Generalized $f$-clean rings

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Abstract. In this paper, we introduce the new notion of $n$-$f$-clean rings as a generalization of $f$-clean rings. Next, we investigate some properties of such rings. We prove that $M_n(R)$ is $n$-$f$-clean for any $n$-$f$-clean ring $R$. We also, get a condition under which the definitions of $n$-cleanness and $n$-$f$-cleanness are equivalent.

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1. Introduction

Throughout this paper, all rings are associative rings with identity. We denote the set of all invertible elements in $R$ by $U(R)$, $Id(R)$ the set of idempotents, $K(R)$ the set of all full elements and $n$ be a positive integer. Nicholson (1977) introduced the notion of clean ring [7]. A ring $R$ is called clean ring if every element of $R$ can be written as the sum of a unit and an idempotent in $R$. Such rings constitute a subclass of exchange rings in the theory of noncommutative rings. Following Nicholson [7], $R$ is an exchange ring if and only if for any element $a$ in $R$ there exist an idempotent $e \in R$ such that $e \in Ra$ and $(1 - e) \in R(1 - a)$. In [8], a ring $R$ is called $n$-clean ring if for any $x \in R$, $x = e + u_1 + ... + u_n$ where $e \in Id(R)$ and $u_i \in U(R) (1 \leq i \leq n)$. Clean rings are 1-clean ring. In [6], they extended clean rings and introduced the concept of $f$-clean rings. Recall that, an element in $R$ is said to be full element if there exist $s, t \in R$ such that $sxt = 1$. A ring $R$ is called a $f$-clean ring if each element in $R$ can be written as the sum of an idempotent and a full element [6]. Now in this paper, we generalize $f$-
clean rings and denote it by $n$-$f$-clean. Example of $n$-$f$-clean rings is given. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$. We study various properties of $n$-$f$-clean rings. We prove that $M_n(R)$ is $n$-$f$-clean for any $n$-$f$-clean ring $R$ and we also show that the notion of $n$-clean and $n$-$f$-clean are equivalent when $R$ is a left quasi-dou ring.

A Morita Context $(A, B, V, W, \psi, \phi)$ consists two rings $A, B$, two bimodules $_AV_B, _BW_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \to A$, $\phi : W \otimes_A V \to B$, such that $\psi(v \otimes w)v' = \psi(v \otimes v')$, $\phi(w \otimes v)w' = w\phi(v \otimes w')$. We can form

$$C = \left\{ \begin{bmatrix} a & v \\ w & b \end{bmatrix} | a \in A, b \in B, v \in V, w \in W \right\}$$

and define a multiplication on $C$ as follows:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' + \psi(v \otimes v') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{bmatrix}.$$

A routine check shows that, with multiplication, $C$ becomes an associative ring. We call $C$ a Morita Context ring. Obviously, the class of the rings of Morita Contexts includes all $2 \times 2$ matrix rings [4], [5]. We get the relationship of $n$-$f$-cleanness between Morita Context ring $C$ and $A, B$.

Throughout the paper, $T_n(R)$ denote the triangular matrix ring over $R$ and, the Jacobson radical is denote by $J(R)$.

## 2. main results

**Definition 2.1** Let $n$ be a positive integer. An element $x$ of $R$ is called $n$-$f$-clean if $x = e + w_1 + ... + w_n$ where $e$ is in $Id(R)$ and $w_1, ..., w_n$ are full elements in $R$. A ring $R$ is called $n$-$f$-clean ring if every element of $R$ is $n$-$f$-clean.

Note that $f$-clean rings are exactly 1-$f$-clean.

**Proposition 2.2** Let $n$ be a positive integer.

(1) Any homomorphic image of a $n$-$f$-clean ring is $n$-$f$-clean.

(2) A direct product $R = \prod R_i$ of rings $\{R_i\}$ is $n$-$f$-clean if and only if the same is true for each $R_i$.

**Proof.** (1) is straightforward.

(2) Suppose that each $R_i$ is a $n$-$f$-clean ring. For any $x = (x_i) \in R$ and each $i$, we write $x_i = e_i + w_i^1 + ... + w_i^n$ with $e_i^2 = e_i$ and $s_i^a t_i^a = 1$ for some $s_i^a, t_i^a \in R$. Then $x = e + w^1 + ... + w^n$, where $e = (e_i)$ is in $Id(\prod R_i)$ and $w^a = (w_i^a) \in \prod R_i$ with $(s_i^a)(w_i^a)(t_i^a) = 1 \in \prod R_i$. Hence, $x$ is $n$-$f$-clean, as required. The converse follows from (1).

**Example 2.3** If $V$ is an infinite dimensional vector space over division ring $D$, then $End_D(V)$ modulo its unique maximal ideal $M$ is purely infinite simple so; it is $f$-clean. [1], [2].

