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Higher rank numerical ranges of rectangular matrix polynomials

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Abstract. In this paper, the notion of rank$-k$ numerical range of rectangular complex matrix polynomials are introduced. Some algebraic and geometrical properties are investigated. Moreover, for $\epsilon > 0$, the notion of Birkhoff-James approximate orthogonality sets for $\epsilon-$higher rank numerical ranges of rectangular matrix polynomials is also introduced and studied. The proposed definitions yield a natural generalization of the standard higher rank numerical ranges.

Keywords: Rank$-k$ numerical range, isometry, numerical range, rectangular matrix polynomials.

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1. Introduction and preliminaries

Let $M_{n\times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $M_{n\times n}$ is denoted by M_n; namely, the algebra of all $n \times n$ complex matrices. Throughout the paper, k, m and n are considered as positive integers and $k \leqslant \min\{m, n\}$. Moreover, I_k denotes the $k \times k$ identity matrix. The set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n,k}$, i.e., $\mathcal{X}_{n,k} = \{X \in M_{n\times k} \ : \ X^*X = I_k\}$. For the case $n = k$, $\mathcal{X}_{n,n}$ is denoted by $\mathcal{U}_n$; namely, the group of all $n \times n$ unitary matrices.
Motivation of our study comes from quantum information science. A quantum channel

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is a trace preserving completely positive map such as $L : M_n \to M_n$. By the structure of completely positive linear maps, e.g., see [3], there are matrices $E_1, \ldots, E_r \in M_n$ with $\sum_{j=1}^{r} E_j E_j^* = I_n$ such that $L(A) = \sum_{j=1}^{r} E_j^* A E_j$. The matrices $E_1, \ldots, E_r$ are interpreted as the error operators of the quantum channel L. Let V be a $k-$dimensional subspace of $\mathbb{C}^n$ and P be the orthogonal projection of $\mathbb{C}^n$ onto V. Then, the $k-$dimensional subspace V is a quantum error correction code for the channel L if and only if there are scalars $\gamma_{ij} \in \mathbb{C}$ with $i, j \in \{1, \ldots, r\}$ such that $PE_i^* E_j P = \gamma_{ij} P$; for more information, see [7] and its references, and also see [11]. In this connection, *the rank–k numerical range of* $A \in M_n$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P, \text{ for some rank} - k \text{ orthogonal projection } P \text{ on } \mathbb{C}^n\}.$$

It is known, see [4, Proposition 1.1], that

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^* A X = \lambda I_k, \text{ for some } X \in \mathcal{X}_{n,k}\}.$$

The sets $\Lambda_k(A)$, where $k \in \{1, \ldots, n\}$, are generally called higher rank numerical ranges of A. Apparently, for k=1, $\Lambda_k(A)$ reduces to the *classical numerical range of* A; namely,

$$\Lambda_1(A) = W(A) := \{x^* A x : x \in \mathbb{C}^n, \; x^* x = 1\},$$

which has been studied extensively for many decades; e.g., see [9] and [10, Chapter 1]. Stampfli and Williams in [14, Theorem 4], and later Bonsall and Duncan in [2, Lemma 6.22.1], observed that the numerical range of $A \in M_n$ can be rewritten as:

$$W(A) = \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geqslant |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C}\},$$

where $\|.\|_2$ denotes the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). By this idea, Chorianopoulos, Karanasios and Psarrakos [5] recently introduced a definition of the numerical range for rectangular complex matrices. For any $A, B \in M_{n\times m}$ with $B \neq 0$, and any vector norm $\|.\|$ on $M_{n\times m}$, they defined the *numerical range of* A *with respect to* B as the compact and convex set:

$$W_{\|.\|}(A; B) = \{\mu \in \mathbb{C} : \|A - \lambda B\| \geqslant |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C}\}. \tag{1}$$

