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New fixed and periodic point results on cone metric spaces

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Abstract. In this paper, several fixed point theorems for T-contraction of two maps on cone metric spaces under normality condition are proved. Obtained results extend and generalize well-known comparable results in the literature.

Keywords: Cone metric space; Fixed point; Property P; Property Q; Normal cone.

1. Introduction

In 1922, Banach proved his famous fixed point theorem [3]. Afterward, other people consider some various definitions of contractive mappings and proved several fixed point theorems in [4, 7, 10, 11, 13, 15] and the references contained therein. In 2007, Huang and Zhang [8] introduced cone metric space and proved some fixed point theorems. Afterward, several fixed and common fixed point results on cone metric spaces proved in [1, 14, 16, 17] and the references contained therein.

Recently, Morales and Rajes [12] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric space and proved some fixed point theorems. Then, Filipović et al. [5] defined T-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work, we prove several fixed and periodic point theorems for T-contraction of two maps on normal cone metric spaces. Our results extend various comparable results of Filipović et al. [5], and Morales and Rajes [12].

2. preliminaries

Let us start by defining some important definitions.

Definition 2.1 (*See [6, 8]*). *Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if*
(a) *P is closed, non-empty and $P \neq \{0\}$;*
(b) *$a, b \in R, a, b \geqslant 0, x, y \in P$ imply that $ax + by \in P$;*
(c) *if $x \in P$ and $-x \in P$, then $x = 0$.*

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Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by
$x \leqslant y \Longleftrightarrow y - x \in P$.
We shall write $x < y$ if $x \leqslant y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is interior of P). If $intP \neq \emptyset$, the cone P is called solid. The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leqslant x \leqslant y \Longrightarrow \|x\| \leqslant K\|y\|. \tag{1}$$

The least positive number satisfying the above is called the normal constant of P.

Example 2.2 (See [14]). Let $E = C_{\mathbf{R}}[0,1]$ with the supremum norm and $P = \{f \in E : f \geqslant 0\}$. Then P is a normal cone with normal constant $K = 1$.

Definition 2.3 (*See [8]*). *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies*
$(d1)$ $0 \leqslant d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
$(d2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;
$(d3)$ $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Example 2.4 (See [8]). Let $E = R^2$, $P = \{(x,y) \in E | x, y \geqslant 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geqslant 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.5 (*See [5]*). *Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then*
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Also, a cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. In the sequel, we always suppose that E is a real Banach space, P is a normal cone in E, and $\leqslant$ is partial ordering with respect to P.

Definition 2.6 (*See [5]*). *Let (X,d) be a cone metric space, P a solid cone and $S : X \to X$. Then*
(i) S is said to be sequentially convergent if we have for every sequence (x_n), if $S(x_n)$ is convergent, then (x_n) also is convergent.
(ii) S is said to be subsequentially convergent if we have for every sequence (x_n) that $S(x_n)$ is convergent, implies (x_n) has a convergent subsequence.
(iii) S is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} S(x_n) = S(x)$, for all (x_n) in X.

Definition 2.7 (*See [5]*). *Let (X,d) be a cone metric space and $T, f : X \to X$ two mappings. A mapping f is said to be a T-Hardy-Rogers contraction, if there exist $\alpha_i \geqslant 0$, $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tfx, Tfy) \leqslant\ & \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) \\ & + \alpha_5 d(Ty, Tfx). \end{aligned} \tag{2}$$

In previous definition, suppose that $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 \neq 0$ (resp. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 \neq 0$). Then we obtain T-Kannan (resp. T-Chatterjea) contraction from [12].

3. Fixed point results

Theorem 3.1 *Suppose that (X, d) be a complete cone metric space, P be a normal cone with normal constant K, and $T : X \to X$ be a continuous and one to one mapping. Moreover, let f and g be two maps of X satisfying*

$$d(Tfx, Tgy) \leqslant \alpha_1 d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tgy)] \\ +\alpha_3[d(Tx, Tgy) + d(Ty, Tfx)], \quad (3)$$

for all $x, y \in X$, where

$$\alpha_i \geqslant 0 \quad for \quad i = 1, 2, 3 \qquad and \qquad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1. \quad (4)$$

That is, f and g be a T-contraction. Then
(1) There exist $u_x \in X$ such that $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = u_x$.
(2) If T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.
(3) There exist a unique $v_x \in X$ such that $fv_x = gv_x = v_x$, that is, f and g have a unique common fixed point.
(4) If T is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to v_x.

