Commuting \( \pi \)-regular rings

Sh. Sahebi* and M. Azadi*

*Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, PO. Code 14168-94351, Tehran, Iran

Abstract. \( R \) is called commuting regular ring (resp. semigroup) if for each \( x, y \in R \) there exists \( a \in R \) such that \( xy = yxax \). In this paper, we introduce the concept of commuting \( \pi \)-regular rings (resp. semigroups) and study various properties of them.

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1. Introduction

Let \( R \) be a ring (resp. semigroup). \( R \) is called Von Neumann regular ring (resp. semigroup) if for each \( x \in R \) there exists \( a \in R \) such that \( xax = x \). Following [2], \( R \) is called \( \pi \)-regular ring if for any \( x \in R \) there exist a positive integer \( n \) and \( a \in R \) such that \( x^nax^n = x^n \). Following [6,1], \( R \) is called a commuting regular ring (resp. semigroup) if for each \( x, y \in R \) there exists \( a \in R \) such that \( xy = yxax \).

In recent years some authors have studied the commuting regular rings (resp. semigroups)[1, 3, 5]. We extend commuting regular rings (resp. semigroups) and introduce the concept of commuting \( \pi \)-regular rings (resp. semigroups) as following:

**Definition 1.1** \( R \) is called a commuting \( \pi \)-regular ring (resp. semigroup) if for each \( x, y \in R \) there exist a positive integer \( n \) and \( a \in R \) such that \( (xy)^n = (yx)^na(yx)^n \).
Since for each $x, y \in R$ we have $(xy)^n = (yx)^n((yx)^{-1}n(xy)^{n1}(yx)^{-1}n)(yx)^n$, then division rings (resp. groups) are commuting $\pi$-regular. Moreover, nil rings (resp. nil semigroups) are commuting $\pi$-regular. Because, for each $x, y \in R$ there exist positive integers $n_1$ and $n_2$ such that $(xy)^{n_1} = 0 = (yx)^{n_2}$ and so, $(xy)^{n_1} = (yx)^{n_1}(yx)^{n_2}(yx)^{n_1}$. In this paper we investigate various properties of commuting $\pi$-regular rings (resp. semigroups). For a ring $R$, we use the notation $C(R)$ for the center of $R$.

2. Basic properties of commuting $\pi$-regular rings

In this section, we get some basic properties of commuting $\pi$-regular rings. Clearly, a commuting $\pi$-regular ring is $\pi$-regular (put $x = y$), but the converse is not true. As the following Remark, $M_2(\mathbb{Z}_2)$ is not commuting $\pi$-regular however it is $\pi$-regular.

**Remark 1** The ring of $n \times n$ matrices over a commuting $\pi$-regular ring is not necessarily commuting $\pi$-regular. For example $\mathbb{Z}_2$ is commuting $\pi$-regular ring but $M_2(\mathbb{Z}_2)$ is not.

Indeed, let $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$ then $xy = (xy)^n = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $yx = (yx)^n = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ for each $n \in \mathbb{N}$. If there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$ such that $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ then $1 = 0$ which is a contradiction.

**Remark 2** The concepts of $\pi$-regular and commuting $\pi$-regular rings are the same in commutative case.

**Proposition 2.1** Every homomorphic image of a commuting $\pi$-regular ring is commuting $\pi$-regular.

**Proof.** Let $R, S$ be rings and $f : R \rightarrow S$ be a ring epimorphism. Suppose that $R$ is commuting $\pi$-regular and let $v, w \in S$. Since $f$ is an epimorphism, there exist $x, y \in R$ such that $f(x) = v, f(y) = w$. Then since $R$ is commuting $\pi$-regular, there exist $a \in R$ and a positive integer $n$ such that $(xy)^n = (yx)^n(a(yx))^n$. It follows that

$$(vw)^n = (f(xy))^n = f((yx)^n(a(yx))^n) = (uv)^n f(a)(uv)^n,$$

this completes the proof.

**Corollary 2.2** Let $R$ be a commuting $\pi$-regular ring. If $I$ is an ideal of $R$, then $R/I$ is commuting $\pi$-regular.

**Proposition 2.3** Let $R$ be a commutative ring with identity. If $R$ is a commuting $\pi$-regular ring, then every prime ideal of $R$ is maximal.

**Proof.** Let $P$ be a prime ideal of $R$, then $R/P$ is commuting $\pi$-regular by corollary 2.2. If $P \neq a + P = \overline{a} \in R/P$ then there exist a positive integer $n$ and $\overline{b} \in R/P$ such that $\overline{a}^{2n} = \overline{a}^{2n} \overline{b}^{2n}$ and so $\overline{a}^{2n}(\overline{a} - \overline{b}^{2n}) = 0$. Therefore $\overline{b}^{2n} = \overline{a}$ and the proof is complete.

