

## Hereditary properties of amenability modulo an ideal of Banach algebras

H. Rahimi<sup>a\*</sup>, E. Tahmasebi<sup>a</sup>

<sup>a</sup>*Department of Mathematics, Islamic Azad University, Central Tehran Branch,  
PO. Code 13185-768, Tehran, Iran.*

Received 8 July 2014; Revised 10 September 2014; Accepted 22 September 2014.

---

**Abstract.** In this paper we investigate some hereditary properties of amenability modulo an ideal of Banach algebras. We show that if  $(e_\alpha)_\alpha$  is a bounded approximate identity modulo  $I$  of a Banach algebra  $A$  and  $X$  is a neo-unital modulo  $I$ , then  $(e_\alpha)_\alpha$  is a bounded approximate identity for  $X$ . Moreover we show that amenability modulo an ideal of a Banach algebra  $A$  can be only considered by the neo-unital modulo  $I$  Banach algebra over  $A$ .

© 2014 IAUCTB. All rights reserved.

---

**Keywords:** Amenability modulo an ideal, Neo-unital modulo an ideal, Approximate identity modulo an ideal.

**2010 AMS Subject Classification:** 43A07, 46H25.

### 1. introduction

The concept of the amenability of (discrete) groups was considered in 1929 by J. von Neumann in [14] for the first time. But in terms of amenability for the topological groups and semigroups was used by Day [3, 4]. A Hausdorff and locally compact group  $G$  is called to be amenable when there exists a left invariant mean on  $L^1(G)$ . This concept comes to the attention of mathematicians and since then they wrote numerous articles about this theory. The concept of amenability of Banach algebras was introduced by Barry Johnson in 1972 [10]. He showed that for a Hausdorff and locally compact group  $G$ ,  $G$  is amenable (in the usual sense) if and only if (resp.  $l^1(G)$ )  $L^1(G)$  is amenable. Duncan and

---

\*Corresponding author.

E-mail address: rahimi@iauctb.ac.ir (H. Rahimi).

Namioka in [6] proved that for the semigroup  $S$ , amenability of Banach algebra  $l^1(S)$  will result in an amenability of  $S$ , but the reverse is not true. They also investigated amenability of inverse semigroups and showed that the inverse semigroup  $S$  is amenable if and only if  $G_S$  is amenable, where  $G_S$  is the maximal subgroup of  $S$ . As in the above it is mentioned, Johnson's theorem fails to be true for discrete semigroups. For semigroup  $S$ , some necessary and sufficient conditions (in especial cases) for amenability of semigroup algebra  $l^1(S)$  was introduced, see [2, 5, 8] for instant. So it seems that the expression of another concept for amenability of Banach algebras in dealing with the concept of amenability of semigroup is essential.

The first author and Amini in [1] introduced and initiated the concept of amenability modulo an ideal. They proved for the semigroup  $S$ , amenability of  $l^1(S)$  modulo ideals that induced by certain classes of group congruences  $\sigma$  on  $S$  is equivalent to the amenability of  $S$ . This could be considered as a restoring the Johnson's theorem for a large class of semigroups. The authors studied in [11, 12], basic properties of amenability modulo an ideal such as virtual and approximate diagonal modulo an ideal, contractible modulo an ideal and showed that for the semigroup  $S$ , the semigroup algebra  $l^1(S)$  is contractible modulo an ideal if and only if  $\frac{S}{\sigma}$  is finite, restoring the Selivanov's theorem for a large class of semigroups [13].

In this paper, we study some hereditary properties of amenability modulo an ideal. we show that if  $A$  is a Banach algebra with a bounded approximate identity modulo  $I$ ,  $(e_\alpha)_\alpha$  and  $X$  is a neo-unital modulo  $I$  Banach algebra (that is, if  $X = (A \setminus I) \cdot X \cdot (A \setminus I) = \{a \cdot x \cdot b \mid a, b \in A \setminus I, x \in X\}$ ) then  $(e_\alpha)_\alpha$  is a bounded approximate identity for  $X$ . We also show that the neo-unital modulo  $I$  Banach algebra  $X$  over a Banach algebra  $A$  plays the main role in order to determine amenability modulo an ideal of  $A$ .