Proof Suppose x_0 is an arbitrary point of X, and define $\{x_n\}$ by
$x_1 = fx_0\ ,\ x_2 = gx_1\ ,\ \cdots\ ,\ x_{2n+1} = fx_{2n}\ ,\ x_{2n+2} = gx_{2n+1}\ for\ n = 0, 1, 2, ...$
First, we prove that $\{Tx_n\}$ is a Cauchy sequence.

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Tfx_{2n}, Tgx_{2n+1}) \\ &\leqslant \alpha_1 d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +\alpha_2[d(Tx_{2n}, Tfx_{2n}) + d(Tx_{2n+1}, Tgx_{2n+1})] \\ &\quad +\alpha_3[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})] \\ &= \alpha_1 d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +\alpha_2[d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})] \\ &\quad +\alpha_3[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3) d(Tx_{2n}, Tx_{2n+1}) \\ &\quad +(\alpha_2 + \alpha_3) d(Tx_{2n+1}, Tx_{2n+2}), \end{aligned}$$

which implies that

$$d(Tx_{2n+1}, Tx_{2n+2}) \leqslant \gamma d(Tx_{2n}, Tx_{2n+1}),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$.
Similarly, we get

$$d(Tx_{2n+3}, Tx_{2n+2}) \leqslant \gamma d(Tx_{2n+2}, Tx_{2n+1}),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$.

Thus, for all n

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leqslant \gamma d(Tx_{n-1}, Tx_n) \leqslant \gamma^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leqslant \cdots \leqslant \gamma^n d(Tx_0, Tx_1). \end{aligned} \tag{5}$$

Now, for any $m > n$

$$\begin{aligned} d(Tx_n, Tx_m) &\leqslant d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \\ &\leqslant (\gamma^n + \gamma^{n+1} + \cdots + \gamma^{m-1}) d(Tx_0, Tx_1) \\ &\leqslant \frac{\gamma^n}{1-\gamma} d(Tx_0, Tx_1). \end{aligned}$$

From (1), we have

$$\|d(Tx_n, Tx_m)\| \leqslant K \frac{\gamma^n}{1-\gamma} \|d(Tx_0, Tx_1)\|.$$

It follows that $\{Tx_n\}$ is a Cauchy sequence by Definition 2.5.(ii). Since cone metric space X is complete, there exist $u_x \in X$ such that $Tx_n \to u_x$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} Tfx_{2n} = u_x, \qquad \lim_{n\to\infty} Tgx_{2n+1} = u_x. \tag{6}$$

Now, if T is subsequentially convergent, $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $v_{x_1} \in X$ and $\{fx_{2n_i}\}$ (resp. $v_{x_2} \in X$ and $\{gx_{2n_i+1}\}$) such that

$$\lim_{n\to\infty} fx_{2n_i} = v_{x_1}, \qquad \lim_{n\to\infty} gx_{2n_i+1} = v_{x_2}. \tag{7}$$

Because of continuity T and by (7), we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tv_{x_1}, \qquad \lim_{n\to\infty} Tgx_{2n_i+1} = Tv_{x_2}. \tag{8}$$

Now, by (6) and (8) and because of injectivity of T, there exist $w_x \in X$ (set $v_x = v_{x_1} = v_{x_2}$) such that $Tv_x = u_x$.
On the other hand, by (d_3) and (3), we have

$$\begin{aligned} d(Tv_x, Tgv_x) &\leqslant d(Tv_x, Tgx_{2n_i+1}) + d(Tgx_{2n_i+1}, Tfx_{2n_i}) + d(Tfx_{2n_i}, Tgv_x) \\ &\leqslant d(Tv_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + \alpha_1 d(Tx_{2n_i}, Tv_x) \\ &\quad + \alpha_2 [d(Tx_{2n_i}, Tx_{2n_i+1}) + d(Tv_x, Tgv_x)] \\ &\quad + \alpha_3 [d(Tx_{2n_i}, Tgv_x) + d(Tv_x, Tx_{2n_i+1})] \\ &\leqslant d(Tv_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + (\alpha_1 + \alpha_3) d(Tx_{2n_i}, Tv_x) \\ &\quad + \alpha_2 d(Tx_{2n_i}, Tx_{2n_i+1}) + \alpha_3 d(Tv_x, Tx_{2n_i+1}) \\ &\quad + (\alpha_2 + \alpha_3) d(Tv_x, Tgv_x). \end{aligned}$$

