Journal of
Linear and Topological Algebra
Vol. 02, No. 02, Summer 2013, 115- 120

Recognition of the group $G_2(5)$ by the prime graph

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Abstract. Let G be a finite group. The prime graph of G is a graph $\Gamma(G)$ with vertex set $\pi(G)$, the set of all prime divisors of $|G|$, and two distinct vertices p and q are adjacent by an edge if G has an element of order pq. In this paper we prove that if $\Gamma(G) = \Gamma(G_2(5))$, then G has a normal subgroup N such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

Keywords: prime graph, recognition, linear group

1. Introduction

Let G be a finite group. The spectrum $\omega(G)$ of G is the set of orders of elements in G, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order G has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of isomorphic classes of finite groups H such that $\omega(G) = \omega(H)$ is denoted by $h(G)$. If $h(G) = k \geqslant 1$ is finite then the group G is called a k-recognizable group by spectrum. If $h(G)$ is not finite, G is called non-recognizable. A 1-recognizable group is usually called a recognizable group. The recognizability of finite groups by spectrum was first considered by W.J.Shi et.al. in [16]. A list of finite simple groups which are known to be or not to be recognizable by spectrum is given in [11].

For $n \in N$, let $\pi(n)$ denote the set of all the prime divisors of n, and for a finite group G let us set $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of a finite group G is a simple graph with vertex set $\pi(G)$ in which two distinct vertices p and q are

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joined by an edge if and only if G has an element of order pq. It is clear that a knowledge of $w(G)$ determines $\Gamma(G)$ completely but not vise-versa in general. Given a finite group G, the number of non-isomorphic classes of finite groups H with $\Gamma(G) = \Gamma(H)$ is denoted by $h_\Gamma(G)$. If $h_\Gamma(G) = 1$, then G is said to be recognizable by prime graph. If $h_\Gamma(G) = k < \infty$, then G is called k-recognizable by prime graph, in case $h_\Gamma(G) = \infty$ the group G is called non-recognizable by prime graph. Obviously a group recognizable by spectra need not to be recognizable by prime graph, for example A_5 is recognizable by spectra but $\Gamma(A_5) = \Gamma(A_6)$.

The number of connected components of $\Gamma(G)$ is denoted by $s(G)$. As a consequence of the classification of the finite simple groups it is proved in [19] and [9], that for any finite simple group G we have $s(G) \leqslant 6$. Let $\pi_i = \pi_i(G)$, $1 \leqslant i \leqslant s$, be the connected components of G. For a group of even order we let $2 \in \pi_1$. Recognizability of groups by prime graph was first studied in [5] where some sporadic simple groups were characterized by prime graph. As another concept we say that a non-abelian simple group G is quasi-recognizable by graph if every finite group whose prime graph is $\Gamma(G)$ has a unique non-abelian composition factor isomorphic to G.

It is proved in [20] that the simple groups $G_2(7)$and ${}^2G_2(q)$, $q = 3^{2m+1} > 3$, are recognizable by prime graph, where both groups have disconnected prime graphs. A series of interesting results concerning recognition of finite simple groups were obtained by B.Khosravi et.al. In particular they have stablished quasi-recognizability of the group $L_{10}(2)$ by graph and the recognizability of $L_{16}(2)$ by graph in [7] and [8], where both groups have connected prime graphs.

Next we introduce useful notation. Let p be a prime number. The set of all non-abelian finite simple groups G such that $p \in \pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathfrak{S}_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes p. The sets $\mathfrak{S}_p$, where p is a prime less than 1000 is given in [21].