## 2. A brief review on amenability modulo an ideal of semigroup algebras

In this section we give a survey of the notion of amenability modulo an ideal, contractibility modulo an ideal of Banach algebras and some their hereditary property and their applications for semigroup algebras.

Let  $A$  be a Banach algebra, a Banach space  $X$  which is also a  $A$ -bimodule is said to be a Banach  $A$ -bimodule if there is  $C > 0$  such that

$$\|a \cdot x\| \leq C\|a\|\|x\|, \|x \cdot a\| \leq C\|a\|\|x\| \quad (a \in A, x \in X)$$

The minimum constant  $C$  that can occur in above inequalities is denoted by  $C_X$ . Let  $A$  be a Banach algebra and  $X$  be a Banach  $A$ -bimodule. A bounded linear map  $D : A \rightarrow X$  is called a *Derivation* if for all  $a, b \in A$ ,  $D(ab) = a \cdot D(b) + D(a) \cdot b$ . A derivation  $D : A \rightarrow X$  is said to be inner if there exists  $x \in X$  such that  $D(a) = a \cdot x - x \cdot a$  for each  $a \in A$ . A Banach algebra  $A$  is called *amenable* if for every  $A$ -bimodule  $X$ , every derivation  $D : A \rightarrow X^*$  is inner.

We now recall some definitions and results that are given in [1, 11], which contains interesting results on characterization on amenable modulo an ideal of Banach algebras, contractible modulo an ideal of Banach algebras in terms of asymptotic version of a projective diagonal.

**Definition 2.1** Let  $I$  be a closed ideal of  $A$ .

(i) A Banach algebra  $A$  is amenable modulo  $I$  if for every Banach  $A$ -bimodule  $E$  such

that  $I \cdot E = E \cdot I = 0$ , and every derivation  $D$  from  $A$  into  $E^*$  there is  $\phi \in E^*$  such that  $D = ad_\phi$  on the set theoretic difference  $A \setminus I := \{a \in A : a \notin I\}$ .

(ii) A Banach algebra  $A$  is contractible modulo  $I$  if for every Banach  $A$ -bimodule  $X$  such that  $I \cdot X = X \cdot I = 0$ , every bounded derivation  $D$  from  $A$  into  $X$  is an inner derivation on the set theoretic difference  $A \setminus I := \{a \in A : a \notin I\}$ .

(iii) A bounded net  $\{u_\alpha\}_\alpha \subseteq A$  is called approximate identity modulo  $I$  if  $\lim_\alpha u_\alpha \cdot a = \lim_\alpha a \cdot u_\alpha = a \quad (a \in A \setminus I)$ .

(iv) An element  $M \in (\frac{A}{I} \hat{\otimes} \frac{A}{I})^{**}$  is a virtual diagonal modulo  $I$  if

$$\begin{aligned} a \cdot \pi_{\frac{A}{I}}^{**} M - \tilde{a} &= 0 \quad (a \in A, \tilde{a} = a + I), \\ a \cdot M - M \cdot a &= 0 \quad (a \in A \setminus I). \end{aligned}$$

(v) A bounded net  $(m_\alpha)_\alpha \subset \frac{A}{I} \hat{\otimes} \frac{A}{I}$  is a approximate diagonal modulo  $I$  if

$$\begin{aligned} a \cdot \pi_{\frac{A}{I}} m_\alpha - \tilde{a} &\rightarrow 0 \quad (a \in A, \tilde{a} = a + I), \\ a \cdot m_\alpha - m_\alpha \cdot a &\rightarrow 0 \quad (a \in A \setminus I). \end{aligned}$$

(vi) An element  $m \in \frac{A}{I} \hat{\otimes} \frac{A}{I}$  is a diagonal modulo  $I$  if

$$\begin{aligned} a \cdot \pi_{\frac{A}{I}} m - \tilde{a} &= 0 \quad (a \in A, \tilde{a} = a + I), \\ a \cdot m - m \cdot a &= 0 \quad (a \in A \setminus I). \end{aligned}$$

**Proposition 2.2** Let  $I$  be a closed ideal of  $A$ .

(i) If  $A/I$  is amenable (contractible) and  $I^2 = I$  then  $A$  is amenable (contractible) modulo  $I$ .