Now, by (4) and (5) we have

$$\begin{aligned} d(Tv_x, Tgv_x) &\leqslant \frac{1}{1-\alpha_2-\alpha_3} d(Tv_x, Tx_{2n_i+2}) + \frac{1}{1-\alpha_2-\alpha_3} d(Tx_{2n_i+2}, Tx_{2n_i+1}) \\ &\quad + \frac{\alpha_1+\alpha_3}{1-\alpha_2-\alpha_3} d(Tx_{2n_i}, Tv_x) + \frac{\alpha_2}{1-\alpha_2-\alpha_3} d(Tx_{2n_i}, Tx_{2n_i+1}) \\ &\quad + \frac{\alpha_3}{1-\alpha_2-\alpha_3} d(Tv_x, Tx_{2n_i+1}) \\ &= A_1 d(Tv_x, Tx_{2n_i+2}) + A_2 \gamma^{2n_i} + A_3 d(Tx_{2n_i}, Tv_x) \\ &\quad + A_4 d(Tv_x, Tx_{2n_i+1}), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{1}{1-\alpha_2-\alpha_3} \quad , \quad A_2 = \frac{\alpha_2+\gamma}{1-\alpha_2-\alpha_3} d(Tx_0, Tx_1) \\ A_3 &= \frac{\alpha_1+\alpha_3}{1-\alpha_2-\alpha_3} \quad , \quad A_4 = \frac{\alpha_3}{1-\alpha_2-\alpha_3}. \end{aligned}$$

From (1), we have

$$\begin{aligned} \|d(Tv_x, Tgv_x)\| &\leqslant A_1 K \|d(Tv_x, Tx_{2n_i+2})\| + A_2 K \gamma^{2n_i} \|d(Tx_0, Tx_1)\| \\ &\quad + A_3 K \|d(Tx_{2n_i}, Tv_x)\| + A_4 K \|d(Tv_x, Tx_{2n_i+1})\| \end{aligned}$$

Now the right hand side of the above inequality approaches zero as $i \to \infty$. The convergence above give us that $\|d(Tv_x, Tgv_x)\| = 0$. Hence $d(Tv_x, Tgv_x) = 0$, that is, $Tv_x = Tgv_x$. Since T is one to one, then $gv_x = v_x$. Now, we shall show that $fv_x = v_x$.

$$\begin{aligned} d(Tfv_x, Tv_x) &= d(Tfv_x, Tgv_x) \\ &\leqslant \alpha_1 d(Tv_x, Tv_x) + \alpha_2 [d(Tv_x, Tfv_x) + d(Tv_x, Tgv_x)] \\ &\quad + \alpha_3 [d(Tv_x, Tgv_x) + d(Tv_x, Tfv_x)] \\ &= (\alpha_2 + \alpha_3) d(Tv_x, Tfv_x). \end{aligned}$$

which, using the definition of partial ordering on E and properties of cone P, gives $d(Tfv_x, Tv_x) = 0$. Hence, $Tfv_x = Tv_x$. Since T is one to one, then $fv_x = v_x$. Thus, $fv_x = gv_x = v_x$, that is, v_x is a common fixed point of f and g. Now, we shall show that v_x is a unique common fixed point. Suppose that v'_x be another common fixed point of f and g, then

$$\begin{aligned} d(Tv_x, Tv'_x) &= d(Tfv_x, Tgv'_x) \\ &\leqslant \alpha_1 d(Tv_x, Tv'_x) + \alpha_2 [d(Tv_x, Tfv_x) + d(Tv'_x, Tgv'_x)] \\ &\quad + \alpha_3 [d(Tv_x, Tgv'_x) + d(Tv'_x, Tfv_x)] \\ &= (\alpha_1 + 2\alpha_3) d(Tv_x, Tv'_x). \end{aligned}$$

By the same arguments as above, we conclude that $d(Tv_x, Tv'_x) = 0$, which implies the equality $Tv_x = Tv'_x$. Since T is one to one, then $v_x = v'_x$. Thus f and g have a unique common fixed point.