It is clear that $W_{\|.\|_2}(A; I_n) = W(A) = \Lambda_1(A)$, where $A \in M_n$. Hence, $W_{\|.\|}(. \, ;.)$ is a direct generalization of the classical numerical range. It is known that $W_{\|.\|}(A; B) \neq \emptyset$ if and only if $\|B\| \geqslant 1$. So, to avoid trivial consideration, we assume that $\|B\| \geqslant 1$.
Suppose

$$P(\lambda) = A_l \lambda^l + A_{l-1} \lambda^{l-1} + \cdots + A_1 \lambda + A_0 \tag{2}$$

is a rectangular matrix polynomial, where $A_i \in M_{n\times m}$ $(i \in \{0, 1, 2, \ldots, l\})$, $A_l \neq 0$, and λ is a complex variable. The study of matrix polynomials has a long history, especially with regard to their applications on higher order linear systems of differential equations; e.g., see [8, 12] and references therein. Let $B \in M_{n\times m}$ and $\| \cdot \|$ be a vector norm on $M_{n\times m}$ such that $\|B\| \geqslant 1$. Moreover, let $P(\lambda)$ be a rectangular matrix polynomial as in (2). Using (1), Chorianopoulos and Psarrakos [6] recently introduced and studied the *numerical range of* $P(\lambda)$ *with respect to* B as:

$$W_{\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in W_{\|\cdot\|}(P(\mu); B)\}. \tag{3}$$

For the case $n = m$, $B = I_n$ and $\|\cdot\| = \|\cdot\|_2$, we have the classical numerical range of the square matrix polynomial $P(\lambda)$; namely,

$$W_{\|\cdot\|_2}[P(\lambda); I_n] = W[P(\lambda)] := \{\mu \in \mathbb{C} : x^*P(\mu)x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\}.$$

Hence, $W_{\|\cdot\|}[.;.]$ is a direct generalization of the classical numerical range of square matrix polynomials, which plays an important role in the study of overdamped vibration systems with a finite number of degrees of freedom, and it also is related to the stability theory; e.g., see [13] and its references. Recently, Aretaki and Maroulas [1] introduced the notion of higher rank numerical ranges of square complex matrix polynomials. Let $P(\lambda)$, as in (2), be a square matrix polynomial; i.e., $n = m$. For a positive integer $k \leqslant n$, they defined the *rank−k numerical range of* $P(\lambda)$ as:

$$\Lambda_k[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ X^*P(\mu)X = 0_k \text{ for some } X \in \mathcal{X}_{n,k}\}, \tag{4}$$

where $0_k \in M_k$ is the zero matrix. It is readily verified that

$$W_{\|\cdot\|_2}[P(\lambda); I_n] = W[P(\lambda)] = \Lambda_1[P(\lambda)] \supseteq \Lambda_2[P(\lambda)] \supseteq \cdots \supseteq \Lambda_n[P(\lambda)].$$

So, the notion of the numerical range of rectangular matrix polynomials is a generalization of the higher rank numerical ranges of square matrix polynomials.
In this paper, we are going to generalize the notion of higher rank numerical ranges of square matrix polynomial to rectangular matrix polynomials. For this, in Section 2, we introduce the notion of rank−k numerical range of a rectangular matrix polynomial, and we investigate some properties of this notion. The emphasis is on the study of the boundedness of this set. In Section 3, we state some additional properties of the higher rank numerical ranges of rectangular matrix polynomials.

2. Main results

In [15], the authors introduced a formula analogous to (1) to propose a definition of the higher rank numerical ranges of rectangular matrices. For any A, $B \in M_{n\times m}$ and any vector norm $\|\cdot\|$ on $M_{(n-k+1)\times(m-k+1)}$, where $1 \leqslant k \leqslant min\{n, m\}$ is a positive integer, they defined the *rank−k numerical range of* A *with respect to* B as

$$\Lambda_{k,\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|X^*(A - \lambda B)Y\| \geqslant |\mu - \lambda|, \ \forall\, \lambda \in \mathbb{C}, \ \forall\, (X, Y) \in \mathcal{X}\}, \tag{5}$$

where

$$\begin{cases} \mathcal{X} = \{(X, Y := \left[\begin{array}{c|c} X & 0 \\ \hline 0 & U \end{array}\right]) \ : \ X \in \mathcal{X}_{n,n-k+1}, \ U \in \mathcal{U}_{m-n}\} & if\ m \geqslant n, \\[2ex] \mathcal{X} = \{(X := \left[\begin{array}{c|c} Y & 0 \\ \hline 0 & U \end{array}\right], Y) \ : \ Y \in \mathcal{X}_{m,m-k+1}, \ U \in \mathcal{U}_{n-m}\} & if\ n \geqslant m. \end{cases} \tag{6}$$