Let $M = M_1 \oplus ... \oplus M_n$ where $M_i$ are $D$-modules for each $i$, and $End_D(M_i)$ modulo its unique maximal ideal $M_i$ and $Hom(M_j, M_i) = 0$ where $i \neq j$. Then, $End_D(M)$ is $n$-$f$-clean.
It was proved by Camillo and Yu [3], that the ring $R$ is a clean ring if and only if $\hat{R} = R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$. Now, we prove next proposition:

**Proposition 2.4** Let $R$ be a ring. If idempotents can be lifted modulo $J(R)$, then $R$ is a $n$-$f$-clean ring if and only if $\hat{R}$ is a $n$-$f$-clean ring.

**Proof.** One direction is trivial by proposition 2.2(1).

Conversely, Suppose that $\hat{R}$ is a $n$-$f$-clean ring. Let $x \in R$ then, $x = \bar{e} + \bar{w}_1 + \ldots + \bar{w}_n$ with $e^2 - e \in J(R)$ and $\bar{s}_i \bar{w}_i \bar{t}_i = 1(1 \leq i \leq n)$ for some $s_i, t_i \in R$. Since idempotents can be lifted modulo $J(R)$, we may assume $e$ is an idempotent and write $x = e + w_1 + \ldots + w_n + r$ for some $r \in J(R)$. Again, we have $s_i w_i t_i = 1 + h \in 1 + J(R) \subseteq U(R)$ for some $h \in J(R)$. Therefore, there exist $s_i', t_i' \in R$ such that $s_i' w_i t_i' = 1$. Hence, $s_i' (w_1 + r) t_i' = 1 + s_i' r t_i' \in 1 + J(R) \subseteq U(R)$. We have $s_i' (w_1 + r) t_i' u^{-1} = 1$ for some $u \in U(R)$, hence $w_1 + r$ and $w_2, \ldots, w_n$ are full elements. Therefore, we get that $x$ is $n$-$f$-clean, as required.

Recall that for a ring $R$ with a ring endomorphism $\alpha : R \to R$, the skew power series ring $R[[x; \alpha]]$ of $R$ is the ring obtained by giving the formal power series ring over $R$ with the new multiplication $x \cdot r = \alpha(r) x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$.

**Proposition 2.5** Let $\alpha$ be an endomorphism of $R$. Then the following statements are equivalent.

1. $R$ is a $n$-$f$-clean ring.
2. The formal power series ring $R[[x]]$ of $R$ is $n$-$f$-clean.
3. The skew power series ring $R[[x; \alpha]]$ of $R$ is $n$-$f$-clean.

**Proof.** (1) $\Rightarrow$ (3) For any $h = a_0 + a_1 x + \ldots \in R[[x, \alpha]]$, write $a_0 = e + u_1 + \ldots + u_n$ where $e \in \text{Id}(R)$ and $u_i \in K(R)$ ($1 \leq i \leq n$). Then $h = e + (u_1 + a_1 x + a_2 x^2 + \ldots) + u_2 + \ldots + u_n$, where $e \in \text{Id}(R) \subseteq \text{Id}(R[[x, \alpha]])$ and $u_i \in K(R) \subseteq K(R[[x, \alpha]])$. Let $h' = u_1 + a_1 x + a_2 x^2 + \ldots$. The equation $u = (s_1 + 0 + \ldots) h'(t_1 + 0 + \ldots) = 1 + s_1 a_1 \alpha(t_1) x + \ldots$ shows that $u \in U(R[[x, \alpha]])$, since $U(R[[x, \alpha]]) = \{a_0 + a_1 x + \ldots : a_0 \in U(R)\}$, without any assumption on the endomorphism $\alpha$. Hence, $h' \in K(R[[x, \alpha]])$, as desired.

(1) $\Rightarrow$ (2) Since $R[[x]] = R[[x, 1_R]]$, the proof is similar to that of (1) $\Rightarrow$ (3), as desired.

**Theorem 2.6** If $R$ is a $n$-$f$-clean ring, then $M_n(R)$ is also a $n$-$f$-clean ring for any $n \geq 1$.

**Proof.** Suppose that $R$ is $n$-$f$-clean. Given any $x \in R$, there exist some $e \in \text{Id}(R)$ and $w_1, \ldots, w_n \in K(R)$ such that $x = e + w_1 + \ldots + w_n$. So for some $s_i, t_i \in R$, $s_i w_i t_i = 1(1 \leq i \leq n)$.

Assume that theorem holds for the matrix ring $M_k(R)$, $K \geq 1$.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{k+1}(R)$$

Where $a_{11} \in R$, $a_{12} \in R^{1 \times k}$, $a_{21} \in R^{k \times 1}$ and $a_{22} \in M_k(R)$.