(ii) If  $A$  is amenable (contractible) modulo  $I$  then  $A/I$  is amenable (contractible).

(iii) If  $A$  is amenable (contractible) modulo  $I$  and  $I$  is amenable (contractible), then  $A$  is amenable (contractible).

**Proposition 2.3** The following conditions are equivalent;

- (i)  $A$  is amenable modulo  $I$ ,
- (ii) There is an approximate diagonal modulo  $I$ ,
- (iii) There is a virtual diagonal modulo  $I$ .

Let  $S$  be a semigroup,  $s^*$  is called an inverse of  $s$  if  $ss^*s = s, s^*ss^* = s^*$ . A semigroup  $S$  is called regular if each  $s \in S$  is regular, i.e. there exists  $t \in T$  such that  $sts = s$ ,  $S$  is called inverse semigroup if  $S$  is regular and every element in  $S$  has a unique inverse,  $S$  is called eventually regular if every element of  $S$  has some power that is regular and  $E(S)$  is semi lattice,  $S$  is called  $E$ -inversive if for all  $x \in S$ , there exists  $y \in S$  such that  $xy \in E(S)$ ,  $S$  is called an  $E$ -semigroup if  $E(S)$  forms a sub-semigroup of  $S$  and  $S$  is called eventually regular if every element of  $S$  has some power that is regular and  $E(S)$  is semi lattice.

In recent years amenability of semigroup algebras was studied by some researchers. In the below we refer some results on the amenability of semigroup algebras are well-known [2, 6].

**Proposition 2.4** Suppose  $S$  is a semigroup and  $l^1(S)$  is amenable, then

- (i)  $S$  is amenable;

- (ii)  $S$  is regular;
- (iii)  $E(S)$  is finite;
- (iv)  $l^1(S)$  has an identity.

Also, for finite semigroup we have the following result [7];

**Proposition 2.5** If  $S$  is finite semigroup, then the following statements are equivalent;

- (i)  $l^1(S)$  is amenable;
- (ii)  $S$  is regular and  $l^1(S)$  is unital;
- (iii)  $S$  is regular and  $l^1(S)$  is semisimple.

And for inverse semigroups, we have:

**Proposition 2.6** Let  $S$  be an inverse semigroup and  $E(S)$  is finite. Then  $l^1(S)$  is amenable if and only if the maximal subgroup of  $S$  is amenable.

It is observed only partial results were known in literature of amenability of semigroup algebras whereas amenability of semigroup algebra is characterized by the notion of amenability modulo an ideal for a wide class of semigroups. In the following we refer some results of amenability modulo an ideal and contractibility modulo an ideal for semigroup algebras [1, 11].

A congruence  $\rho$  on semigroup  $S$  is called a group congruence if  $\frac{S}{\rho}$  is a group and the least group congruence on  $S$  is denoted by  $\sigma$ , as used in [9].

**Proposition 2.7** (i) Let  $S$  be an  $E$ -inverse  $E$ -semigroup with commutative idempotents. Then  $S$  is amenable if and only if  $l^1(S)$  is amenable modulo  $I_{\sigma^*}$ ;

(ii) Let  $S$  be an eventually inverse semigroup then  $S$  is amenable if and only if  $l^1(S)$  is amenable modulo  $I_{\sigma'}$ ;

(iii) Let  $S$  be a semigroup,  $\rho$  be a congruence on  $S$ . If  $Ker(\rho)$  is central and  $I_\rho$  has an approximate identity, then  $S$  is amenable if and only if  $l^1(S)$  is amenable modulo  $I_\sigma$ ;

**Theorem 2.8** If  $S$  is either

- (i)  $E$ -inverse  $E$ -semigroup with commutative idempotents, or
  - (ii) eventually inverse semigroup with commutative idempotents,
- then  $l^1(S)$  is contractible modulo  $I_\sigma$  if and only if  $\frac{S}{\sigma}$  is finite.