The sets $\Lambda_{k,\|\cdot\|}(.;.)$, where $k \in \{1, 2, \ldots, \min\{m, n\}\}$, are generally called the higher rank numerical ranges of rectangular matrices.
At first, we state some results from [15] which are useful in our discussion. Recall that, in a complex normed space $(X, \|.\|)$, for any $\epsilon \in [0, 1)$, two vectors ϕ and ψ are said to be Birkhoff-James $\epsilon-$orthogonal, denoted by $\phi\perp^{\epsilon}_{BJ}\psi$, if $\|\phi+\lambda\psi\| \geqslant \sqrt{1-\epsilon^2}\|\phi\|$ for all $\lambda \in \mathbb{C}$. For the case $\epsilon = 0$, we write $\phi\perp_{BJ}\psi$ instead $\phi\perp^{0}_{BJ}\psi$. Also, let $1 \leqslant k_2 \leqslant k_1 \leqslant \min\{n, m\}$ be two positive integers. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k_2+1)\times(m-k_2+1)}$. Define $|||\cdot|||$ on $M_{(n-k_1+1)\times(m-k_1+1)}$ by

$$|||Z||| = \| \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & 0_{k_1-k_2} \end{array}\right) \|, \tag{7}$$

where $Z \in M_{(n-k_1+1)\times(m-k_1+1)}$, and $0_{k_1-k_2} \in M_{k_1-k_2}$ is the zero matrix.

Theorem 2.1 Let A, $B \in M_{n\times m}$ and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $\mathcal{X}$ be the set as in (6). Then the following assertions are true:
(i) $\Lambda_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} W_{\|\cdot\|}(X^*AY; X^*BY)$. Consequently, $\Lambda_{k,\|\cdot\|}(A; B)$ is a compact and convex set in $\mathbb{C}$. For the case $k = 1$, if the vector norm $\|\cdot\|$ is unitarily invariant, then

$$\Lambda_1(A; B) = W_{\|\cdot\|}(A; B);$$

(ii) For the case $n = m$, $\Lambda_{k,\|\cdot\|}(A; B) = \bigcap_{X\in\mathcal{X}_{n,n-k+1}} W_{\|\cdot\|}(X^*AX; X^*BX)$. Consequently, if $B = I_n$, then

$$\Lambda_{k,\|\cdot\|_2}(A; I_n) = \Lambda_k(A);$$

(iii) $\Lambda_{k,\|\cdot\|}(UAV; UBV) = \Lambda_{k,\|\cdot\|}(A; B)$, where for the case $m \geqslant n$, $U \in \mathcal{U}_n$ and $V = \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_m$, and for the other case, i.e., $n \geqslant m$, $V \in \mathcal{U}_m$ and $U = \left(\begin{array}{c|c} V^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_n$;
(iv) Let $1 \leqslant k_2 \leqslant k_1 \leqslant \min\{n, m\}$ be two positive integers, $\|\cdot\|$ be a unitarily invariant norm on $M_{(n-k_2+1)\times(m-k_2+1)}$ and $|||\cdot|||$ be the vector norm on $M_{(n-k_1+1)\times(m-k_1+1)}$ as in (7). Then

$$\Lambda_{k_1,|||\cdot|||}(A; B) \subseteq \Lambda_{k_2,\|\cdot\|}(A; B);$$

(v) If $\|X^*BY\| > 1$ for all $(X, Y) \in \mathcal{X}$, then

$$\Lambda_{k,\|\cdot\|}(A; B) \supseteq \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*(A - \mu B)Y\},$$

and the equality holds if $\|X^*BY\| = 1$ for all $(X, Y) \in \mathcal{X}$;
(vi) For any nonzero $b \in \mathbb{C}$,

$$\begin{cases} \text{if } |b| = 1, \ then\ \Lambda_{k,\|\cdot\|}(A; bB) = b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \\ \text{if } |b| < 1, \ then\ \Lambda_{k,\|\cdot\|}(A; bB) \subseteq b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \\ \text{if } |b| > 1, \ then\ \Lambda_{k,\|\cdot\|}(A; bB) \supseteq b^{-1}\Lambda_{k,\|\cdot\|}(A; B); \end{cases}$$

(vii) $\Lambda_{k,\|\cdot\|}(aA + bB; B) = a\Lambda_{k,\|\cdot\|}(A; B) + b$, where $a, b \in \mathbb{C}$.

