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# Fixed point theorems for $\alpha$-$\psi$-$\phi$-contractive integral type mappings

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**Abstract.** In this paper, we introduce a new concept of $\alpha$-$\psi$-$\phi$-contractive integral type mappings and establish some new fixed point theorems in complete metric spaces.



**Keywords:** Fixed point, $\alpha$-$\beta$-$\psi$-contractive integral type mapping, complete metric space.

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## 1. Introduction and Preliminaries

Fixed point theory became one of the most interesting area of research in the last forty years for instance research about control theory, differential equations, integral equations, economics, and etc. The fixed point theorem , generally known as the Banach contraction mapping principle, appeared by Banach in 1922 [1]. Later, the Banach contraction principle has been widely generalized and extended (see, for example, [2, 3, 5–15]).

In 2002, Branciari [2], introduced the integral contraction as follow.

**Theorem 1.1** Let $(X, d)$ be a complete metric space, $c \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$ ,

$$\int_0^{d(fx,fy)} \phi(t)dt \leqslant c \int_0^{d(x,y)} \phi(t)dt;$$

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where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$ , $lim_{n\to\infty} f^n x = a$.

Recently, Samet et al.[14] introduced $\alpha$-$\psi$-contractive type mappings and established some fixed point theorems in complete metric spaces. In this paper, we introduce a new $\alpha$-$\psi$-$\phi$-contractive integral type mapping and establish some new fixed point theorems in complete metric spaces. The presented theorems generalized and extended Banach contraction, Branciari integral contraction and Samet et al. $\alpha$-$\psi$-contraction and those contained therein.

We start by recalling a few definitions and lemmas.

**Definition 1.2** [14] Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that $f : X \to X$ is $\alpha$-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geqslant 1 \Longrightarrow \alpha(f(x), f(y)) \geqslant 1. \tag{1}$$

**Definition 1.3** [14] Let $\Psi$ be a family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi$ and $t > 0$, $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, where $\psi^n$ is the n-th iterate of $\psi$.

**Lemma 1.4** ([14]) If $\psi : [0, \infty) \to [0, \infty)$ is nondecreasing function and for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ then $\psi(t) < t$.

**Definition 1.5** Let $\Phi$ be a collection of mappings $\phi : [0, \infty) \to [0, \infty)$ which are Lebesgue-integrable, summable on each compact subset of $[0, \infty)$ and satisfying following condition:

$$\int_0^\epsilon \phi(t)dt > 0,$$

for each $\epsilon > 0$ and all $t \in [0, +\infty)$.

**Lemma 1.6** ([5]) Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$, then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

**Lemma 1.7** ([5]) Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = 0 \Longleftrightarrow \lim_{n\to\infty} r_n = 0.$$

## 2. Fixed Point Theorems

We start the main section by presenting the new notion of $\alpha$-$\psi$-$\phi$-contractive integral type mapping.

**Definition 2.1** Let $(X, d)$ be a metric space. We say that $f : X \to X$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping if there exist three functions $\alpha : X \times X \to [0, +\infty)$,

$\phi \in \Phi$ and $\psi \in \Psi$ such that,

$$\int_0^{\alpha(x,y)d(fx,fy)} \phi(t)dt \leqslant \psi\left(\int_0^{d(x,y)} \phi(t)dt\right); \tag{2}$$

for all $x, y \in X$ and all $t \in [0, +\infty)$.

***Remark 1*** *If $f : X \to X$ satisfies the Banach contraction principle, then $f$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$, $\phi(t) = 1$ and $\psi(t) = ct$ for all $t \geqslant 0$ and some $c \in [0, 1)$.*

**Theorem 2.2** Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a continuous mapping satisfying following conditions:

(i) $f$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;
(iii) $f$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping,

then, $f$ has a fixed point $a \in X$ such that for $x_0 \in X$ , $lim_{n\to\infty} f^n x_0 = a$.

