On the construction of symmetric nonnegative matrix with prescribed Ritz values

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Abstract. In this paper for a given prescribed Ritz values that satisfy in the some special conditions, we find a symmetric nonnegative matrix, such that the given set be its Ritz values.

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1. Introduction

The Ritz values of a \( n \times n \) matrix are the eigenvalues of its leading principal submatrices of order \( m = 1, 2, \ldots, n \). In the symmetric case, the Ritz values are real and interlacing, i.e. the ordered values \( \lambda_i^{(j)} \) at level \( j \) and \( \lambda_i^{(j+1)} \) at the next level \( j + 1 \) satisfy

\[
\lambda_1^{(j+1)} \leq \lambda_1^{(j)} \leq \lambda_2^{(j+1)} \leq \lambda_2^{(j)} \leq \ldots \leq \lambda_{j+1}^{(j+1)}.
\] (1)

Kostant and Wallach \cite{1,2}, construct the unique unit Hessenberg matrix \( H \) with given \( R_B \), where \( R_B \) is the list of Ritz values. Beresford Parlett and Gilbert Strang \cite{3} solved the problem for symmetric arrow and tridiagonal matrices.
In the section 2 of this paper similar Theorem 2.1 of [4], we prove a theorem which can be used to construction a nonnegative matrix for a given set of Ritz values in a systematic method. The algorithm of this method is presented in section 3.

2. Main Result

We will use some standard basic concepts and results about square nonnegative matrices such as reducible, irreducible, Frobenius normal form of a reducible matrix, irreducible component and Frobenius Theorem about the spectral structure of an irreducible matrix as are described in [1].

**Theorem 2.1** Let $B$ be an $n \times n$ ($n \geq 2$) irreducible symmetric nonnegative matrix and let

$$R_B = \{\lambda_1^{(1)}, \lambda_2^{(2)}; \ldots; \lambda_1^{(n)}, \lambda_2^{(n)}; \ldots, \lambda_{n-1}^{(n)}\}$$

be the Ritz values of $B$ that satisfy in the interlacing condition (1) for $j = 1, 2, \ldots, n-1$.

If $\lambda_n^{(n)}$ is the Perron eigenvalue of $B$ and $A = \begin{pmatrix} \lambda & a \\ a & \lambda_n^{(n)} \end{pmatrix}$ is a $2 \times 2$ nonnegative symmetric matrix with Ritz values $\{\mu_1; \mu_2; \mu_3\}$, then there exist an $(n + 1) \times (n + 1)$ nonnegative symmetric matrix $C = \begin{pmatrix} B & sa \\ s^T & \delta \end{pmatrix}$ with the following Ritz values

$$M = \{\lambda_1^{(1)}; \lambda_1^{(2)}; \lambda_2^{(2)}; \ldots; \lambda_1^{(n)}; \lambda_2^{(n)}; \ldots; \lambda_n^{(n)}, \lambda_1^{(n)}; \lambda_1^{(n)}; \lambda_2^{(n)}; \ldots, \lambda_{n-1}^{(n)}, \mu_2, \mu_3\},$$

where $s$ is the normalized eigenvector corresponding to the Perron eigenvalue $\lambda_n^{(n)}$ and $\delta = \mu_3 + \mu_2 - \lambda_n^{(n)}$.

**Proof.** Let $s$ be the normalized eigenvector associated to the Perron eigenvalue of $B$. We find the $n \times (n - 1)$ matrix $V_1$ such that $Y_1 = (s, V_1)$ be unitary and $BY_1 = (\lambda_n^{(n)} s, BV_1)$. Therefore

$$B_1 = Y_1^* BY_1 = \begin{pmatrix} \lambda_n^{(n)} s s^* s^* BV_1 \\ \lambda_n^{(n)} V_1^* s V_1^* BV_1 \end{pmatrix} = \begin{pmatrix} \lambda_n^{(n)} * * \ldots * \\ 0 \\ \vdots \hat{B} \\ 0 \end{pmatrix},$$

where $\hat{B} = V_1^* BV_1$ is an $(n - 1) \times (n - 1)$ matrix and $\{\lambda_1^{(1)}, \ldots, \lambda_n^{(n)}\}$ is set of eigenvalues of $\hat{B}$. On the other hand, by Schur decomposition theorem there exist the unitary matrix $V_2$ of order $(n - 1)$, such that $V_2^* \hat{B} V_2 = \hat{T}_B$, where $\hat{T}_B$ is upper triangular matrix with the eigenvalues of $\hat{B}$ on its main diagonal. If

$$Y_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & V_2 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$
whereas $V_2$ is a unitary matrix, then $Y_2$ is also unitary. Thus

$$Y_2^* B_1 Y_2 = Y_2^* (Y_1^* B Y_1) Y_2 = (Y_1 Y_2)^* B (Y_1 Y_2) = Y^* B Y,$$

(if $Y = Y_1 Y_2$)

$$Y = Y_1 Y_2 = (s \ V_1 V_2) = (s \ T) \implies Y^* = \begin{pmatrix} s^* \\ T^* \end{pmatrix}.$$

