over R and, the Jacobson radical is denote by $J(R)$.

2. main results

Definition 2.1 Let n be a positive integer. An element x of R is called n-f-clean if $x = e + w_1 + ... + w_n$ where e is in $Id(R)$ and $w_1, ..., w_n$ are full elements in R. A ring R is called n-f-clean ring if every element of R is n-f-clean.

Note that f-clean rings are exactly 1-f-clean.

Proposition 2.2 Let n be a positive integer.

(1) Any homomorphic image of a n-f-clean ring is n-f-clean.
(2) A direct product $R = \prod R_i$ of rings $\{R_i\}$ is n-f-clean if and only if the same is true for each R_i.

Proof. (1) is straightforward.
(2) Suppose that each R_i is a n-f-clean ring. For any $x = (x_i) \in R$ and each i, we write $x_i = e_i + w_i^1 + ... + w_i^n$ with $e_i^2 = e_i$ and $s_i^\alpha w_i^\alpha t_i^\alpha = 1$ for some $s_i^\alpha, t_i^\alpha \in R$. Then $x = e + w^1 + ... + w^n$, where $e = (e_i)$ is in $Id(\prod R_i)$ and $w^\alpha = (w_i^\alpha) \in \prod R_i$ with $(s_i^\alpha)(w_i^\alpha)(t_i^\alpha) = (1) \in \prod R_i$. Hence, x is n-f-clean, as required. The converse follows from (1).
■

Example **2.3** If V is an infinite dimentional vector space over division ring D, then $End_D(V)$ modulo its uniqe maximal ideal M is purely infinite simple so; it is f-clean. [1], [2].
Let $M = M_1 \oplus ... \oplus M_n$ where M_i are D-modules for each i, and $End_D(M_i)$ modulo its uniqe maximal ideal M_i and $Hom(M_j, M_i) = 0$ where $i \neq j$. Then, $End_D(M)$ is n-f-clean.

It was proved by Camillo and Yu [3], that the ring R is a clean ring if and only if $\bar{R} = R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$. Now, we prove next proposition:

Proposition 2.4 Let R be a ring. If idempotents can be lifted modulo $J(R)$, then R is a n-f-clean ring if and only if $\bar{R}$ is a n-f-clean ring.

Proof. One direction is trivial by proposition 2.2(1).
Conversely, Suppose that $\bar{R}$ is a n-f-clean ring. Let $x \in R$ then, $\bar{x} = \bar{e} + \bar{w_1} + ... + \bar{w_n}$ with $e^2 - e \in J(R)$ and $\bar{s}_i \bar{w}_i \bar{t}_i = \bar{1} (1 \leqslant i \leqslant n)$ for some $s_i, t_i \in R$. Since idempotents can be lifted modulo $J(R)$, we may assume e is an idempotent and write $x = e + w_1 + ... + w_n + r$ for some $r \in J(R)$. Again, we have $s_i w_i t_i = 1 + h \in 1 + J(R) \subseteq U(R)$ fore some $h \in J(R)$. Therefore, there exist $s'_i, t'_i \in R$ such that $s'_i w_i t'_i = 1$. Hence, $s'_1(w_1 + r)t'_1 = 1 + s'_1 r t'_1 \in 1 + J(R) \subseteq U(R)$. We have $s'_1(w_1 + r)t'_1 u^{-1} = 1$ fore some $u \in U(R)$, hence $w_1 + r$ and $w_2, ..., w_n$ are full elements. Therefore, we get that x is n-f-clean, as reqiured.
■

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$, the skew power series ring $R[[x; \alpha]]$ of R is the ring obtained by giving the formal power series ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$.

Proposition 2.5 Let α be an endomorphism of R. Then the following statements are equivalent.
(1) R is a n-f-clean ring.
(2) The formal power series ring $R[[x]]$ of R is n-f-clean.
(3) The skew power series ring $R[[x; \alpha]]$ of R is n-f-clean.

Proof. (1) ⇒ (3) For any $h = a_0 + a_1 x + ... \in R[[x, \alpha]]$, write $a_0 = e + u_1 + ... + u_n$ where $e \in Id(R)$ and $u_i \in K(R)$ $(1 \leqslant i \leqslant n)$. Then $h = e + (u_1 + a_1 x + a_2 x^2 + ...) + u_2 + ... + u_n$, where $e \in Id(R) \subseteq Id(R[[x, \alpha]])$ and $u_i \in K(R) \subseteq K(R[[x, \alpha]])$. Let $h' = u_1 + a_1 x + a_2 x^2 + ...$ The equation $u = (s_1 + 0 + ...)h'(t_1 + 0 + ...) = 1 + s_1 a_1 \alpha(t_1)x + ...$ shows that $u \in U(R[[x, \alpha]])$, since $U(R[[x, \alpha]]) = \{a_0 + a_1 x + ... : a_0 \in U(R)\}$, without any assumption on the endomorphism α. Hence, $h' \in K(R[[x, \alpha]])$, as desired.
(1) ⇒ (2) Since $R[[x]] = R[[x, 1_\alpha]]$, the proof is similar to that of (1) ⇒ (3), as desired.
■

Theorem 2.6 If R is a n-f-clean ring, then $M_n(R)$ is also a n-f-clean ring for any $n \geqslant 1$.