Now we are ready, by using a formula analogous to (3), to propose a definition of the higher rank numerical ranges of rectangular matrix polynomials.

Definition 2.2 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. The rank$-k$ numerical range of $P(\lambda)$ with respect to B is defined and denoted by

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in \Lambda_{k,\|\cdot\|}(P(\mu); B)\}.$$

The sets $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$, where $k \in \{1, 2, \ldots, \min\{n, m\}\}$, are generally called the *higher rank numerical ranges of $P(\lambda)$ with respect to B*.

Theorem 2.3 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY],$$

where $\mathcal{X}$ is the set as in (6) and $X^*P(\lambda)Y = (X^*A_lY)\lambda^l + \cdots + (X^*A_1Y)\lambda + (X^*A_0Y)$. Consequently, if $k = 1$ and the vector norm $\|\cdot\|$ is unitarily invariant, then

$$\Lambda_{1,\|\cdot\|}[P(\lambda); B] = W_{\|\cdot\|}[P(\lambda); B].$$

Proof. Using Definition 2.2 and Theorem 2.1(i), the first equality is easy to verify. If $k = 1$ and the vector norm $\|\cdot\|$ is unitarily invariant on $M_{n\times m}$, then by Theorem 2.1(i), the second equality can be also easily verify by the first result. So, the proof is complete. ■

Theorem 2.4 Let $B \in M_n$, $P(\lambda)$, as in (2), be a square matrix polynomial (i.e., n=m), and $1 \leqslant k \leqslant n$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on M_{n-k+1}. Then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{X\in\mathcal{X}_{n,n-k+1}} W_{\|\cdot\|}[X^*P(\lambda)X; X^*BX].$$

Consequently, for the case $B = I_n$,

$$\Lambda_{k,\|\cdot\|_2}[P(\lambda); I_n] = \Lambda_k[P(\lambda)].$$

Proof. The results follow directly from Definition 2.2, relation (4) and Theorem 2.1(ii). ■

Remark 1 *Theorems 2.3 and 2.4 show that the notion of rank$-k$ numerical range of rectangular matrix polynomials can be considered as generalizations of the numerical range of rectangular matrix polynomials and the rank$-k$ numerical range of square matrix polynomials.*

Noe, we are going to state some basic properties of the higher rank numerical ranges of rectangular matrix polynomials. For this, we need the following lemma.

Lemma 2.5 [6, Proposition 10] Let $B \in M_{n\times m}$ and $P(\lambda)$ be a rectangular matrix polynomial as in (2). Moreover, let $\|\cdot\|$ be a vector norm on $M_{n\times m}$ and $0 \neq \alpha \in \mathbb{C}$. Then the following assertions are true:
(i) $W_{\|\cdot\|}[\alpha P(\lambda); B] = W_{\|\cdot\|}[P(\lambda); B]$, $W_{\|\cdot\|}[P(\alpha\lambda); B] = \alpha^{-1} W_{\|\cdot\|}[P(\lambda); B]$ and $W_{\|\cdot\|}[P(\lambda + \alpha); B] = W_{\|\cdot\|}[P(\lambda); B] - \alpha$;
(ii) If $R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l$ is the reversal matrix polynomial of $P(\lambda)$, then

$$W_{\|\cdot\|}[R(\lambda); B] \setminus \{0\} = \{\mu \in \mathbb{C} : \frac{1}{\mu} \in W_{\|\cdot\|}[P(\lambda); B],\ \mu \neq 0\}.$$

