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Numerical solution of functional integral equations by using B-splines

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Abstract. This paper describes an approximating solution, based on Lagrange interpolation and spline functions, to treat functional integral equations of Fredholm type and Volterra type. This method can be extended to functional differential and integro-differential equations. For showing efficiency of the method we give some numerical examples.

Keywords: Lagrange interpolation, B-spline functions, Functional integral equation

1. Introduction

In recent years there has been a growing interest in the numerical treatment of the functional differential equations,

$$a_0 y^{'}(x) + a_1 y^{'}(h(x)) + b_0 y(x) + b_1 y(h(x)) = g(x) \tag{1}$$

which is said to be of retarded type if $a_0 \neq 0$ and $a_1 = 0$. It is said to be natural type if $a_0 \neq 0$ and $a_1 \neq 0$. If $a_0 = 0$ and $a_1 = 0$ it is said to be advanced type. For more general functional equations by Arndt [2]. For further details, See, El-Gendi [4] , Zennaro [11], Fox, et al., [5].

Rashed introduced new interpolation method for functional integral equations and functional integro-differential equations [7]. In this paper we approximate the numerical solution $y_n(x)$ of the following functional integral equations:

$$y(x) + A(x)y(h(x)) + \lambda \int_a^x k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{2}$$

$$y(x) + A(x)y(h(x)) + \lambda \int_a^b k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{3}$$

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$$y(x) + \lambda \int_a^x k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{4}$$

$$y(h(x)) + \lambda \int_a^{h(x)} k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{5}$$

For this approximation we first interpolate $y(x)$ with the following interpolation formula:

$$y(x) \approx \sum_{j=0}^{n} y(x_j) l_j(x) \tag{6}$$

where

$$l_j(x) = \prod_{\substack{k=0,\\ k\neq j}}^{n} \left(\frac{x - x_k}{x_j - x_k}\right) \tag{7}$$

and

$$y(h(x)) \approx \sum_{j=0}^{n} y(x_j) l_j(h(x)) \tag{8}$$

where

$$l_j(h(x)) = \prod_{\substack{k=0,\\ k\neq j}}^{n} \left(\frac{h(x) - x_k}{x_j - x_k}\right) \tag{9}$$

where $h(x)$ is a well known function.

Then for computing $y(x_j)$ we use B-Spline approximation that we present its details in the next section. In the section 3 we apply our method for functional integral equations. Also the section 4 is devoted to some examples of both Fredholm and Volterra type equations with numerical solutions. Of course for computing integrals in the third section we used Clenshaw-Curtis rule [3, 6].

1.1 *B-splines functions*

We choose cubic spline functions in finite linear space as our approximate functions. Let Δ be a partition of the interval $[a, b]$, defined by the knots $\{t_i\}$, such that

$$a = t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1} = b.$$

In this partition, together with the extended knots $t_{-2} \leqslant t_{-1} \leqslant t_0 \leqslant t_1$ and $t_{n-1} \leqslant t_n \leqslant t_{n+1} \leqslant t_{n+2}$, we can construct the $n+1$ cubic B-splines denoted by $\{B_i; i = 0, \cdots, n\}$ where B_i is nonzero over the interval (t_{i-2}, t_{i+2}), and B-splines is a cubic polynomial over each sub-interval $(t_{i+j}, t_{i+j+1}), j = -2, \cdots, 1$. If, however, multiple knots are introduced, then loss of continuity ensues. If for some j, $t_j = t_{j+1}$, then at that point C^1 is obtained, while for $t_{j-1} = t_j = t_{j+1}$ C^0 continuity is obtained. In all such cases $\{B_i; \quad i = 0, \cdots, n\}$ is a basis for the linear space S_3 of a cubic splines with partition Δ. For more details see [9, 10].

Using this linear space, we approximate exact solution $y(t)$ of $(1)-(6)$ by a linear combination given by

$$y(t) \approx \sum_{i=0}^{n} \alpha_i B_i(t). \tag{10}$$

Here this coefficients is unknown and must be determined. For finding this values we solve a linear system.

1.2 *computation of B-splines*

Let $r \geqslant 1$ be an integer and $\mathbf{t} = \{t_i\}_{i\in\mathbb{Z}}$ any infinite nondecreasing unbounded sequence of reals t_i with
$\inf t_i = -\infty, \quad \sup t_i = +\infty \quad$ and $t_i < t_{i+r} \quad$ for all $i \in \mathbb{Z}$. Then the i-th B-splines of order r associated with $\mathbf{t}$ is satisfied in the following recurrence formula:

$$B_{i,r} = \frac{x - t_i}{t_{i+r-1} - t_i} B_{i,r-1}(x) + \frac{t_{i+r} - x}{t_{i+r} - t_{i+1}} B_{i+1,r-1}(x) \tag{11}$$

that represents $B_{i,r}(x)$ directly as a positive linear combination of $B_{i,r-1}(x)$ and $B_{i+1,r-1}(x)$. It can be used to compute the values of all B-splines $B_{i,r}(x) = B_{i,r,\mathbf{t}}(x)$ for a fixed value of x. By definition $B_{j,1} = B_{j,1}(x) = 1$ for $t_j \leqslant x < t_{j+1}$, the following tableau can be computed consecutively using (11).