Proof. Suppose that R is n-f-clean. Given any $x \in R$, there exist some $e \in Id(R)$ and $w_1, ..., w_n \in K(R)$ such that $x = e + w_1 + ... + w_n$. So for some $s_i, t_i \in R$, $s_i w_i t_i = 1 (1 \leqslant i \leqslant n)$.
Assume that theorem holds for the matrix ring $M_k(R)$, $K \geqslant 1$.
Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{k+1}(R)$$

Where $a_{11} \in R$, $a_{12} \in R^{1\times k}$, $a_{21} \in R^{k\times 1}$ and $a_{22} \in M_k(R)$.

We have $a_{11} = e + w_1 + ... + w_n$ with $e = e^2$ and $s_i w_i t_i = 1$ for some $s_i, t_i \in R (1 \leqslant i \leqslant n)$. There aslo exist an idempotent matrix E and full matrices $W_1, ..., W_n$ such that $a_{22} - a_{21} t s a_{12} = E + W_1 + ... + W_n$. By hypothesis, we write $S_i W_i T_i = I_k$ for some $S_i, T_i \in M_{k\times k}(R) (1 \leqslant i \leqslant n)$.

Therefore, we have

A= diag (e, E)+ $\begin{bmatrix} w & a_{12} \\ a_{21} & W_1 + ... + W_n + a_{21}tsa_{12} \end{bmatrix}$

= diag(e, E)+ $\begin{bmatrix} w_1 & a_{12} \\ a_{21} & W_1 + a_{21}tsa_{12} \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & W_2 \end{bmatrix} + ... + \begin{bmatrix} w_n & 0 \\ 0 & W_n \end{bmatrix}$

Obviously, diag(e,E) is an idempotent matrix and $\begin{bmatrix} w_2 & 0 \\ 0 & W_2 \end{bmatrix}, ..., \begin{bmatrix} w_n & 0 \\ 0 & W_n \end{bmatrix}$ are full matrices in $M_{k+1}(R)$. Let

$$P = \begin{bmatrix} s & 0 \\ -S_1 a_{21} ts & S_1 \end{bmatrix}, Q = \begin{bmatrix} t & -tsa_{12}T_1 \\ 0 & T_1 \end{bmatrix} \in M_{k+1}(R),$$

and the equation

$$P \begin{bmatrix} w & a_{12} \\ a_{21} & W_1 + a_{21}tsa_{12} \end{bmatrix} Q = \begin{bmatrix} 1 & 0 \\ 0 & I_k \end{bmatrix} = I_{k+1}$$

shows that $\begin{bmatrix} w & a_{12} \\ a_{21} & W_1 + a_{21}tsa_{12} \end{bmatrix}$ is a full matrix, hence A is n-f-clean, as desired.
■

Proposition 2.7 Let n be odd positive integear. If $a \in R$ is a n-f-clean element, then $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is always n-f-clean in $M_2(R)$ for any $b \in R$.

Proof.
Suppose $a = e + w_1 + ... + w_n$ where $e \in Id(R)$ and $s_i w_i t_i = 1$ for some $s_i, t_i \in R (1 \leqslant i \leqslant n)$. Then we can write A as

$$A = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & 1 \end{bmatrix} + ... + \begin{bmatrix} w_n & 0 \\ 0 & -1 \end{bmatrix}.$$

We also have

$$\begin{bmatrix} s_1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix} \begin{bmatrix} t_1 & -t_1 s_1 b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

hence $\begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix}$ is a full element and abviously, $\begin{bmatrix} w_2 & 0 \\ 0 & 1 \end{bmatrix}, ..., \begin{bmatrix} w_n & 0 \\ 0 & -1 \end{bmatrix}$ are full elements. therefore, A is a n-f-clean element.

■

We are going to investigate the n-f-cleanness between Morita Context ring C and A, B. Our concern here is the Morita Context rings with zero homomorphism.

Theorem 2.8 Let $C = \begin{bmatrix} A & V \\ W & B \end{bmatrix}$ be the Morita Context with $\psi, \varphi = 0$. Then C is n-f-clean iff A and B are n-f-clean.