Proposition 2.6 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:
(i) $\Lambda_{k,\|\cdot\|}[P(\alpha\lambda); B] = \alpha^{-1}\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ and $\Lambda_{k,\|\cdot\|}[\alpha P(\lambda); B] = \Lambda_{k,\|\cdot\|}[P(\lambda); B]$, where $\alpha \in \mathbb{C}$ is nonzero;
(ii) $\Lambda_{k,\|\cdot\|}[P(\lambda + \alpha); B] = \Lambda_{k,\|\cdot\|}[P(\lambda); B] - \alpha$, where $\alpha \in \mathbb{C}$.
(iii) If $R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l$, then

$$\Lambda_{k,\|\cdot\|}[R(\lambda); B] \setminus \{0\} = \{\frac{1}{\mu} : \mu \in \Lambda_{k,\|\cdot\|}[P(\lambda); B],\ \mu \neq 0\}.$$

Proof. Let $\mathcal{X}$ be the set as in (6) and $(X, Y) \in \mathcal{X}$ be given. By setting

$$Q(\lambda) := X^*P(\lambda)Y = (X^*A_lY)\lambda^l + \cdots + (X^*A_1Y)\lambda + (X^*A_0Y),$$

and using Lemma 2.5, we have

$$\begin{aligned} W_{\|\cdot\|}[X^*P(\alpha\lambda)Y; X^*BY] &= W_{\|\cdot\|}[Q(\alpha\lambda); X^*BY] \\ &= \alpha^{-1}W_{\|\cdot\|}[Q(\lambda); X^*BY] \\ &= \alpha^{-1}W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY], \end{aligned}$$

$$\begin{aligned} W_{\|\cdot\|}[X^*(\alpha P(\lambda))Y; X^*BY] &= W_{\|\cdot\|}[\alpha Q(\lambda); X^*BY] \\ &= W_{\|\cdot\|}[Q(\lambda); X^*BY] \\ &= W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY], \end{aligned}$$

and

$$\begin{aligned} W_{\|\cdot\|}[X^*P(\lambda+\alpha)Y; X^*BY] &= W_{\|\cdot\|}[Q(\lambda+\alpha); X^*BY] \\ &= W_{\|\cdot\|}[Q(\lambda); X^*BY] - \alpha \\ &= W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY] - \alpha. \end{aligned}$$

Now, the results in (i) and (ii) follow from Theorem 2.3.

By Theorem 2.3 and Lemma 2.5(ii), we have:

$$\begin{aligned}\mu \neq 0,\ \mu \in \Lambda_{k,\|\cdot\|}[R(\lambda); B] &\Longleftrightarrow \forall (X,Y) \in \mathcal{X},\ \mu \in W_{\|\cdot\|}[X^*R(\lambda)Y; X^*BY],\ \mu \neq 0 \\ &\Longleftrightarrow \forall (X,Y) \in \mathcal{X},\ \frac{1}{\mu} \in W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY],\ \mu \neq 0 \\ &\Longleftrightarrow \frac{1}{\mu} \in \Lambda_{k,\|\cdot\|}[P(\lambda); B],\ \mu \neq 0.\end{aligned}$$

So, the set equality in (ii) also holds. ■

In the following proposition, we investigate the closeness of the rank$-k$ numerical range of rectangular matrix polynomials.

Proposition 2.7 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is a closed set in $\mathbb{C}$.

Proof. In view of Theorem 2.3, it is enough to show that for every $(X,Y) \in \mathcal{X}$, where $\mathcal{X}$ is the set as in (6), $W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$ is closed. Let $(X,Y) \in \mathcal{X}$ and $\{\mu_t\}_{t=1}^{\infty} \subseteq W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$ with $\lim_{t\to\infty} \mu_t = \mu$ be given. We will show that $\mu \in W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$. For this, let $\lambda \in \mathbb{C}$ be arbitrary. By (3) and (1), we have

$$\|X^*P(\mu_t)Y - \lambda X^*BY\| \geqslant |\lambda|$$

for all $t \in \mathbb{N}$. Since $\|\cdot\|$ and $P(\cdot)$ are continuous functions, the above inequality shows that

$$\|X^*P(\mu)Y - \lambda X^*BY\| \geqslant |\lambda|.$$

So, by (3) and (1), $\mu \in W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$, and hence, the result holds. ■

The following example shows that the rank$-k$ numerical range of rectangular matrix polynomials need not be a bounded set, and so a compact set in $\mathbb{C}$.