**Proof.** Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$. Since $f$ is $\alpha$-admissible then, we get

$$\alpha(x_0, fx_0) \geqslant 1 \to \alpha(fx_0, f^2x_0) \geqslant 1 \to \ldots \to \alpha(f^n x_0, f^{n+1}x_0) \geqslant 1,$$

hence, for all $n \in \mathbb{N}$,

$$\alpha(f^n x_0, f^{n+1}x_0) \geqslant 1. \tag{3}$$

We will show that

$$\int_0^{d(f^n x_0, f^{n+1}x_0)} \phi(t)dt \leqslant \psi^n\left(\int_0^{d(x_0,fx_0)} \phi(t)dt\right). \tag{4}$$

Since $f$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping then, by induction, we have

$$\begin{aligned}\int_0^{d(fx_0,f^2x_0)} \phi(t)dt &\leqslant \int_0^{\alpha(x_0,fx_0)d(fx_0,f^2x_0)} \phi(t)dt \\ &\leqslant \psi\left(\int_0^{d(x_0,fx_0)} \phi(t)dt\right),\end{aligned}$$

now, assuming that (4) holds for an $n \in \mathbb{N}$, then, from (3) and the hypothesis (iii) we get

$$\begin{aligned}\int_0^{d(f^{n+1}x_0,f^{n+2}x_0)} \phi(t)dt &\leqslant \int_0^{\alpha(f^nx_0,f^{n+1}x_0)d(f^{n+1}x_0,f^{n+2}x_0)} \phi(t)dt\\ &\leqslant \psi\left(\int_0^{d(f^nx_0,f^{n+1}x_0)} \phi(t)dt\right)\\ &\leqslant \psi\left(\psi^n\left(\int_0^{d(x_0,fx_0)} \phi(t)dt\right)\right)\\ &= \psi^{n+1}\left(\int_0^{d(x_0,fx_0)} \phi(t)dt\right).\end{aligned}$$

Therefore (4) holds for all $n \in \mathbb{N}$. Now, as $n \to +\infty$, in (4) then, from Lemma 1.4 we obtain

$$\int_0^{d(f^nx_0,f^{n+1}x_0)} \phi(t)dt \longrightarrow 0,$$

hence, by Lemma 1.7

$$d(f^nx_0, f^{n+1}x_0) \longrightarrow 0.$$

In the following, we show that $\{f^nx_0\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose that it is not. Then, there exists an $\epsilon > 0$ such that for each $k \in \mathbb{N}$ there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k > k$, such that

$$d(f^{m_k}x_0, f^{n_k}x_0) \geqslant \epsilon, \quad d(f^{m_k-1}x_0, f^{n_k}x_0) < \epsilon. \tag{5}$$

Hence,

$$\begin{aligned}\epsilon \leqslant d(f^{m_k}x_0, f^{n_k}x_0) &\leqslant d(f^{m_k}x_0, f^{m_k-1}x_0) + d(f^{m_k-1}x_0, f^{n_k}x_0)\\ &< d(f^{m_k}x_0, f^{m_k-1}x_0) + \epsilon.\end{aligned}$$

Letting $k \to +\infty$ then, we have

$$d(f^{m_k}x_0, f^{n_k}x_0) \longrightarrow \epsilon^+, \tag{6}$$

this implies that there exists $l \in \mathbb{N}$, such that

$$k > l \Longrightarrow d(f^{m_l+1}x_0, f^{n_l+1}x_0) < \epsilon. \tag{7}$$

Actually, if there exists a subsequence $\{k_l\} \subset \mathbb{N}$, $k_l > l$,

$$d(f^{m_{k_l}+1}x_0, f^{n_{k_l}+1}x_0) \geqslant \epsilon,$$

as $l \to +\infty$, we get

$$\begin{aligned}\epsilon &\leqslant d(f^{m_{k_l}+1}x_0, f^{n_{k_l}+1}x_0)\\ &\leqslant d(f^{m_{k_l}+1}x_0, f^{m_{k_l}}x_0) + d(f^{m_{k_l}}x_0, f^{n_{k_l}}x_0) + d(f^{n_{k_l}}x_0, f^{n_{k_l}+1}x_0)\\ &\to \epsilon\end{aligned}$$

therefore, from (2) and lemma 1.4, we have

$$\begin{aligned}\int_0^{d(f^{m_{k_l}+1}x_0, f^{n_{k_l}+1}x_0)} \phi(t)dt &\leqslant \psi\left(\int_0^{d(f^{m_{k_l}}x_0, f^{n_{k_l}}x_0)} \phi(t)dt\right)\\ &< \int_0^{d(f^{m_{k_l}}x_0, f^{n_{k_l}}x_0)} \phi(t)dt.\end{aligned}$$