We now characterize amenability of semigroup algebra when  $S$  is an inverse semigroup. A subset  $H$  of a semigroup  $S$  is called (respectively) full, reflexive and dense if  $E(S) \subseteq H$ ,  $\forall a, b \in S [ab \in H \Rightarrow ba \in H]$  and  $\forall s \in S \exists x, y \in S [sx, ys \in H]$ .  $H$  is called closed ( in  $S$ ) if  $H\omega = H$  where  $H\omega = \{s \in S \mid \exists a \in H [as \in H]\}$ . By a normal subsemigroup  $H$  of semigroup  $S$  we mean a full, dense, reflexive and closed subsemigroup and we write  $H \triangleleft S$ .

We state the same result of Lemma 2 [1] for inverse semigroup by different method.

**Lemma 2.9** Let  $S$  be a semigroup, then the following statements hold.

(i) If  $\rho$  is a group congruence on  $S$ . Then  $l^1(\frac{S}{\rho}) \simeq \frac{l^1(S)}{I}$  where  $I$  is a closed ideal of  $l^1(S)$ .

(ii) If  $S$  is an inverse semigroup and  $\rho$  is the least group congruence on  $S$ . Then  $l^1(\frac{S}{\rho}) \simeq \frac{l^1(S)}{I}$  and  $I^2 = I$ .

**Proof.** (i) Let  $\pi : S \rightarrow \frac{S}{\rho}$  be the quotient map and  $\hat{\pi} : l^1(S) \rightarrow l^1(\frac{S}{\rho})$  by  $\hat{\pi}(\delta_s) = \delta_{\pi(s)}$  be the induced epimorphism by  $\pi$ . It is not far to see that  $l^1(\frac{S}{\rho}) \simeq \frac{l^1(S)}{I}$ .

ii) By (i),  $l^1(\frac{S}{\rho}) \simeq \frac{l^1(S)}{I}$ , we show that  $I^2 = I$ . Let  $J$  be the closed span of  $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E(S)\}$  and define the congruence  $\sigma$  on  $S$  as  $x\sigma y$  if and only if  $\delta_x - \delta_y \in J(x, y \in S)$ . It is easy to see that  $\sigma$  is a congruence on  $S$ . Since  $\rho$  is the least group congruence so  $\rho \subseteq \sigma$ . On the other hand if  $x\sigma y$ , then  $\delta_x - \delta_y \in J \subseteq \ker \hat{\pi} = I$ . Thus  $[x]_\rho = [y]_\rho$  so  $x\rho y$ . This implies that  $\sigma \subseteq \rho$ . Then  $\rho = \sigma$  and  $l^1(\frac{S}{\sigma}) = l^1(\frac{S}{\rho})$ . Thus  $I = J$ . Now define  $x\eta y$  if and only if  $\delta_x - \delta_y \in I^2(x, y \in S)$ . Clearly  $\eta$  is a congruence on  $S$ . Let  $e, f \in E(S)$ . Now  $(\delta_e - \delta_f)(\delta_e - \delta_f) = (\delta_f - \delta_e)$ , then  $[e]_\eta = [f]_\eta$ . Also, for each  $s \in S$  there exists  $s^* \in S$  such that  $s = ss^*$ . Let  $e = ss^*$ , so  $s = se$  and  $\delta_s = \delta_{se}$ . Thus  $\delta_s - \delta_{se} = 0 \in I^2$  and  $[se]_\eta = [s]_\eta$ . Thus  $\frac{S}{\eta}$  is a group and  $\eta$  is a group congruence. Clearly  $\rho \subseteq \eta$ . let  $x\eta y(x, y \in S)$ . Then  $\delta_x - \delta_y \in I^2 \subseteq I$ . Thus  $x\rho y$  and  $\eta \subseteq \rho$ . This implies that  $\rho = \eta$  and  $l^1(\frac{S}{\rho}) = l^1(\frac{S}{\eta})$ . Hence  $I = I^2$ . ■

We recall the following result of [1]

**Proposition 2.10** Let  $S$  be an inverse semigroup and  $\rho$  be the least group congruence on  $S$ . Then  $S$  is amenable if and only if  $\frac{S}{\rho}$  is amenable.