2. Preliminary results

Let G be a finite group with disconnected prime graph. The structure of G is given in [19] which is stated as a lemma here. Let G be a finite group with disconnected prime graph. Then G satisfies one of the following conditions:
$s(G) = 2$, $G = KC$ is a Frobenius group with kernel K and complement C, and the two connected components of G are $\Gamma(K)$ and $\Gamma(C)$. Moreover K is nilpotent, and here $\Gamma(K)$ is a complete graph.
$s(G) = 2$ and G is a 2-Frobeuius group, i.e. , $G = ABC$ where $A, AB \trianglelefteq G$, $B \trianglelefteq BC$, and AB, BC are Frobenius groups.

There exists a non-abelian simple group P such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ for some nilpotent normal $\pi_1(G)$-subgroup N of G and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geqslant s(G)$. If a group G satisfies condition(c) of the above lemma we may write $P = B/N$, $B \leqslant G$, and $\overline{G}/P = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where N is a nilpotent normal $\pi_1(G)$-subgroup of G and A is a $\pi_1(G)$-group.

The above structure lemma was extended to groups with connected prime graphs satisfying certain conditions [17]. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$ and $t(2, G)$ the maximal number of primes in $\pi(G)$ nonadjacent to 2. Let G be a finite group satisfying the following conditions: There exist three pairwise distinct primes in $\pi(G)$ nonadjacent in $\Gamma(G)$, i.e. , $t(G) \geqslant 3$.
There exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to 2, i.e. ,$t(2, G) \geqslant 2$. Then, there is a finite non-abelian simple group S such that $S \leqslant \overline{G} = G/K \leqslant Aut(S)$ for the maximal normal solvable subgroup K of G. Furthermore $t(S) \geqslant t(G) - 1$ and one of the following statements holds:

(1) $S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, G) = 3$.
(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroups of G is isomorphic to a Sylow p-subgroup of S. In particular $t(2, S) \geqslant t(2, G)$.

In the following we list some properties of the Frobenius group where some of its proof can be found in [15]. Let G be a Frobenius group with kernel K and complement H, then:
K is nilpotent and $|H| \mid (|K| - 1)$.
The connected components of G are $\Gamma(K)$ and $\Gamma(H)$.
$|\mu(K)| = 1$ and $\Gamma(K)$ is a complete graph.
If $|H|$ is even, then K is abelian.
Every subgroup of H of order pq, p and q not necessary distinct primes, is cyclic. In particular if H is abelian, then it would be cyclic.
If H is non-solvable, then there is a normal subgroup H_0 of H such that $[H : H_0] \leqslant 2$ and $H_0 \cong SL_2(5) \times Z$, where every Sylow subgroup of Z is cyclic and $|Z|$ is prime to $2, 3$ and 5. A Frobenius group with cyclic kernel of order m and cyclic complement of order n is denoted by $m : n$.

The following result is also used in this paper whose proof is included in [3]. Every 2-Frobenius group is solvable. [[6]] Let G be a finite solvable group all of whose elements are of prime power order, then the order of G is divisible by at most two distinct primes. [[12]] Let G be a finite group, $K \trianglelefteq G$, and let G/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |K|) = 1$ and F dose not lie in $(K \cdot C_G(K))/K$, then $r \cdot |C| \in w(G)$ for some prime divisor r of $|K|$. [[18]]
If there exists a primitive prime divisor r of $q^n - 1$, then $L_n(q)$ has a Frobenius subgroup with kernel of order r and cyclic complement of order n.

$L_n(q)$ contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1}-1)/(n, q-1)$. Using [**?**], we can find $\mu(G_2(5)) = \{20, 21, 24, 25, 30, 31\}$.Therefore, the prime graph of $G_2(5)$ is as a follows.