Letting $l \to +\infty$, we obtain

$$\int_0^{\epsilon} \phi(t)dt < \int_0^{\epsilon} \phi(t)dt,$$

which is a contradiction. So, (7) holds. Now, we prove that there exists $\sigma_\epsilon \in (0, \epsilon), k_\epsilon \in \mathbb{N}$ such that

$$k > k_\epsilon \Longrightarrow d(f^{m_k+1}x_0, f^{n_k+1}x_0) < \epsilon - \sigma_\epsilon. \tag{8}$$

Assume that (8) is not true, from (7) there exists a subsequence $\{k_l\} \subset \mathbb{N}$,

$$d(f^{m_{k_l}+1}x_0, f^{n_{k_l}+1}x_0) \to \epsilon^-, \quad as \quad l \to +\infty.$$

Then, we get

$$\int_0^{d(f^{m_{k_l}+1}x_0, f^{n_{k_l}+1}x_0)} \phi(t)dt \leqslant \psi\left(\int_0^{d(f^{m_{k_l}}x_0, f^{n_{k_l}}x_0)} \phi(t)dt\right) < \int_0^{d(f^{m_{k_l}}x_0, f^{n_{k_l}}x_0)} \phi(t)dt,$$

if $l \to +\infty$ then, we have

$$\int_0^{\epsilon} \phi(t)dt < \int_0^{\epsilon} \phi(t)dt,$$

which is a contradiction. Therefore, (8) holds. So, $\{f^n x_0\}_{n=1}^{\infty}$ is a Cauchy sequence. In fact, for each $k > k_\epsilon$ we have

$$\begin{aligned}\epsilon \leqslant d(f^{m_k}x_0, f^{n_k}x_0) &\leqslant d(f^{m_k}x_0, f^{m_k+1}x_0) + d(f^{m_k+1}x_0, f^{n_k+1}x_0)\\ &\quad + d(f^{n_k+1}x_0, f^{n_k}x_0)\\ &< d(f^{m_k}x_0, f^{m_k+1}x_0) + (\epsilon - \sigma_\epsilon) + d(f^{n_k+1}x_0, f^{n_k}x_0)\\ &\quad \to \epsilon - \sigma_\epsilon \quad as \quad k \to \infty.\end{aligned}$$

which is a contradiction. Since $(X, d)$ is a complete metric space, then there exists $a \in X$ such that $\lim_{n\to\infty} f^n x_0 = a$. Now, we show that $a$ is a fixed point of $f$. Suppose $\epsilon > 0$ is given. Since $f$ is a continuous function, then there exists $\delta > 0$ such that, for each $z \in X$, $d(z, x) < \delta$ implies that $d(fz, fx) < \frac{\epsilon}{2}$. Given $\eta = min\{\frac{\epsilon}{2}, \delta\}$, now by convergence of $\{f^n x_0\}_{n=1}^{\infty}$ to $a$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n \geqslant n_0$,

$$d(f^n x_0, a) < \eta.$$

Taking $n \in \mathbb{N}$, $n \geqslant n_0$, we get

$$\begin{aligned} d(fa, a) &\leqslant d(fa, f(f^n x_0)) + d(f^{n+1}x_0, a) \\ &< \frac{\epsilon}{2} + \eta \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

therefore, $d(fa, a) = 0$ and so $fa = a$, which means $a$ is a fixed point of $f$. ■

In the next theorem, the continuity hypothesis of $f$ has been omitted.

**Theorem 2.3** Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a mapping satisfying following conditions:

(i) $f$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;
(iii) $f$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping,
(iv) if $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ then, $\alpha(x_n, x) \geqslant 1$;

Then, $f$ has a fixed point $a \in X$ such that for $x_0 \in X$, $lim_{n\to\infty} f^n x_0 = a$.