By using Lemma 2.10, Propositions 2.2 and 2.10, we have the following result;

**Theorem 2.11** Let  $S$  be an inverse semigroup, then  $S$  is amenable if and only if  $l^1(S)$  is amenable modulo  $I_\rho$ .

We now continue this section to recall some concrete examples. In these examples by  $\rho$  we mean the least group congruence on semigroup  $S$ .

(i) Let  $S = \{p^m q^n : m, n \geq 0\}$  be the bicyclic semigroup generated by  $p, q$ , then  $S$  is a  $E$ -unitary inverse semigroup with  $E(S) = \{p^n q^n : n = 0, 1, 2, \dots\}$ . We have that  $S$  is amenable but  $l^1(S)$  is not amenable [6]. Whereas amenability of  $S$  is equivalent to amenability of  $l^1(S)$  modulo  $I_{\sigma'}$ , where  $I_{\sigma'}$  is the corresponding ideal to the least group congruence on  $S$ .

(ii) Suppose  $X$  is a singleton, then  $l^1(FI(X))$  is not amenable but  $l^1(FI(X))$  is amenable modulo  $I_{\sigma'}$  for some the least group congruence  $\sigma'$ .

(iii) Let  $T = (\mathbb{N}_0, +) \times (\mathbb{N}, max)$ , ( $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ) and  $S = G \times T$  where  $G$  be an amenable group with identity 1. Then  $E(S) = \{(1, e) : e \in E(T)\}$  is infinite. Then  $l^1(S)$  is amenable modulo  $I_\sigma$ . We note that, since the semigroup algebra  $l^1(T)$  is not amenable,  $l^1(S)$  could not be amenable.

(iv) Let  $\mathbb{N}$  be the commutative semigroup of positive integers with maximum operation. Then  $l^1(\mathbb{N})$  is contractible modulo  $I_\sigma$  where  $I_\sigma$  is the corresponding ideal to a least group congruence  $\sigma$  but  $l^1(\mathbb{N})$  is not contractible.

### 3. Some hereditary properties

All over this paper we fix  $A$  and  $I$  as above, unless they are otherwise specified.

An  $A$ -bimodule  $X$  is neo-unital modulo  $I$  if

$$X = A \setminus I \cdot X \cdot A \setminus I = \{a \cdot x \cdot b \mid a, b \in A \setminus I, x \in X\}$$

and  $X$  is essential modulo  $I$  if

$$X = \overline{A \setminus I \cdot X \cdot A \setminus I} = \text{Closed span}\{a \cdot x \cdot b \mid a, b \in A \setminus I, x \in X\}.$$

Let  $I$  be a closed ideal of Banach algebra  $A$ ,  $X$  be a Banach  $A$ -bimodule such that  $I \cdot X = X \cdot I = 0$ . We denote the space of all continuous derivation from  $A$  into  $X$  by  $\mathcal{Z}_I^1(A, X)$ , the space of all inner derivation on  $A \setminus I$  by  $\mathcal{B}_I^1(A, X)$  and the first Hochschild cohomology group of  $A$  modulo  $I$  with coefficients in  $X$  by

$$\mathcal{H}_I^1(A, X) := \mathcal{Z}_I^1(A, X) / \mathcal{B}_I^1(A, X).$$

**Definition 3.1** ([11], Definition 2.3) Let  $A$  be a Banach algebra. A bounded net  $\{u_\alpha\}_\alpha \subseteq A$  is a right (left) approximate identity modulo  $I$  if  $\lim_\alpha a \cdot u_\alpha = a$  ( $\lim_\alpha u_\alpha \cdot a = a$ ) for each  $a \in A \setminus I$ . An approximate identity modulo  $I$  is a net  $\{u_\alpha\}_\alpha \subseteq A$  which is both a left and right approximate identity modulo  $I$ .