***Example* 2.8** Let $P(\lambda) = \lambda A - I_2$, where $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2$. By Theorem 2.4, we have:

$$\begin{aligned}\Lambda_{1,\|\cdot\|_2}[P(\lambda); I_2] &= W[P(\lambda)] \\ &= \{\mu \in \mathbb{C} :\ (x^*Ax)\mu = 1,\ \ \textit{for some } x \in \mathbb{C}^2 \textit{and } x^*x = 1\} \\ &= \{\mu \in \mathbb{C} :\ t\mu = 1 \ \text{ for some } t \in [-1,1]\} \\ &= \{\frac{1}{t} :\ t \in [-1,1],\ t \neq 0\} \\ &= (-\infty, -1] \cup [1, +\infty).\end{aligned}$$

So, $\Lambda_{1,\|\cdot\|_2}[P(\lambda); I_2]$ is an unbounded and disconnected set in $\mathbb{C}$.

At the end of this section, we investigate the boundedness of $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$. For this, we need the following Lemma.

Lemma 2.9 [6, Theorem 12] Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), and $\|\cdot\|$ be a vector norm on $M_{n\times m}$. Then the following assertions are true:
(i) If $0 \notin W_{\|\cdot\|}(A_l; B)$, then $W_{\|\cdot\|}[P(\lambda); B]$ is bounded.
(ii) Suppose $0 \in W_{\|\cdot\|}(A_l; B)$ and 0 is not an isolated point of $W_{\|\cdot\|}[R(\lambda); B]$, where

$$R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l.$$

Then $W_{\|\cdot\|}[P(\lambda); B]$ is unbounded.

Theorem 2.10 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:
(i) If $0 \notin \Lambda_{k,\|\cdot\|}(A_l; B)$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is bounded.
(ii) Suppose $0 \in \Lambda_{k,\|\cdot\|}(A_l; B)$ and 0 is not an isolated point of $\Lambda_{k,\|\cdot\|}[R(\lambda); B]$, where

$$R(\lambda) = \lambda^l P(\frac{1}{\lambda}) := A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l.$$

Then $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is unbounded.

Proof. (i); Since $0 \notin \Lambda_{k,\|\cdot\|}(A_l; B)$, by Theorem 2.1$(i)$, there exists a $(X, Y) \in \mathcal{X}$ such that

$$0 \notin W_{\|\cdot\|}(X^*A_lY; X^*BY),$$

where $\mathcal{X}$ is the set as in (6). Using Lemma 2.9(i), $W_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY]$ is a bounded set in $\mathbb{C}$, and hence, by Theorem 2.3, $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ is also bounded.
To prove (ii), since $0 \in \Lambda_{k,\|\cdot\|}(A_l; B)$, by Definition 2.2, it follows that $0 \in \Lambda_{k,\|\cdot\|}[R(\lambda); B]$. Moreover, since 0 is not an isolated point of the set $\Lambda_{k,\|\cdot\|}[R(\lambda); B]$, there is a sequence $\{\mu_t\}_{t\in\mathbb{N}} \subseteq \Lambda_{k,\|\cdot\|}[R(\lambda); B] \setminus \{0\}$ such that $\lim_{t\to\infty} \mu_t = 0$. So, by Proposition 2.6$(iii)$, we have

$$\{\mu_t^{-1}\}_{t\in\mathbb{N}} \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B],$$

and hence, the result in (ii) follows from this fact that the range of the sequence $\{\mu_k^{-1}\}_{k\in\mathbb{N}}$ is unbounded. ■

3. Additional results

In this section, we investigate some algebraic properties of the higher rank numerical range of rectangular matrix polynomials.