**Proof.** Following the proof of Theorem 2.2, there exists $a \in X$ such that $\lim_{n\to\infty} f^n x_0 = a$. To prove $a$ is a fixed point of $f(x)$, we have

$$d(a, fa) \leqslant d(f(f^n x_0), fa) + d(f^{n+1}x_0, a),$$

since, $d(f^{n+1}x_0, a) \longrightarrow 0$, it is sufficient to show that $d(f(f^n x_0), fa) \longrightarrow 0$. So, from (2), Lemma 1.4 and the hypotheses $(iii)$ and $(iv)$, we get

$$\begin{aligned} 0 < \int_0^{d(f(f^n x_0), fa)} \phi(t)dt &\leqslant \int_0^{\alpha(f^n x_0, a) d(f(f^n x_0), fa)} \phi(t)dt \\ &\leqslant \psi\left(\int_0^{d(f^n x_0, a)} \phi(t)dt\right) \\ &\leqslant \int_0^{d(f^n x_0, a)} \phi(t)dt, \end{aligned}$$

as $n \longrightarrow +\infty$, we get

$$\int_0^{d(f(f^n x_0), fa)} \gamma(t)dt \longrightarrow 0,$$

consequently,

$$d(f(f^n x_0), fa) \longrightarrow 0.$$

Therefore, $d(fa, a) = 0$ and so $fa = a$, which means $a$ is a fixed point of $f$. ■

In the next theorem, we discuss about the uniqueness of fixed points of above mentioned theorems.

**Theorem 2.4** Suppose all the hypotheses of Theorems 2.2 and 2.3 are satisfied. If there exists $z \in X$ such that for all $x, y \in X$,

$$\alpha(x, z) \geqslant 1 \quad and \quad \alpha(y, z) \geqslant 1, \tag{9}$$

then, $f$ has a unique fixed point.

**Proof.** Suppose $x^\star$ and $y^\star$ are two fixed points of $f$, then $f(x^\star) = x^\star$ and $f(y^\star) = y^\star$. From (9), there exists $z \in X$ such that

$$\alpha(x^\star, z) \geqslant 1 \quad and \quad \alpha(y^\star, z) \geqslant 1. \tag{10}$$

Since $f$ is an $\alpha$-admissible function then

$$\alpha(fx^\star, fz) \geqslant 1 \quad and \quad \alpha(fy^\star, fz) \geqslant 1,$$

therefore,

$$\alpha(x^\star, fz) \geqslant 1 \quad and \quad \alpha(y^\star, fz) \geqslant 1,$$

Continuing this process, we get

$$\alpha(x^\star, f^n z) \geqslant 1 \quad and \quad \alpha(y^\star, f^n z) \geqslant 1, \tag{11}$$

for all $n \in \mathbb{N}$. Using (2) and the first part of (11), we have

$$\begin{aligned}
\int_0^{d(x^\star, f^n z)} \phi(t)dt &= \int_0^{d(fx^\star, f(f^{n-1}z))} \phi(t)dt \\
&\leqslant \int_0^{\alpha(x^\star, f^{n-1}z)d(fx^\star, f(f^{n-1}z))} \phi(t)dt \\
&\leqslant \psi\left(\int_0^{d(x^\star, f^{n-1}z))} \phi(t)dt\right) \\
&\leqslant \psi\left(\psi\left(\int_0^{d(x^\star, f^{n-2}z))} \phi(t)dt\right)\right) \\
&\vdots \\
&\leqslant \psi^n\left(\int_0^{d(x^\star, z)} \phi(t)dt\right),
\end{aligned}$$

for all $n \in \mathbb{N}$. Now, letting $n \to \infty$ from lemmas 1.4 and 1.7, $\lim_{n\to\infty} f^n(z) = x^\star$. Similarly, $\lim_{n\to\infty} f^n(z) = y^\star$. Therefore, $x^\star = y^\star$. Hence $f$ has a unique fixed point. ■

In the following, we define subclass of integrals and we prove the existence of fixed point of $f$ by applying these integrals.