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & B_{j-3,4} & \cdots \\
0 & 0 & B_{j-2,3} & B_{j-2,4} & \cdots \\
0 & B_{j-1,2} & B_{j-1,3} & B_{j-1,4} & \cdots \\
B_{j,1} & B_{j,2} & B_{j,3} & B_{j,4} & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

This method is numerically very stable because only nonnegative multiples of nonnegative numbers are added together.

Example 1.1. For $t_i = i, i = 0, 1, \cdots$ and $x = 3.5 \in [t_3, t_4]$ the following tableau of values $B_{i,r} = B_{i,r}(x)$ is obtained:

r=	1	2	3	4
i=0	0	0	0	1/48
i=1	0	0	1/8	23/48
i=2	0	1/2	6/8	23/48
i=3	1	1/2	1/8	1/48
i=4	0	0	0	0

For instance, $B_{2,4}$ is obtained from

$$B_{2,4} = B_{2,4}(3.5) = \frac{3.5-2}{5-2}.\frac{6}{8} + \frac{6-3.5}{6-3}\frac{1}{8} = \frac{23}{48}.$$

1.3 *Functional linear integral equations of the second kind*

In this paper we use spline function with Lagrange interpolation to compute the numerical solution $y_n(x)$ of functional linear integral equations of the second kind:

$$y(x) + A(x)y(h(x)) + \lambda \int_a^x k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{12}$$

$$y(x) + A(x)y(h(x)) + \lambda \int_a^b k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{13}$$

$$y(x) + \lambda \int_a^x k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{14}$$

$$y(x) + \lambda \int_a^{h(x)} k(x,t)y(t)dt = g(x), \quad a \leqslant x \leqslant b, \tag{15}$$

In fact, We seek to find an approximation to $y(x)$ which satisfies some interpolation property or variational principle.

In this functional integral equations $(2)-(5)$, we may use Lagrange interpolation [8] of $y(x)$ by

$$y(x) \approx \sum_{i=0}^{n} \alpha_i \Phi_i(x), \tag{16}$$

where

$$\Phi_i(x) = \sum_{j=0}^{n} B_i(x_j) l_j(x) \tag{17}$$

and

$$l_j(x) = \prod_{\substack{k=0,\\ k\neq j}}^{n} \left(\frac{x - x_k}{x_j - x_k}\right) \tag{18}$$

and also

$$y(h(x)) \approx \sum_{i=0}^{n} \alpha_i \Phi_i(h(x)), \tag{19}$$

where

$$\Phi_i(h(x)) = \sum_{j=0}^{n} B_i(x_j) l_j(h(x)), \tag{20}$$

and

$$l_j(h(x)) = \prod_{\substack{k=0,\\ k\neq j}}^{n} (\frac{h(x) - x_k}{x_j - x_k}), \tag{21}$$

where $h(x)$ is well known function. Also

$$k(x,t)y(t) \approx \sum_{i=0}^{n} \alpha_i \Psi_i(x,t). \tag{22}$$

$$\Psi_i(x,t) = \sum_{j=0}^{n} k(x,x_j)B_i(x_j)l_j(t) \tag{23}$$

The integral part of each functional equations $(2)-(5)$ is given as follows: Integrating (22) with respected to t from a to x we have

$$\int_a^x k(x,t)y(t)dt \approx \sum_{i=0}^{n} \alpha_i b_i(x). \tag{24}$$

$$b_i(x) = \int_a^x \Psi_i(x,t)dt \tag{25}$$

Integrating (22) with respected to t from a to b

$$\int_a^b k(x,t)y(t)dt \approx \sum_{i=0}^{n} \alpha_i b_i(x). \tag{26}$$

$$b_i(x) = \int_a^b \Psi_i(x,t)dt \tag{27}$$

Integrating (22) with respected to t from a to $h(x)$

$$\int_a^{h(x)} k(x,t)y(t)dt \approx \sum_{i=0}^{n} \alpha_i b_i(h(x)). \tag{28}$$

where

$$b_i(h(x)) = \int_a^{h(x)} \Psi_i(x,t)dt \approx \sum_{j=0}^{n} k(x,x_j)B_i(x_j)b_j^*(h(x)) \tag{29}$$

$$b_j^*(h(x)) = \int_a^{h(x)} \prod_{\substack{k=0,\\ k\neq j}}^{n} (\frac{t - x_k}{x_j - x_k})dt. \tag{30}$$