(if $T = V_1 V_2$)

Since $T$ is a unitary matrix, we can write

$$Y Y^* = s s^* + T T^* = I_n, \quad Y^* Y = \begin{pmatrix} s^* s \\ T^* s T^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \quad (2)$$

and

$$Y^* B Y = \begin{pmatrix} s^* B s \\ T^* B T \end{pmatrix} = \begin{pmatrix} \lambda^{(n)}_n \\ 0 \end{pmatrix} = T_B. \quad (3)$$

Consequently, $T_B$ is an upper triangular matrix with eigenvalues of $B$ on its main diagonal. On the other hand by Schur decomposition theorem there exist a unitary matrix $X$, such that $X^* A X = T_A$, where $T_A$ is an upper triangular matrix so that eigenvalues of $A \ (\{\mu_2, \mu_3\})$ lie on its main diagonal. Assume that the matrices $X$ and $X^*$ have the following partitions:

$$X = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \quad X^* = \begin{pmatrix} K_1^* & K_2^* \end{pmatrix},$$

where $K_1$ and $K_2$ are $1 \times 2$ vectors. Since $X$ is a unitary matrix we have

$$X X^* = \begin{pmatrix} K_1 K_1^* & K_1 K_2^* \\ K_2 K_1^* & K_2 K_2^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

$$X X^* = K_1 K_2 + K_2 K_2 = I_2.$$

By (4) we have

$$T_A = X^* A X = K_1^* \lambda K_1 + K_2^* a K_1 + K_1^* a K_2 + \lambda^{(n)}_n K_2^* K_2 = \lambda K_1^* K_1 + a (K_2^* K_1 + K_1^* K_2) + \lambda^{(n)}_n K_2^* K_2. \quad (5)$$

Now we consider two matrices $Z$ and $Z^*$ and a nonnegative matrix $C$ of order $(n + 2 - 1) \times (n + 2 - 1)$ in the following forms:

$$Z = \begin{pmatrix} s K_2 \\ K_1 \end{pmatrix}, \quad Z^* = \begin{pmatrix} K_2^* s^* \\ T^* \end{pmatrix}, \quad C = \begin{pmatrix} B & s a \\ a s^* & \delta \end{pmatrix}.$$
Using relations (2) and (4), it is easy to show that $Z$ is a unitary matrix. Now by relations (2) – (5), we can calculate $Z^*CZ$ as follows:

\[
Z^*CZ = \begin{pmatrix} K^*_1\lambda K_1 + a(K^*_2K_1 + K^*_1K_2) + \lambda_n^p K^*_2 K^*_1 s s^* T + K^*_2 s^* B T \\ T^* s a K_1 + T B s K_2 \\ T^* B T \end{pmatrix} = \begin{pmatrix} T_A^* & * \\ 0 & T_B \end{pmatrix} = T_C,
\]

where $T_C$ is an upper triangular matrix and the elements of its main diagonal are the elements of the last level of $M$. On the other hand by the above relation, $C$ and $T_C$ are similar, therefore $C$ solves the problem which completes the proof.

3. An Algorithm for construction of a symmetric nonnegative matrix

Let

\[
R_A = \{\lambda_1^{(1)}; \lambda_1^{(2)}; \lambda_2^{(2)}; ...; \lambda_1^{(n)}; \lambda_2^{(n)}; ..., \lambda_n^{(n)}\},
\]

where $\lambda_1^{(j)}, \lambda_2^{(j)}, ..., \lambda_j^{(j)}$ are the eigenvalues of leading principal submatrix $A_j$ for $j = 1, 2, \ldots, n$ and assume that $R_A$ satisfies the interlacing condition (1). Assume the eigenvalues of $A_{j+1}$ are only different with eigenvalues of $A_j$ in $\lambda_j^{(j+1)}, \lambda_{j+1}^{(j+1)}$ where $\lambda_j^{(j+1)}$ is the Perron eigenvalue of $A_{j+1}$ and $\lambda_i^{(j)} = \lambda_i^{(j+1)}$, for $i = 1, 2, ..., j - 1$ and $j = 1, 2, ..., n$. By following algorithm we construct a nonnegative symmetric $n \times n$ matrix $A$ with Ritz values $R_A$ and leading principal submatrices.