**Definition 2.5** Let $\Gamma$ be a collection of mappings $\gamma : [0,\infty) \to [0,\infty)$ which are Lebesgue-integrable, summable on each compact subset of $[0,\infty)$ and satisfying following conditions:

(i) $\int_0^\epsilon \gamma(t)dt > 0$, for each $\epsilon > 0$
(ii) $\int_0^{a+b} \gamma(t)dt \leqslant \int_0^a \gamma(t)dt + \int_0^b \gamma(t)dt$, for each $a, b \geqslant 0$

***Remark 2*** *If we replace $\Gamma$ instead of $\Phi$ at the definition of 2.1, then we can say $f$ is an $\alpha$-$\psi$-$\gamma$-contractive integral type mapping if there exist three functions $\alpha : X \times X \to [0,+\infty)$, $\gamma \in \Gamma$ and $\psi \in \Psi$ such that*

$$\int_0^{\alpha(x,y)d(fx,fy)} \gamma(t)dt \leqslant \psi\left(\int_0^{d(x,y)} \gamma(t)dt\right), \tag{12}$$

*for all $x, y \in X$ and all $t \in [0,+\infty)$.*

**Theorem 2.6** Let $(X,d)$ be a complete metric space. Let $f : X \to X$ be a continuous mapping satisfying following conditions:

(i) $f$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;
(iii) $f$ is an $\alpha$-$\psi$-$\gamma$-contractive integral type mapping,

then $f$ has a fixed point $a \in X$ such that for $x_0 \in X$, $lim_{n\to\infty} f^n x_0 = a$.

**Proof.** Similar to the proof of Theorem 2.2, we have

$$\int_0^{d(f^n x_0, f^{n+1} x_0)} \gamma(t)dt \leqslant \psi^n\left(\int_0^{d(x_0,fx_0)} \gamma(t)dt\right). \tag{13}$$

In the following, we will show that $\{f^n x_0\}_{n=1}^\infty$ is a Cauchy sequence. Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geqslant n(\epsilon)} \psi^n\left(\int_0^{d(x_0,fx_0)} \gamma(t)dt\right) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n(\epsilon)$, we get

$$\begin{aligned}
\int_0^{d(f^n x_0, f^m x_0)} \gamma(t)dt &\leqslant \int_0^{d(f^n x_0, f^{n+1}x_0)+\cdots+d(f^{m-1}x_0, f^m x_0)} \gamma(t)dt \\
&\leqslant \int_0^{d(f^n x_0, f^{n+1}x_0)} \gamma(t)dt + \cdots + \int_0^{d(f^{m-1}x_0, f^m x_0)} \gamma(t)dt \\
&\leqslant \psi^n \left( \int_0^{d(x_0, fx_0)} \gamma(t)dt \right) + \cdots + \psi^{m-1} \left( \int_0^{d(x_0, fx_0)} \gamma(t)dt \right) \\
&= \sum_{k=n}^{m-1} \psi^k \left( \int_0^{d(x_0, fx_0)} \gamma(t)dt \right) \\
&\leqslant \sum_{n \geqslant n(\epsilon)} \psi^n \left( \int_0^{d(x_0, fx_0)} \gamma(t)dt \right) < \epsilon,
\end{aligned}$$

which means

$$\int_0^{d(f^n x_0, f^m x_0)} \gamma(t)dt \longrightarrow 0,$$

therefore,

$$d(f^n x_0, f^m x_0) \longrightarrow 0,$$

hence, $\{f^n x_0\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X, d)$ is a complete metric space, then there exists $a \in X$ such that, $\lim_{n\to\infty} f^n x_0 = a$. The rest of the proof is similar to the proof of Theorem 2.2. ■

In the next theorem, the continuity hypothesis of $f$ has been removed.

**Theorem 2.7** Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a mapping satisfying following conditions:

(i) $f$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;
(iii) $f$ is an $\alpha$-$\psi$-$\gamma$-contractive integral type mapping,
(iv) if $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ then, $\alpha(x_n, x) \geqslant 1$.