**Proposition 3.2** Let  $A$  be a Banach algebra,  $I$  be a closed ideal of  $A$  and  $X$  be a Banach  $A$ -bimodule such that  $A \cdot X = X \cdot I = 0$ . If  $A$  has a bounded right approximate identity modulo  $I$ , then  $\mathcal{H}_I^1(A, X^*) = 0$ .

**Proof.** Clearly  $X^* \cdot I = 0$ . Let  $D : A \rightarrow X^*$  be an arbitrary derivation then  $D(ab) = a \cdot D(b)$  ( $a, b \in A$ ). Let  $(e_\alpha)_\alpha$  be a bounded right approximate identity modulo  $I$  and  $\phi \in X^*$  be a  $w^*$ -accumulation point of  $(D(e_\alpha))_\alpha$ . We may suppose that  $\phi = w^*\text{-}\lim_\alpha D(e_\alpha)$ . It follows that  $D(a) = \lim_\alpha D(ae_\alpha) = \lim_\alpha a \cdot D(e_\alpha) = a \cdot \phi$  ( $a \in A \setminus I$ ), so  $D$  is inner on  $A \setminus I$ . ■

**Lemma 3.3** Let  $A$  be a Banach algebra with a bounded approximate identity modulo  $I$  and  $X$  be a Banach  $A$ -bimodule. Then  $X_1 := \{xa \mid a \in A \setminus I, x \in X\}$  is a closed right submodule and  $X_0 := (A \setminus I) \cdot X_1$  is a closed right and left submodule of  $X$ .

**Proof.** Let  $B$  denote  $\text{span } X_1$ . Clearly  $B$  is a right submodule of  $X$  and so is  $\overline{B}$ . Now If  $x_n \rightarrow x$  for  $x_n \in B$ , then  $x_n a \rightarrow xa$  for all  $a \in A \setminus I$ , so that  $xa \in \overline{B}$ . Let  $(e_\alpha)_\alpha$  be a bounded approximate identity modulo  $I$  and  $K > 0$  be the constant bounding  $(e_\alpha)_\alpha$ . Since  $\overline{B}$  is closed,  $\overline{B}$  is a Banach right  $A$ -module. For all  $\xi = \sum_n x_n a_n \in B$ ,  $\xi e_\nu = \sum_n x_n (a_n e_\alpha) \rightarrow \xi$ , and for any  $\zeta \in \overline{B}$  and any  $\epsilon > 0$ , there exists  $\eta \in B$  such that  $\|\zeta - \eta\| < \epsilon$ , so that

$$\|\zeta e_\nu - \zeta\| \leq \|(\zeta - \eta)e_\alpha\| + \|\eta e_\alpha - \eta\| + \|\eta - \zeta\| < \epsilon(K + 2)$$

for  $\alpha \geq \beta$ , where  $\beta \in J$  is chosen such that  $\|\eta e_\alpha - \eta\| < \epsilon$ . By Cohen's factorization theorem modulo an ideal ([11, Theorem 2.5])  $\overline{B} = \overline{B}(A \setminus I)$  and so  $X_1 \subseteq \overline{B} = \overline{B}(A \setminus I) \subseteq X_1$ . Thus  $X_1 = \overline{B}$ .

Let  $C = \text{span } X_0$ . Clearly  $C$  is a left and right submodule of  $X$ ; the closure is as well and thus  $\overline{C}$  is a Banach  $A$ -bimodule. Now by the same method as above,  $\overline{C} \subseteq \overline{C}(A \setminus I) = (A \setminus I)\overline{C} \subseteq (A \setminus I)\overline{C}(A \setminus I)$ , so that  $X_0 \subseteq \overline{C} = (A \setminus I)\overline{C}(A \setminus I) \subseteq (A \setminus I)X(A \setminus I) = X_0$ . ■

**Theorem 3.4** Let  $A$  be a Banach algebra with a bounded approximate identity modulo  $I$ . Then the following are equivalent:

- (i)  $\mathcal{H}_I^1(A, X^*) = 0$  for each Banach  $A$ -bimodule  $X$ .
- (ii)  $\mathcal{H}_I^1(A, X^*) = 0$  for each neo-unital modulo  $I$  Banach  $A$ -bimodule  $X$ .