Proof. Assume that C is n-f-clean with $\psi, \varphi = 0$, let $I = \begin{bmatrix} 0 & V \\ W & B \end{bmatrix}$, $J = \begin{bmatrix} A & V \\ W & 0 \end{bmatrix}$. Clearly I, J are ideals of C and $C/I \cong A$, $C/J \cong B$. From proposition 2.1, A, B are n-f-clean.

Conversely, let A, B are both n-f-clean rings. For any $r = \begin{bmatrix} a & v \\ w & b \end{bmatrix} \in C$, we have $a = e_1 + w_1 + ... + w_n$ and $b = e_2 + u_1 + ... + u_n$ for some idempotents $e_1, e_2 \in R$ and $w_1, ..., w_n, u_1, ..., u_n \in K(R)$. Assume that $s_i w_i t_i = 1$ and $s'_i u_i t'_i = 1$ for some $s_i, t_i, s'_i, t'_i \in R$ $(1 \leqslant i \leqslant n)$. Let

$$r = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} + \begin{bmatrix} w_1 & v \\ w & u_1 \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & u_2 \end{bmatrix} + ... + \begin{bmatrix} w_n & 0 \\ 0 & u_n \end{bmatrix} = E + K_1 + ... + K_n.$$

Obviously, $E \in Id(R)$ and the equation

$$\begin{bmatrix} s_1 & 0 \\ -s'_1 w t_1 s_1 & s'_1 \end{bmatrix} \begin{bmatrix} w_1 & v \\ w & u_1 \end{bmatrix} \begin{bmatrix} t_1 & -t_1 s_1 v t'_1 \\ 0 & t'_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

shows that K_1 is a full matrix. Clearly, $K_2, ..., K_n$ are full matrices. Hence, r is n-f-clean. ■

Proposition 2.9 (1) Let R, S be two rings, and M be an (R, S)-bimodule. Let $E = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ be the formal triangular matrix ring. Then, E is n-f-clean ring iff R and S are n-f-clean rings.
(2) For any $n \geqslant 1$, R is n-f-clean ring iff $T_n(R)$ is n-f-clean.

Proof. The proof is similar to [[6], proposition 2.8]. ■

Next, we will investigate the equivalence of n-f-cleanness and n-cleanness. In [9], Yu call a ring R to be a left quasi-duo ring if every maximal left ideal of R is a two-sided ideal. Commutative rings and local rings are all belong to this class of rings.

Theorem 2.10 For a left quasi-duo ring R, the following are equivalent:
(1) R is a n-clean ring;
(2) R is a n-f-clean ring.

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (1)It is suffices to show that $w_i \in K(R)$ implies that $w_i \in U(R)$ for any $(1 \leqslant i \leqslant n)$. By using the proof of [[6], theorem 2.9], the result is clear. ■

Corollary 2.11 Let R be a left quasi-duo ring and e be an idempotent element of R. If eRe and $(1-e)R(1-e)$ are both n-f-clean rings, then so is R.

Proof. The result follows from [[8], theorem 2.10] and 2.10. ■

Corollary 2.12 Every abelian n-f-clean ring is a n-clean ring.

Proof. It suffices to prove that every abelian ring is Dedekind finite by the proof of theorem 2.10. Suppose $ab = 1$, then ba is an idempotent and hence central by assumption. $ba = ba.ab = ab.ab = 1$ shows that R is Dedekind finite. Then the result follows from the proof of theorem 2.10. ■

For an idempotent e, we do not know whether the corner ring eRe is again f-clean for a f-clean ring R. But when e is a central idempotent, we can get an affirmative answer:

Proposition 2.13 Let R be a n-f-clean ring and e be a central idempotent in R. Then eRe is also n-f-clean.

Proof. We can view eRe as a homomorphic image of R since e is central, hence the result follows from proposition 2.2. ■

Let R be a ring in which 2 is invertible. Camillo and Yu [3], showed that R is clean iff every element of R is the sum of a unit and square root of 1. We extend this result for n-f-clean rings.

Proposition 2.14 Let R be a ring in which 2 is invertible, then R is n-f-clean iff every element of R is a sum of n full elements and a square root of 1.

Proof. Suppose R is n-f-clean and $x \in R$. Then $(x+1)/2 = e + u_1 + ... + u_n$ for some $e \in Id(R)$ and $u_1, ..., u_n \in K(R)$. So $x = (2e-1) + 2u_1 + ...2u_n$ with $(2e-1)^2 = 1$ and $2u_1, ..., 2u_n \in K(R)$.
Conversely, if $x \in R$ then $2x - 1 = f + u_1 + ... + u_n$ where $f^2 = 1$ and $u_1, ..., u_n \in K(R)$. Thus $x = (f+1)/2 + (u_1 + ... + u_n)/2$. It is easy to show that $e = (f+1)/2 \in Id(R)$ and $u_1/2, ..., u_n/2 \in K(R)$. Hence, x is n-f-clean. Thus R is a n-f-clean ring. ■

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