Although, the subring of a commuting $\pi$-regular ring is not necessarily commuting $\pi$-regular (for example, $\mathbb{Z}$ as a subring of $\mathbb{Q}$ is not commuting $\pi$-regular). But we have the following:

**Proposition 2.4** The center $C(R)$ of every commuting $\pi$-regular ring $R$ is again commuting $\pi$-regular.
Proof. Let $x, y \in C(R)$. Then there exist $a \in R$ and a positive integer $n$ such that

$$(xy)^n = (yx)^n a (yx)^n = (yx)^{2n} a = (yx)^{2n}$$

and so

$$(xy)^n a = (yx)^{2n} a^2 = a^2 (yx)^{2n}.$$ 

Let $z = (yx)^n a^2$ then

$$(yx)^n z (yx)^n = (yx)^{2n} a^2 (yx)^n = (yx)^n a (yx)^n = (xy)^n.$$ 

Now it is enough to show that $z \in C(R)$. First note that $(yx)^n a \in C(R)$, because for any $r \in R$ we have

$$(yx)^n ar = ar (yx)^n = ar (yx)^n a (yx)^n = a (yx)^{2n} ra = (yx)^n ra = r (yx)^n a$$

and so

$$zr = ((yx)^n a)r = ar (yx)^n a = (yx)^n ara = r (yx)^n a^2 = rz$$

and the proof is complete. ■

The following shows that the corner of a commuting $\pi$-regular ring $R$ (i.e. $eRe$ for some idempotent $e$ of $R$) is also commuting $\pi$-regular.

**Proposition 2.5** Let $R$ be a commuting $\pi$-regular ring. Then for any $e^2 = e$, $eRe$ is commuting $\pi$-regular.

**Proof.** Let $x, y \in eRe$. Since $R$ is commuting $\pi$-regular we have $(xy)^n = (yx)^n a (yx)^n$ for some $a \in R$ and a positive integer $n$. Note that $(yx)^n = (yx)^n e = e (yx)^n$. Thus $(yx)^n = (yx)^n e ae (yx)^n$ and it follows that $eRe$ is commuting $\pi$-regula. ■

3. Commuting $\pi$-regular semigroups

Recall the following definition from [4]:

**Definition 3.1** Let $S$ be a semigroup. A relation $E$ on the set $S$ is called compatible if:

$$(\forall s, t, a \in S)((s, t) \in E, (s', t') \in E) \Rightarrow (ss', tt') \in E.$$ 

A compatible equivalence relation is called congruence.

Let $\rho$ be a congruence on a semigroup $S$ and $S/\rho$ be the set of $\rho$-classes, whose elements are the subsets $x\rho$, then we can define a binary operation on the quotient set $S/\rho$, in a natural way as follows:

$$(a\rho)(b\rho) = (ab)\rho$$

It is easy to check that $S/\rho$ with the above operation is a semigroup.
Proposition 3.2 Let $\rho$ be a congruence on a commuting $\pi$-regular semigroup $S$. Then $S/\rho$ is commuting $\pi$-regular.

Proof. Let $x, y \in S$, so there exist $c \in S$ and a positive integer $n$ such that $(xy)^n = (yx)^n c (yx)^n$ and therefore

$$(xy)^n = \rho = ((yx)^n c (yx))^n \rho = ((y, \rho)(x, \rho))^n c \rho((y, \rho)(x, \rho))^n$$

Thus $S$ is commuting $\pi$-regular semigroup.

Definition 3.3 Let $S$ be a semigroup. The left map $\lambda : S \rightarrow S$ is called a left translation of $S$ if $s(\lambda t) = (\lambda s)t$, for all $s, t \in S$. The right map $\rho : S \rightarrow S$ is called a right translation of $S$ if $(st)\rho = s(t\rho)$, for all $s, t \in S$. A left translation $\lambda$ and a right translation $\rho$ are said to be linked if $s(\lambda t) = (sp)t$ for all $s, t \in S$.

The set of all linked pairs $(\lambda, \rho)$ of left and right translation is called the translation hull of $S$ and will be denoted by $\Omega(S)$. $\Omega(S)$ is a semigroup under the obvious multiplication $(\lambda, \rho)(\lambda', \rho') = (\lambda' \lambda, \rho \rho')$ where $\lambda' \lambda$ denote the composition of the left maps $\lambda$ and $\lambda'$, while $\rho \rho'$ denotes the composition of the right maps $\rho$ and $\rho'$.

Proposition 3.4 Let $S$ be a commuting $\pi$-regular semigroup. For every $a \in S$ define $\lambda_0(a) = as$ and $s\rho_0 = sa$. Then $(\lambda_0, \rho_0)$ is a linked pair in $\Omega(S)$ and the set of every $(\lambda_0, \rho_0)$, where $a \in S$, with multiplication of link translations is a commuting $\pi$-regular semigroup.

Proof. It is easy to verify that, for all $a, b \in S$, $(\lambda_0(a), \rho_0(b))(\lambda_0(b), \rho_0(a)) = (\lambda_0(ab), \rho_0(ab))$. Therefore the set of $(\lambda_0, \rho_0)$'s, is a semigroup. On the other hand for $a, b \in S$ there exist $t \in S$ and a positive integer $n$ such that $(ab)^n = (ba)^n t(ba)^n$ and so

$$(\lambda_0(ab), \rho_0(ab)) = ((\lambda_0(ab))^n, \rho_0(ab))^n = (\lambda_0(ab)^n t(ba)^n, \rho_0(ab)^n t(ba)^n)$$

$$= ((\lambda_0(b), \rho_0(b))(\lambda_0(a), \rho_0(a))^n (\lambda_0, \rho_0)(\lambda_0(b), \rho_0(a))^n).$$

References