**Proof.** (i)  $\Rightarrow$  (ii) is straightforward.

(ii)  $\Rightarrow$  (i) Let  $X$  be a Banach  $A$ -bimodule such that  $I \cdot X = X \cdot I = 0$ . Let  $X_1 := \{x \cdot a \mid$

$a \in A \setminus I, x \in X$  and  $X_0 := (A \setminus I)X_1$ . By Lemma 3.3,  $X_0$  and  $X_1$  are closed submodule of  $X$ . Let  $\tilde{D} : A \rightarrow X_1^*$  be a derivation and  $\pi_0 : X_1^* \rightarrow X_0^*$  be the restriction map defined by  $\pi_0(f) = f|_{X_0}$ . Clearly  $\pi_0$  is an  $A$ -bimodule homomorphism,  $I \cdot X_0 = X_0 \cdot I = 0$  and  $\pi_0 \circ \tilde{D} : A \rightarrow X_0^*$  is a derivation. Since  $\mathcal{H}_I^1(A, X_0^*) = 0$ , there is a  $f_0 \in X_0^*$  such that  $\pi_0 \circ \tilde{D}|_{A \setminus I} = ad_{f_0}$ . By Hahn-Banach's theorem there exists  $f_1 \in X_1^*$  such that  $f_1|_{X_0} = f_0$ . We have

$$\begin{aligned} \langle x, \tilde{D}(a) \rangle &= \langle x, \pi_0 \circ \tilde{D}(a) \rangle = \langle x, af_0 - f_0a \rangle, \quad (a \in A \setminus I, x \in X_0) \\ &= \langle x, af_1 - f_1a \rangle = \langle x, ad_{f_1}(a) \rangle \end{aligned}$$

Thus  $(\tilde{D} - ad_{f_1})(a)$  vanishes on  $X_0$  ( $a \in A \setminus I$ ). This implies that  $(\tilde{D} - ad_{f_1})(a) \in X_1^* \cap X_0^\perp$  ( $a \in A \setminus I$ ). Set

$$D_1(a) := \begin{cases} (\tilde{D} - ad_{f_1})(a) & a \in A \setminus I \\ 0 & a \in I \end{cases}$$

Now  $(\frac{X_1}{X_0})^* \simeq X_1^* \cap X_0^\perp$  and  $A \cdot (\frac{X_1}{X_0}) = 0$  so  $(\frac{X_1}{X_0})^* \cdot A = 0$ . Let  $(e_\alpha)_\alpha$  be the bounded approximate identity modulo  $I$  of  $A$  and  $\Delta : A \rightarrow (\frac{X_1}{X_0})^*$  be a bounded derivation. Banach-Alaglu's theorem implies that  $(\Delta(e_\alpha))$  has a convergent subnet  $(\Delta(e_\beta))$ . Let  $f := \lim_\beta \Delta(e_\beta)$  then  $\Delta(a) = \lim_\beta \Delta(ae_\beta) = \lim_\beta a \Delta(e_\beta) = a \cdot f = a \cdot f - f \cdot a$  ( $a \in A \setminus I$ ). Thus  $\Delta$  is inner on  $A \setminus I$ . By Proposition 3.2, there exists  $f_2 \in X_1^* \cap X_0^\perp$  such that  $D_1|_{A \setminus I} = ad_{f_2}$  and  $\tilde{D}|_{A \setminus I} = ad_{f_1+f_2}$ . Then  $\mathcal{H}_I^1(A, X_1^*) = 0$ .