**Step 1**: Let $A_1 = (\lambda_1^{(1)})$.

**Step 2**: Let

\[
A_2 = \begin{pmatrix} \lambda_1^{(1)} \\ \sqrt{(\lambda_2^{(2)} - \lambda_1^{(1)})(\lambda_1^{(1)} - \lambda_2^{(1)})} \\ \lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(1)} \end{pmatrix}.
\]

By interlacing condition (1) the elements of $A_2$ are nonnegative.

**Step $j$**: We construct a nonnegative symmetric matrix with prescribed Ritz values

\[
\{\lambda_1^{(1)}; \lambda_1^{(2)}; \lambda_2^{(2)}; ...; \lambda_1^{(j)}; \lambda_2^{(j)}; ..., \lambda_j^{(j)}\}
\]

that satisfy in interlacing condition (1) and $\lambda_1^{(j-1)}, ..., \lambda_{j-2}^{(j-1)}$ lies in the set of eigenvalues of $A_j$. At first we construct the $2 \times 2$ symmetric nonnegative matrix $A'$ with Ritz values $\{\mu_1; \mu_2, \mu_3\}$, where

\[
\mu_3 = \lambda_j^{(j)}, \quad \mu_2 = \lambda_j^{(j-1)}, \quad \mu_1 = \mu_2 + \mu_3 - \lambda_j^{(j-1)}
\]

and $A' = \begin{pmatrix} \mu_1 & a \\ a & \lambda_j^{(j-1)} \end{pmatrix}$ and $a = \sqrt{(\mu_3 - \mu_1)(\mu_1 - \mu_2)}$. The condition on $\mu_1$ is the solvability condition for this problem. Now by Theorem 2.1 we construct $A_j$ by combining
two matrices $A_{j-1}$ and $A'$ in the form $A_j = \begin{pmatrix} A_{j-1} & as \\ as^T & \mu_1 \end{pmatrix}$, where $s$ is the eigenvector corresponding to the Perron eigenvector of $A'$.

**Example 3.1** We construct a matrix with the following prescribed Ritz values \{1; -1, 3; -1, 2, 4; -1, 2, 3, 5; -1, 2, 2, 4, 6\}.

$A_1 = (1)$,

$A_2 = \begin{pmatrix} \lambda_1^{(1)} & a \\ a & \delta_1 \end{pmatrix}$,

where

$a = \sqrt{(\lambda_1^{(1)} - \lambda_1^{(2)})(\lambda_2^{(2)} - \lambda_1^{(1)})} = 2$,

$\delta_1 = \lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(1)} = 1$.

The construction of the matrix $A_3$ needs the normalized eigenvector to associated eigenvalue 3. This vector is

$s = \left( \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right)$.

We must construct 2 × 2 matrix $A'$ with Ritz values $\{\mu_1; \mu_2, \mu_3\} = \{\mu_1; 2, 4\}$, so that $\lambda_2^{(2)} = 3$ is lies on the main diagonal of $A'$.

$A' = \begin{pmatrix} \mu_1 & a \\ a & \lambda_2^{(2)} \end{pmatrix}$,

where

$\mu_1 = \mu_2 + \mu_3 - \lambda_2^{(2)} = 3$,

$a = \sqrt{(\mu_1 - \mu_2)(\mu_3 - \mu_1)} = 1$.

Now by Theorem 2.1, we combine $A'$ and $A_2$ to construct $A_3$ as

$A_3 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 \\ 2 & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 \end{pmatrix}$.

Similarly we can construct the matrices $A_4$ and $A_5$.

$A_4 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 & 1/2 \\ 2 & 1 & \sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} & 4 \end{pmatrix}$,

$A_5 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 & 1/2 & \sqrt{2}/4 \\ 2 & 1 & \sqrt{2}/2 & 1/2 & \sqrt{2}/4 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 & 1/\sqrt{2} & 1/2 \\ 1/2 & 1/2 & 1/\sqrt{2} & 4 & \sqrt{2}/2 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 & \sqrt{2}/2 & 5 \end{pmatrix}$.

**References**