Ultimately, if T is sequentially convergent, then we replace n for n_i. Thus, we have

$$\lim_{n\to\infty} fx_{2n} = v_x, \qquad \lim_{n\to\infty} gx_{2n+1} = v_x.$$

Therefore if T is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to v_x. ■

The following results is obtained from Theorem 3.1.

Corollary 3.2 *Let (X, d) be a complete cone metric space, P be a normal cone and $T : X \to X$ be a continuous and one to one mapping. Moreover, let mapping f be a map of X satisfying*

$$d(Tfx, Tfy) \leqslant \alpha_1 d(Tx, Ty) + \alpha_2[d(Tx, Tfx) + d(Ty, Tfy)] + \alpha_3[d(Tx, Tfy) + d(Ty, Tfx)], \tag{9}$$

for all $x, y \in X$, where

$$\alpha_i \geqslant 0 \quad \textit{for} \quad i = 1, 2, 3 \qquad \textit{and} \qquad \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1. \tag{10}$$

That is, f be a T-contraction. Then,
(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a cauchy sequence.*
(2) *There exist $u_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.*
(3) *If T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exist a unique $v_{x_0} \in X$ such that $fv_{x_0} = v_{x_0}$, that is, f has a unique fixed point.*
(5) *If T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{f^n x_0\}$ converges to v_{x_0}.*

Recently, Fillipović et al. prove that the Corollary 3.2 for a non-normal cone.

Corollary 3.3 *Let (X, d) be a complete cone metric space, P be a solid cone and $T : X \to X$ be a continuous and one to one mapping. Moreover, let mapping f be a T-Hardy-Rogers contraction. Then, the results of previous Corollary hold.*

Proof See [5]. ■

4. Periodic point results

Obviously, if f is a map which has a fixed point z, then z is also a fixed point of f^n for each $n \in \mathbf{N}$. However the converse is not true [2]. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbf{N}$, where $Fix(f)$ stands for the set of fixed points of f [9], then f is said to have property P. Recall also that two mappings $f, g : X \to X$ is said to have property Q if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$. The following results extend some theorems of [2].

Theorem 4.1 *Let (X, d) be a cone metric space, P be a normal cone and $T : X \to X$ be a one to one mapping. Moreover, let mapping f be a map of X satisfyiing*
(i) $d(fx, f^2x) \leqslant \lambda d(x, fx)$ for all $x \in X$, where $\lambda \in [0, 1)$ and or (ii) with strict inequality, $\lambda = 1$ for all $x \in X$ with $x \neq fx$. If $Fix(f) \neq \emptyset$, then f has property P.

Proof See [5]. ■

THEOREM 4.2 *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose that mappings $f, g : X \to X$ satisfy all the conditions of Theorem 3.1. Then f and g have property Q.*

Proof From Theorem 3.1, f and g have a unique common fixed point in X. Suppose that $z \in Fix(f^n) \bigcap Fix(g^n)$, thus we have

$$\begin{aligned} d(Tz, Tgz) &= d(Tf(f^{n-1}z), Tg(g^n z)) \\ &\leqslant \alpha_1 d(Tf^{n-1}z, Tg^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tg^n z, Tg^{n+1}z)] \\ &\quad +\alpha_3[d(Tf^{n-1}z, Tg^{n+1}z) + d(Tg^n z, Tf^n z)] \\ &= \alpha_1 d(Tf^{n-1}z, Tz) + \alpha_2[d(Tf^{n-1}z, Tz) + d(Tz, Tgz)] \\ &\quad +\alpha_3 d(Tf^{n-1}z, Tgz) \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3) d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3) d(Tz, Tgz), \end{aligned}$$

which implies that

$$d(Tz, Tgz) \leqslant \gamma d(Tf^{n-1}z, Tz),$$

where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$ (by relation (4)). Now, we have

$$d(Tz, Tgz) = d(Tf^n z, Tg^{n+1}z) \leqslant \gamma d(Tf^{n-1}z, Tz) \leqslant \cdots \leqslant \gamma^n d(Tfz, Tz).$$