Proposition 3.1 Let $B \in M_{n\times m}$ and $P(\lambda) = q(\lambda)B$, where $q(\lambda)$ is a scalar polynomial. Moreover, let $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer and $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : q(\mu) = 0\}.$$

Proof. Using Definition 2.2 and Theorem 2.1(vii), we have:

$$\begin{aligned}\mu \in \Lambda_{k,\|\cdot\|}[P(\lambda); B] &\Longleftrightarrow 0 \in \Lambda_{k,\|\cdot\|}(P(\mu); B) = \Lambda_{k,\|\cdot\|}(q(\mu)B; B) = \{q(\mu)\}\\ &\Longleftrightarrow q(\mu) = 0.\end{aligned}$$

So, the result holds. ■

In the following theorem, we show that the rank$-k$ numerical range of rectangular matrix polynomials is invariant under some unitary matrices.

Theorem 3.2 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then

$$\Lambda_{k,\|\cdot\|}[UP(\lambda)V; UBV] = \Lambda_{k,\|\cdot\|}[P(\lambda); B],$$

where for the case $m \geqslant n$, $U \in \mathcal{U}_n$ and $V = \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_m$, and for the other case, i.e., $n \geqslant m$, $V \in \mathcal{U}_m$ and $U = \left(\begin{array}{c|c} V^* & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{U}_n$. Also, $UP(\lambda)V = (UA_lV)\lambda^l + \cdots + (UA_1V)\lambda + (UA_0V)$.

Proof. Using Definition 2.2 and Theorem 2.1(iii), the result is easy to verify. ■

In the following theorem, we state the relationship between higher rank numerical ranges of rectangular matrix polynomials.

Theorem 3.3 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k_2 \leqslant k_1 \leqslant \min\{n, m\}$ be two positive integers. Moreover, let $\|\cdot\|$ be a unitarily invariant norm on $M_{(n-k_2+1)\times(m-k_2+1)}$ and $|||\cdot|||$ be the vector norm on $M_{(n-k_1+1)\times(m-k_1+1)}$ as in (7). Then

$$\Lambda_{k_1,|||\cdot|||}[P(\lambda); B] \subseteq \Lambda_{k_2,\|\cdot\|}[P(\lambda); B].$$

Proof. Let $\mu \in \Lambda_{k_1,|||\cdot|||}[P(\lambda); B]$ be given. So, by Definition 2.2, $0 \in \Lambda_{k_1,|||\cdot|||}(P(\mu); B)$, and hence, by Theorem 2.1(iv), $0 \in \Lambda_{k_2,|||\cdot|||}(P(\mu); B)$. So, $\mu \in \Lambda_{k_2,|||\cdot|||}[P(\lambda); B]$. Hence, the proof is complete. ■

Using Definition 2.2 and Theorem 2.1(v), we have the following proposition.

Proposition 3.4 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $\mathcal{X}$ be the set as in (6). Then the following assertions are true:
(i) If $\|X^*BY\| = 1$ for all $(X, Y) \in \mathcal{X}$, then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*P(\mu)Y\};$$

(ii) If $\|X^*BY\| > 1$ for all $(X, Y) \in \mathcal{X}$, then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] \supseteq \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp_{BJ} X^*P(\mu)Y\}.$$

The following proposition follows from Definition 2.2 and Theorem 2.1(vi).

Proposition 3.5 Let $B \in M_{n\times m}$, $0 \neq b \in \mathbb{C}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. Then the following assertions are true:
(i) If $|b| = 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] = \Lambda_{k,\|\cdot\|}[P(\lambda); B]$;
(ii) If $|b| < 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B]$;
(iii) If $|b| > 1$, then $\Lambda_{k,\|\cdot\|}[P(\lambda); bB] \supseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B]$.