The integral in the relation (30) is approximated as:

$$b_j^*(h(x)) \approx \frac{h(x)-a}{2}\int_{-1}^{1}\prod_{\substack{k=0\\k\neq j}}^{n}\Big(\frac{\frac{h(x)-a}{2}t+\frac{h(x)+a}{2}-x_k}{x_j-x_k}\Big)dt \tag{31}$$

$$= \frac{h(x)-a}{2}\sum_{s=0}^{N} d_s \prod_{\substack{k=0\\k\neq j}}^{n}\Big(\frac{\frac{h(x)-a}{2}t+\frac{h(x)+a}{2}-x_k}{x_j-x_k}\Big)dt,$$

where

$$d_s = \frac{2\delta}{N}\sum_{k=0}^{N}\delta_k I_k cos(\frac{ks\pi}{N}),$$

$$I_k = \begin{cases} 2, & k=0,\\ 0, & k \quad is \quad odd\\ \frac{2}{1-k^2}, & k \quad is \quad even,\end{cases}$$

$$\delta_s = \begin{cases} 0.5, & s=0,N,\\ 1, & 0<s<N.\end{cases}$$

Finally, the functional integral equation is approximated by system of n linear equations. Also, the method is extended to treat the functional equations of advanced type ($\lambda = 0$ in (2) or (4))

$$y(x) + A(x)y(h(x)) = g(x), \quad a \leqslant x \leqslant b. \tag{32}$$

2. Numerical examples

We compared our results with Rashed results [7]. We consider here the following examples on the functional integral equation for comparison. The computed errors d_n in these tables are defined to be

$$d_n = \left\{\sum_{j=0}^{n} e_n^2(x_j)/n\right\}^{1/2} \approx \left\{\int_{-1}^{1} e_n^2(x)dx\right\}^{1/2}, \quad e_n = y_n - y_{exact}.$$

Example 2.1. Volterra integral equation of the second kind

$$y(x) + A(x)y(h(x)) + \lambda\int_a^x k(x,t)y(t)dt = g(x),$$

$$A(x) = e^x, \quad k(x,t) = e^{x+t}, \quad g(x) = e^{-x} + e^{x-h(x)} + xe^x,$$

n	(1)	(2)
2	0/0241809	0/00477333
3	0/00215269	0/000170489
4	0/000193117	0/0000557323
5	0/0000212866	$5/0907 \times 10^{-6}$
6	$2/2262 \times 10^{-6}$	$1/40704 \times 10^{-7}$
7	$2/74542 \times 10^{-7}$	$5/40375 \times 10^{-9}$
8	$2/4736 \times 10^{-8}$	$1/72177 \times 10^{-10}$
9	$2/92672 \times 10^{-9}$	$5/59269 \times 10^{-12}$
10	$2/45301 \times 10^{-10}$	$2/77894 \times 10^{-12}$

Table 1. the solution of equation for some n

$$y(x) = e^x, \quad a = 0, \quad b = 1.1, \quad h(x) = \ln(x + 0.5) \quad \lambda = 1.$$

Example 2.2. Fredholm integral equation of the second kind

$$y(x) + A(x)y(h(x)) + \lambda \int_a^b k(x,t)y(t)dt = g(x),$$

$$A(x) = e^{-x}, \quad k(x,t) = e^{x-t},$$

$$g(x) = e^{-x} + e^{h(x)-x} + e^{x(b-a)}$$

$$y(x) = e^x, \quad a = 0, \quad b = 1.1, \quad h(x) = \frac{x^2}{2} \quad \lambda = 1.$$

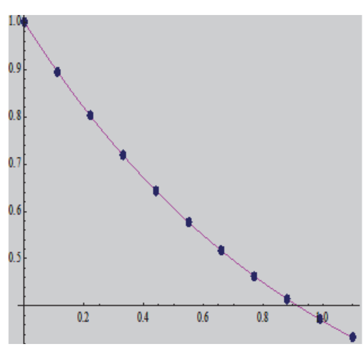

Figure 1. Function with it's approximation for $h(x) = \ln(x + 0/5)$.

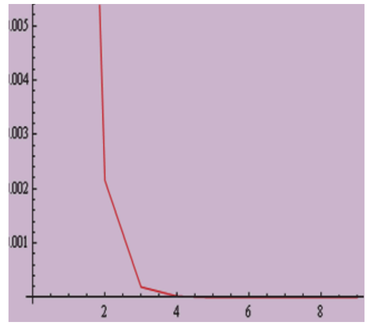

Figure 2. Error of method for $h(x) = \ln(x + 0/5)$.

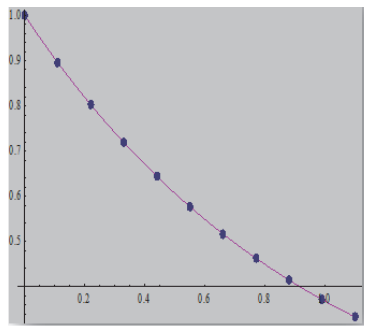

Figure 3. Function with it's approximation for $h(x) = \frac{x^2}{2}$.

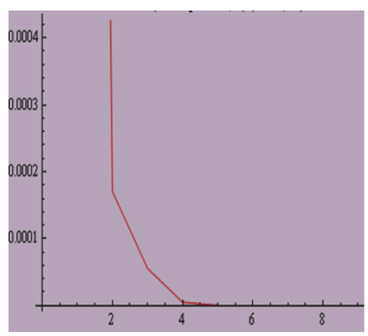

Figure 4. Error of method for $h(x) = \frac{x^2}{2}$.

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