Then, $f$ has a fixed point $a \in X$ such that for $x_0 \in X$, $lim_{n\to\infty} f^n x_0 = a$.

**Proof.** The first part of the proof is similar to the proof of Theorem 2.6 and the second part is similar to the Theorem 2.3. ■

***Remark 3*** *The uniqueness of the fixed points of Theorems 2.6 and 2.7 comes from theorem 2.4.*

## 3. Examples and Final Remarks

***Example* 3.1** Let $X = \mathbb{R}$ endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ by $fx = \frac{2x}{5}$ and $\alpha : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} \frac{3}{2} & \text{if } 0 \leqslant y \leqslant x; \\ 0 & \text{o.w.} \end{cases}$$

Clearly, $f$ is continuous. $f$ is $\alpha$-admissible, since for $\alpha(x,y) \geqslant 1$, we have

$$\alpha(fx, fy) = \alpha(\frac{2x}{5}, \frac{2y}{5}) \geqslant 1.$$

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t^2$ then $\phi \in \Phi$. There exists $x_0 = 0 \in \mathbb{R}$ such that $\alpha(0, f0) = \alpha(0,0) = 3/2 \geqslant 1$. Let $\psi(t) = \frac{t}{3}$ where $\psi : [0,\infty) \to [0,\infty)$ then, for all $x, y \in \mathbb{R}$

$$\begin{aligned} \int_0^{\alpha(x,y)d(fx,fy)} \phi(t)dt &= \int_0^{\frac{3}{2}d(\frac{2x}{5},\frac{2y}{5})} t^2 dt \\ &\leqslant \frac{\int_0^{d(x,y)} t^2 dt}{3} \\ &= \psi\left(\int_0^{d(x,y)} \phi(t)dt\right). \end{aligned}$$

Then, $f$ is an $\alpha$-$\psi$-$\phi$-contractive integral type mapping, hence all the hypotheses of Theorems 2.2 are satisfied, consequently $f$ has a fixed point. Here, 0 is a fixed point of $f(x)$.

***Example* 3.2** Let $X = \mathbb{R}$ endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ and $\alpha : X \times X \to [0, +\infty)$, respectively by

$$f(x) = \begin{cases} 2x - \frac{1}{2} & \text{if } x \geqslant \frac{1}{2}; \\ \frac{x}{2} & \text{if } 0 \leqslant x < \frac{1}{2}; \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{1}{2}]; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f$ is discontinuous. Define $\gamma : [0,\infty) \to [0,\infty)$ by $\gamma(t) = 1$ then $\gamma \in \Gamma$. Let $\psi(t) = \frac{t}{2}$ for each $t > 0$ then $f$ is an $\alpha$-$\psi$-$\gamma$-contractive integral type mapping. For all

$x, y \in \mathbb{R}$, if $\alpha(x,y) \geqslant 1$ then we get

$$\begin{aligned}\int_0^{\alpha(x,y)d(fx,fy)} \gamma(t)dt &\leqslant \int_0^{d(fx,fy)} 1dt \\ &= d(f(x), f(y)) \\ &= \frac{|x-y|}{2} \\ &= \psi(d(x,y)) \\ &= \psi\left(\int_0^{d(x,y)} 1dt\right) \\ &= \psi\left(\int_0^{d(x,y)} \gamma(t)dt\right).\end{aligned}$$

Moreover, there exists $x_0 \in \mathbb{R}$ such that $\alpha(x_0, fx_0) \geqslant 1$. Let $x_0 = 0$ then

$$\alpha(x_0, fx_0) = \alpha(0, f0) = \alpha(0,0) = 1 \geqslant 1.$$

Now, let $x, y \in \mathbb{R}$ such that $\alpha(x,y) \geqslant 1$, which implies that $x, y \in [0, \frac{1}{2}]$, then

$$\alpha(x,y) = 1 \geqslant 1 \quad \to \quad \alpha(fx, fy) = \alpha(\frac{x}{2}, \frac{y}{2}) = 1 \geqslant 1.$$

Therefore, $f$ is $\alpha$-admissible. Finally, if $\{x_n\}$ be a nondecreasing sequence in $\mathbb{R}$ such that $\alpha(x_{n+1}, x_n) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ then, by definition of $\alpha$, $x_n \in [0, \frac{1}{2}]$ consequently, $x \in [0, \frac{1}{2}]$. Then $\alpha(x_n, x) = 1 \geqslant 1$. Therefore, all the required hypotheses of Theorem 2.3 are satisfied, then $f$ has a fixed point. Here, 0 and $\frac{1}{2}$ are two fixed points of $f$.