From (1), we have

$$\|d(Tz, Tgz)\| \leqslant \gamma^n K \|d(Tfz, Tz)\|.$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence, $\|d(Tz, Tgz)\| = 0$. It follows that $d(Tz, Tgz) = 0$, that is, $Tgz = Tz$. Since T is one to one, then $gz = z$. Also, Theorem 3.1 implies that $fz = z$ and $z \in Fix(f) \bigcap Fix(g)$. ■

THEOREM 4.3 *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that mapping $f : X \to X$ satisfies all the conditions of Corollary 3.2. Then f has property P.*

Proof From Corollary 3.2, f has a unique common fixed point in X. Suppose that $z \in Fix(f^n)$, we have

$$\begin{aligned} d(Tz, Tfz) &= d(Tf(f^{n-1}z), Tf(f^n z)) \\ &\leqslant \alpha_1 d(Tf^{n-1}z, Tf^n z) + \alpha_2[d(Tf^{n-1}z, Tf^n z) + d(Tf^n z, Tf^{n+1}z)] \\ &\quad +\alpha_3[d(Tf^{n-1}z, Tf^{n+1}z) + d(Tf^n z, Tf^n z)] \\ &\leqslant (\alpha_1 + \alpha_2 + \alpha_3) d(Tf^{n-1}z, Tz) + (\alpha_2 + \alpha_3) d(Tz, Tfz), \end{aligned}$$

which implies that
$d(Tz, Tfz) \leqslant \gamma d(Tf^{n-1}z, Tz)$ where $\gamma = \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} < 1$, (by relation (10)). Hence,

$$d(Tz, Tfz) = d(Tf^n z, Tf^{n+1}z) \leqslant \gamma d(Tf^{n-1}z, Tz) \leqslant \cdots \leqslant \gamma^n d(Tfz, Tz).$$

Therefore, we have $d(Tz, Tfz) \leqslant \gamma^n d(Tfz, Tz) \leqslant \gamma d(Tfz, Tz)$. By the same arguments as Theorem 4.2, we conclude that $d(Tfz, Tz) = 0$, that is, $Tfz = Tz$. Since T is one to one, then $fz = z$ and proof is complete. ■

Corollary 4.4 *Let (X, d) be a complete cone metric space, and P be a solid cone. Suppose that mapping $f : X \to X$ satisfies all the conditions of Corollary 3.3. Then f has property P.*

Proof See [5]. ■

References

[1] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. 341 (2008), pp. 416–420.
[2] M. Abbas, B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. 22 (2009), pp. 511–515.
[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. J. 3 (1922), pp. 133–181.
[4] L.B. Ćirić , *A generalization of Banach contraction principle*, Proc. Amer. Math. Soc. 45 (1974), pp. 267–273.
[5] M. Filipović, L. Paunović, S. Radenović, M. Rajović, *Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings"*, Math. Comput. Modelling. 54 (2011), pp. 1467–1472.
[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
[7] G.E. Hardy, T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. 1 (6) (1973), pp. 201–206.
[8] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), pp. 1467–1475.
[9] G.S. Jeong, B.E. Rhoades, *Maps for which $F(T) = F(T^n)$*, Fixed Point Theory Appl. 6 (2005), pp. 87–131.
[10] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly 83 (1976), pp. 261–263.
[11] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 60 (1968), pp. 71–76.
[12] J.R. Morales, E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, Int. J. Math. Anal. 4 (4) (2010), pp. 175–184.
[13] H. Rahimi, Gh. Soleimani Rad, *Some fixed point results in metric type space*, J. Basic Appl. Sci. Res. 2 (9) (2012), pp. 9301–9308.
[14] S. Rezapour, R. Hamlbarani, *Some note on the paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 345 (2008), pp. 719–724.
[15] B.E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. 266 (1977), pp. 257–290.
[16] G. Song, X. Sun, Y. Zhao, G. Wang, *New common fixed point theorems for maps on cone metric spaces*, Appl. Math. Lett. 23 (2010), pp. 1033–1037.
[17] S. Wang, B. Guo, *Distance in cone metric spaces and common fixed point theorems*, Appl. Math. Lett. 24 (2011), pp. 1735-1739.