We have $a_{11} = e + w_1 + \ldots + w_n$ with $e = e^2$ and $s_i w_i t_i = 1$ for some $s_i, t_i \in R(1 \leq i \leq n)$. There also exist an idempotent matrix $E$ and full matrices $W_1, \ldots, W_n$ such that $a_{22} - a_{21} s a_{12} = E + W_1 + \ldots + W_n$. By hypothesis, we write $S_i W_i T_i = I_k$ for some $S_i, T_i \in M_{k \times k}(R)(1 \leq i \leq n)$. 


Therefore, we have

\[ A = \text{diag} \left( e, E \right) + \begin{bmatrix} w & a_{12} \\ a_{21}W_1 + \ldots + W_n + a_{21}tsa_{12} \end{bmatrix} \]

\[ = \text{diag}(e, E) + \begin{bmatrix} w_1 & a_{12} \\ a_{21}W_1 + a_{21}tsa_{12} \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & W_2 \end{bmatrix} + \ldots + \begin{bmatrix} w_n & 0 \\ 0 & W_n \end{bmatrix} \]

Obviously, \( \text{diag}(e,E) \) is an idempotent matrix and \( \begin{bmatrix} w_2 & 0 \\ 0 & W_2 \end{bmatrix}, \ldots, \begin{bmatrix} w_n & 0 \\ 0 & W_n \end{bmatrix} \) are full matrices in \( M_{k+1}(R) \). Let

\[ P = \begin{bmatrix} s & 0 \\ -S_1a_{21}ts & S_1 \end{bmatrix}, Q = \begin{bmatrix} t & -tsa_{12}T_1 \\ 0 & T_1 \end{bmatrix} \in M_{k+1}(R), \]

and the equation

\[ P \begin{bmatrix} w & a_{12} \\ a_{21}W_1 + a_{21}tsa_{12} \end{bmatrix} Q = \begin{bmatrix} 1 & 0 \\ 0 & I_k \end{bmatrix} = I_{k+1} \]

shows that \( \begin{bmatrix} w & a_{12} \\ a_{21}W_1 + a_{21}tsa_{12} \end{bmatrix} \) is a full matrix, hence \( A \) is \( n \)-f-clean, as desired.

**Proposition 2.7** Let \( n \) be odd positive integer. If \( a \in R \) is a \( n \)-f-clean element, then \( A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \) is always \( n \)-f-clean in \( M_2(R) \) for any \( b \in R \).

**Proof.**

Suppose \( a = e + w_1 + \ldots + w_n \) where \( e \in \text{Id}(R) \) and \( s_iw_i = 1 \) for some \( s_i, t_i \in R(1 \leq i \leq n) \). Then we can write \( A \) as

\[ A = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & 1 \end{bmatrix} + \ldots + \begin{bmatrix} w_n & 0 \\ 0 & -1 \end{bmatrix} \]

We also have

\[ \begin{bmatrix} s_1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix} \begin{bmatrix} t_1 & -tsa_{12}T_1 \\ 0 & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

hence \( \begin{bmatrix} w_1 & b \\ 0 & -1 \end{bmatrix} \) is a full element and obviously, \( \begin{bmatrix} w_2 & 0 \\ 0 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} w_n & 0 \\ 0 & -1 \end{bmatrix} \) are full elements. Therefore, \( A \) is a \( n \)-f-clean element.

We are going to investigate the \( n \)-f-cleanness between Morita Context ring \( C \) and \( A, B \). Our concern here is the Morita Context rings with zero homomorphism.

**Theorem 2.8** Let \( C = \begin{bmatrix} A & V \\ W & B \end{bmatrix} \) be the Morita Context with \( \psi, \varphi = 0 \). Then \( C \) is \( n \)-f-clean iff \( A \) and \( B \) are \( n \)-f-clean.

**Proof.** Assume that \( C \) is \( n \)-f-clean with \( \psi, \varphi = 0 \), let \( I = \begin{bmatrix} 0 & V \\ W & B \end{bmatrix}, J = \begin{bmatrix} A & V \\ W & 0 \end{bmatrix} \). Clearly \( I, J \) are ideals of \( C \) and \( C/I \cong A, C/J \cong B \). From proposition 2.1, \( A, B \) are \( n \)-f-clean.
Conversely, let $A, B$ be both $n$-f-clean rings. For any $r = \begin{bmatrix} a & v \\ w & b \end{bmatrix} \in C$, we have $a = e_1 + w_1 + \ldots + w_n$ and $b = e_2 + u_1 + \ldots + u_n$ for some idempotents $e_1, e_2 \in R$ and $w_1, \ldots, w_n, u_1, \ldots, u_n \in K(R)$. Assume that $s_i u_i t_i = 1$ for some $s_i, t_i, s_i'$, $t_i' \in R (1 \leq i \leq n)$. Let

\[ r = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} + \begin{bmatrix} w_1 & 0 \\ w & u_1 \end{bmatrix} + \begin{bmatrix} w_2 & 0 \\ 0 & u_2 \end{bmatrix} + \ldots + \begin{bmatrix} w_n & 0 \\ 0 & u_n \end{bmatrix} = E + K_1 + \ldots + K_n. \]