Corollary 3.6 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2) and $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer. Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$. If $\|B\| > 1$, then

$$\Lambda_{k,\|\cdot\|}[P(\lambda); \|B\|^{-1}B] \subseteq \Lambda_{k,\|\cdot\|}[P(\lambda); B].$$

Remark 2 *Let $A, B \in M_{n\times m}$ and $P(\lambda) = \lambda B - A$. Using Definition 2.2 and Theorem 2.1(vii), we have*

$$\Lambda_{k,\|\cdot\|}[P(\lambda); B] = \Lambda_{k,\|\cdot\|}(A; B).$$

Now, if $\|\cdot\|$ is unitarily invariant, then by Theorem 2.3, we have $\Lambda_{1,\|\cdot\|}[P(\lambda); B] = W_{\|\cdot\|}(A; B)$, and so, one can find some numerical examples from [5] or [6] to see the shape of $\Lambda_{1,\|\cdot\|}[P(\lambda); B]$. But in general, it is interesting if we have a MATLAB program to plotting the shape of $\Lambda_{k,\|\cdot\|}[P(\lambda); B]$ for any k and for any rectangular matrix polynomial.

Let $A, B \in M_{n\times m}$, $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer, and $\mathcal{X}$ be the set as in (6). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leqslant \epsilon < 1$. The *Birkhoff-James* $\epsilon-$orthogonality set of A *with respect to* B is defined and denoted, [6, Definition 1], by

$$W^{\epsilon}_{\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|A - \lambda B\| \geqslant \sqrt{1-\epsilon^2}\|B\||\mu - \lambda|, \ \forall\lambda \in \mathbb{C}\}.$$

Also, the *rank*$-k, \epsilon$ *numerical range of* A *with respect to* B is defined and denoted, e.g., see [15, Definition 2.13], by

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : \|X^*(A - \lambda B)Y\| \geqslant \sqrt{1-\epsilon^2}\|X^*BY\||\mu - \lambda|, \ \forall\lambda \in \mathbb{C}, \ \forall(X, Y) \in \mathcal{X}\},$$

and by [15, Theorem 2.14 and Proposition 2.15], we have

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} W^{\epsilon}_{\|\cdot\|}(X^*AY; X^*BY), \tag{8}$$

$$\Lambda^{\epsilon}_{k,\|\cdot\|}(A; B) = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp^{\epsilon}_{BJ} X^*(A - \mu B)Y\}. \tag{9}$$

Moreover, let $P(\lambda)$ be a rectangular matrix polynomial as in (2). The *Birkhoff-James* $\epsilon-$orthogonality set of $P(\lambda)$ *with respect to* B is defined and denoted, e.g., see [6, Relation

(11)], by

$$W^{\epsilon}_{\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in W^{\epsilon}_{\|\cdot\|}(P(\mu); B)\}. \tag{10}$$

By this idea, at the end of this section, we introduce and study the notion of rank$-k, \epsilon$ numerical range of rectangular matrix polynomials.

Definition 3.7 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer, and $\mathcal{X}$ be the set as in (6). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leqslant \epsilon < 1$. The *rank*$-k, \epsilon$ *numerical range of* $P(\lambda)$ *with respect to* B is defined and denoted by

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : 0 \in \Lambda^{\epsilon}_{k,\|\cdot\|}(P(\mu); B)\}.$$

It is clear that:

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \{\mu \in \mathbb{C} : \|X^*(P(\mu) - \lambda B)Y\| \geqslant \sqrt{1-\epsilon^2}\|X^*BY\||\lambda|, \ \forall\lambda \in \mathbb{C}, \ \forall(X, Y) \in \mathcal{X}\}.$$

Using Definition 3.7, Relations (8), (9), (10), and Theorem 2.3, we have the following theorem.

Theorem 3.8 Let $B \in M_{n\times m}$, $P(\lambda)$ be a rectangular matrix polynomial as in (2), $1 \leqslant k \leqslant \min\{n, m\}$ be a positive integer and $\mathcal{X}$ be the set as in (6). Moreover, let $\|\cdot\|$ be a vector norm on $M_{(n-k+1)\times(m-k+1)}$ and $0 \leqslant \epsilon < 1$. Then

$$\Lambda^{\epsilon}_{k,\|\cdot\|}[P(\lambda); B] = \bigcap_{(X,Y)\in\mathcal{X}} W^{\epsilon}_{\|\cdot\|}[X^*P(\lambda)Y; X^*BY] = \bigcap_{(X,Y)\in\mathcal{X}} \{\mu \in \mathbb{C} : X^*BY \perp^{\epsilon}_{BJ} X^*P(\mu)Y\}.$$

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