Let now  $D \in \mathcal{H}_I^1(A, X^*)$  and  $\pi : X^* \rightarrow X_1^*$  be the restriction maps defined by  $\pi(f) = f|_{X_1}$ . Then  $I \cdot X_1 = X_1 \cdot I = 0$ ,  $\mathcal{H}_I^1(A, X_1^*) = 0$  and  $\pi \circ D$  is inner on  $A \setminus I$ . Since  $\pi$  is an  $A$ -bimodule homomorphism,  $\pi \circ D : A \rightarrow X_1^*$  is a derivation. Let  $g_0 \in X_1^*$  such that  $\pi \circ D|_{A \setminus I} = ad_{g_0}$ . By Hahn-Banach's theorem there exists  $g \in X^*$  such that  $g|_{X_1} = g_0$ . In the same manner as above,  $(D - ad_g)(a)$  vanishes on  $X_1$  ( $a \in A \setminus I$ ). Thus  $D - ad_g : A \rightarrow X_1^\perp$  is a derivation. We have  $xa \in X_1$  ( $a \in A \setminus I, x \in X$ ) and  $xa = 0$  ( $a \in I, x \in X$ ). Hence  $(\frac{X}{X_1}) \cdot A = 0$  and  $A \cdot (\frac{X}{X_1})^* = 0$ . Let  $\Lambda : A \rightarrow (\frac{X}{X_1})^*$  be a derivation and  $h$  be the  $w^*$ -convergent subnet  $(\Lambda(e_\beta))$  then

$$\Lambda(a) = \lim_\beta \Lambda(ae_\beta) = \lim_\beta \Lambda(e_\beta) a = h \cdot a = a \cdot (-h) - (-h) \cdot a \quad (a \in A \setminus I).$$

Thus  $\Lambda$  is inner on  $A \setminus I$ . Since  $X_1^\perp \cong (\frac{X}{X_1})^*$ , there exists  $k \in X_1^\perp$  such that  $(D - ad_f)|_{A \setminus I} = ad_k$ , i.e.  $D|_{A \setminus I} = ad_{f+k}$  (by Proposition 3.2). ■

### Acknowledgements

The authors would like to sincerely thank the referee for his/her valuable comments and useful suggestions.

### References

[1] M. Amini and H. Rahimi, *Amenability of semigroups and their algebras modulo a group congruence*, Acta Mathematica Hungarica, Vol 144, Issue 2, (2014), pp 407-415.

- [2] G. K. Dales, A.T.M. Lau and D. Strauss, *Banach Algebras on Semigroups and their Compactifications*, Memoirs American Mathematical Society, American Mathematical Society, Providence, (2010), Vol. 205, No. 966.
- [3] M. Day, *Amenable groups*, Bull. Amer. Math. Soc, (1950), 56: 46-57.
- [4] M. Day, *Amenable semigroups*, Illinois J. Math, (1957), 1: 509-544.
- [5] J. Duncan and A. L. T Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scandinavica, (1990), 66: 141-146.
- [6] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh Sect, (1978), A 80: 309321.
- [7] G. H. Eslamzadeh, *Ideals and representations of certain semigroup algebras*, Semigroup Forum 69 (2004), 51-62.
- [8] N. Groenbaek, *Amenability of discrete convolution algebras, the commutative case*, Pacic J. Math, (1990), 143: 243-249.
- [9] J. M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford (1995).
- [10] B. E. Johnson, *Cohomology in Banach Algebras*, American Mathematical Society, Providence, RI, 1972.
- [11] H. Rahimi and E. Tahmasebi, *Amenability and Contractibility modulo an ideal of Banach algebras*, Abstract and Applied Analysis, (2014), 514761.
- [12] H. Rahimi and E. Tahmasebi, *A note on amenability modulo an ideal of unital Banach algebras*, J. Mathematical Extension, In press.
- [13] Yu. V. Selivanov, *Banach algebras of small global dimension zero*, Uspekhi Mat. Nauk, (1976), 31: 2(188), 227228.
- [14] J. Von Neumann, *Zur allgemeinen Theorie des Mases*, J. Funct. Anal, (1929), 13: 73-116.