Let $(X, d)$ be a complete metric space, we will show that the presented results extend and generalize some fixed point theorems in the literature.

***Remark 4*** *In theorems 2.2, 2.3, 2.6 and 2.7 if $\alpha(x,y) = 1$ for all $x, y \in X$ with $\psi(t) = ct$ for all $t > 0$ and some $c \in (0,1)$ and $\phi(t) = 1$ or $\gamma(t) = 1$, then we get the Banach contraction principle. In fact, we have*

$$d(fx, fy) = \int_0^{\alpha(x,y)d(fx,fy)} 1dt \leqslant c \int_0^{d(x,y)} 1dt = cd(x,y).$$

***Remark 5*** *In theorems 2.2, 2.3, 2.6 and 2.7 if $\alpha(x,y) = 1$ for all $x, y \in X$ and $\psi(t) = ct$ for all $t > 0$ and $c \in (0,1)$ then, we obtain the Branciari theorem(Theorem 1.1).*

***Remark 6*** *Let $\phi(t) = 1$ and $\gamma(t) = 1$ at the Definition of 2.1 and Remark 3 we obtain the $\alpha$-$\psi$-contractive mapping which is introduced by Samet et al. and playing main role in [14]. Actually, we have*

$$\alpha(x,y)d(fx,fy) = \int_0^{\alpha(x,y)d(fx,fy)} 1dt \leqslant \psi\left(\int_0^{d(x,y)} 1dt\right) = \psi(d(x,y)).$$

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## References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fund. Math 3 (1922) 133-181.
[2] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci 29 (2002) 531-536.
[3] A. Djoudi, F. Merghadi, *Common fixed point theorems for maps under a contractive condition of integral type*, J. Math. Anal. Appl. 341 (2) (2008) 953-960.
[4] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 60 (1968), 71-76.
[5] Z. Liu, X. Li, S. M. Kang, S. Y. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory Appl No. 64 (2011) 18 pages doi: 10.1186/1687-1812-2011-64.
[6] J. Meszros, *A comparison of various definitions of contractive type mappings*, Bull. Calcutta Math. Soc. 84 (1992), no. 2, 167-194.
[7] Mocanu, M, Popa, V, *Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces*, Libertas Math 28 (2008) 1-13.
[8] H. Rahimi, G.Soleimani Rad, *Fixed point theory in various spaces*, Lambert Academic Publishing, Germany, 2013.
[9] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), 257-290.
[10] B.E. Rhoades, *A fixed point theorem of integral type*, J. Adv. Math. Stud. 5 (2) (2012) 98-100.
[11] B. E. Rhoades, *Contractive definitions revisited, Topological Methods in Nonlinear Functional Analysis* (Toronto, Ont., 1982), Contemp. Math., vol. 21, American Mathematical Society, Rhode Island, 1983, 189-205.
[12] B. E. Rhoades, *Contractive definitions*, Nonlinear Analysis, World Science Publishing, Singapore, 1987, 513-526.
[13] V. L. Rosa, P. Vetro, *Common fixed points for $\alpha$-$\psi$-$\phi$-contractions in generalized metric spaces*, Nonlinear Analysis: Modeling and Control 19 No. 1 (2014) 43-54.
[14] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha$-$\psi$-contractive type mappings*, Nonlinear Anal 75 (2012) 4341-4349.
[15] B. Samet, H. Yazidi, *Fixed point theorems with respect to a contractive condition of integral type*, Rend. Circ. Mat. Palermo 60 (2011) 181-190.