Obviously, $E \in Id(R)$ and the equation

\[ \begin{bmatrix} s_1 & 0 \\ -s_i' w t_i s_1 & s_i' \end{bmatrix} \begin{bmatrix} w_1 & v \\ w & u_1 \end{bmatrix} \begin{bmatrix} t_1 & -t_1 s_i v t_i' \\ 0 & t_i' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

shows that $K_1$ is a full matrix. Clearly, $K_2, \ldots, K_n$ are full matrices. Hence, $r$ is $n$-f-clean.

**Proposition 2.9** (1) Let $R, S$ be two rings, and $M$ be an $(R, S)$-bimodule. Let $E = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ be the formal triangular matrix ring. Then, $E$ is $n$-f-clean ring iff $R$ and $S$ are $n$-f-clean rings.

(2) For any $n \geq 1$, $R$ is $n$-f-clean ring iff $T_n(R)$ is $n$-f-clean.

**Proof.** The proof is similar to [6], proposition 2.8.

Next, we will investigate the equivalence of $n$-f-cleanness and $n$-cleanness. In [9], Yu call a ring $R$ to be a left quasi-duo ring if every maximal left ideal of $R$ is a two-sided ideal. Commutative rings and local rings are all belong to this class of rings.

**Theorem 2.10** For a left quasi-duo ring $R$, the following are equivalent:

(1) $R$ is a $n$-clean ring;

(2) $R$ is a $n$-f-clean ring.

**Proof.** (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) It is suffices to show that $w_i \in K(R)$ implies that $w_i \in U(R)$ for any $(1 \leq i \leq n)$. By using the proof of [6], theorem 2.9, the result is clear.

**Corollary 2.11** Let $R$ be a left quasi-duo ring and $e$ be an idempotent element of $R$. If $eRe$ and $(1 - e)R(1 - e)$ are both $n$-f-clean rings, then so is $R$.

**Proof.** The result follows from [8], theorem 2.10 and 2.10.

**Corollary 2.12** Every abelian $n$-f-clean ring is a $n$-clean ring.

**Proof.** It suffices to prove that every abelian ring is Dedekind finite by the proof of theorem 2.10. Suppose $ab = 1$, then $ba$ is an idempotent and hence central by assumption. $ba = ba.ab = ab.ab = 1$ shows that $R$ is Dedekind finite. Then the result follows from the proof of theorem 2.10.

For an idempotent $e$, we do not know whether the corner ring $eRe$ is again $f$-clean for a $f$-clean ring $R$. But when $e$ is a central idempotent, we can get an affirmative answer:

**Proposition 2.13** Let $R$ be a $n$-f-clean ring and $e$ be a central idempotent in $R$. Then $eRe$ is also $n$-f-clean.
Proof. We can view $eRe$ as a homomorphic image of $R$ since $e$ is central, hence the result follows from proposition 2.2. ■

Let $R$ be a ring in which 2 is invertible. Camillo and Yu [3], showed that $R$ is clean iff every element of $R$ is the sum of a unit and square root of 1. We extend this result for $n$-f-clean rings.

**Proposition 2.14** Let $R$ be a ring in which 2 is invertible, then $R$ is $n$-f-clean iff every element of $R$ is a sum of $n$ full elements and a square root of 1.

**Proof.** Suppose $R$ is $n$-f-clean and $x \in R$. Then $(x + 1)/2 = e + u_1 + \ldots + u_n$ for some $e \in Id(R)$ and $u_1, \ldots, u_n \in K(R)$. So $x = (2e - 1) + 2u_1 + \ldots 2u_n$ with $(2e - 1)^2 = 1$ and $2u_1, \ldots, 2u_n \in K(R)$.

Conversely, if $x \in R$ then $2x - 1 = f + u_1 + \ldots + u_n$ where $f^2 = 1$ and $u_1, \ldots, u_n \in K(R)$. Thus $x = (f + 1)/2 + (u_1 + \ldots + u_n)/2$. It is easy to show that $e = (f + 1)/2 \in Id(R)$ and $u_1/2, \ldots, u_n/2 \in K(R)$. Hence, $x$ is $n$-f-clean. Thus $R$ is a $n$-f-clean ring. ■

**References**