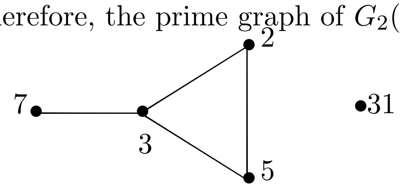

Figure 1: The prime graph of $G_2(5)$

Our main results are the following: If G is a finite group such that $\Gamma(G) = \Gamma(G_2(5))$, then G has a normal subgroup N such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

3. Proof of the theorem

We assume G is a group with $\Gamma(G) = \Gamma(G_2(5))$. By Figure 1, we have $s(G) = 2$, hence, G has disconnected prime graph and we can use Lemma 2.1 here:
G is non-solvable.
If G is solvable, then consider a $\{5, 7, 31\}$-Hall subgroup of G and call it H. By Figure 1, H dose not contain elements of order $5 \cdot 7$, $7 \cdot 31$, $5 \cdot 31$, and since it is solvable, by [5] we deduce $|t(H)| \leqslant 2$, a contradiction.
G is neither a Frobenius nor a 2-Frobenius group.
By (a) and Lemma 2.4, G is not a 2-Frobenius group. If G is a Frobenius group, then by lemma 2.1, $G = KC$ with Frobenius kernel K and Frobenius complement C with connected components $\Gamma(K)$ and $\Gamma(C)$. Obviously $\Gamma(K)$ is a graph with vertex $\{31\}$ and $\Gamma(C)$ with vertex set $\{2, 3, 5, 7\}$. Since G is non-solvable, by Lemma 2.3(a) C must be non-solvable. Therefore, by Lemma 2.3(f) C has a subgroup isomorphic to H_0 and $[C : H_0] \leqslant 2$, where $H_0 \cong SL_2(5) \times Z$ with Z cyclic of order prime to $2, 3, 5$. But $\mu(SL_2(5)) = \{4, 6, 10\}$ from which we can observe that H_0 has no element of order 15. This implies that C has no element of order 15, contradicting Figure 1.

(a) and (b) imply that case (c) of Lemma 2.1 holds for G. Hence, there is a non-abelian simple group P such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ where N is a nilpotent normal $\pi_1(G)$-subgroup of G and $\overline{G}/P$ is a $\pi_1(G)$-group and $s(P) \geqslant 2$. We have $\pi_1(G) = \{2, 3, 5, 7\}$ and $\pi(G) = \{2, 3, 5, 7, 31\}$. Therefore, P is a simple group with $\pi(P) \subseteq \{2, 3, 5, 7, 31\}$, i.e. , $P \in \mathfrak{S}_p$ where p is a prime number satisfying $p \leqslant 31, p \neq 11, 13, 17, 19, 23, 29$. Using [21] we list the possibilities for P in Table I.

Table I: Simple groups in $\mathfrak{S}_p$, $p \leqslant 31, p \neq 11, 13, 17, 19, 23, 29$.

P	$\|P\|$	$\|out(P)\|$	P	$\|P\|$	$\|out(P)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
A_6	$2^3 \cdot 3^2 \cdot 5$	4	$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4

P	$\|P\|$	$\|out(P)\|$	P	$\|P\|$	$\|out(P)\|$
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	6
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2

$\{31\} \subseteq \pi(P)$
By Table I, $|Out(P)|$ is a number of the form $2^\alpha \cdot 3^\beta$, therefore, if $G/N = P \cdot S$ where $S \leqslant Out(P)$, then $|P|_p = |G/N|_p/|S|_p$ for all $p \in \pi(G)$, where n_p denotes the p-part of the integer $n \in N$. Hence, $|N|_p = \frac{|G|_p}{|P|_p \cdot |S|_p}$, from which the claim follows because $\pi(N) \subseteq \{2, 3, 5, 7\}$.

Therefore only the following possibilities arise for P: $L_2(31)$, $L_5(2)$, $L_6(2)$, $L_3(5)$, $L_2(5^3)$ and $G_2(5)$.
$P \cong G_2(5)$
By [4], we have $\mu(L_5(2)) = \{8, 12, 14, 15, 21, 31\}$ and $\mu(L_6(2)) = \{8, 12, 28, 30, 31, 63\}$. Therefore, if $P \cong L_5(2)$ or $L_6(2)$, then, we have $2 \sim 7$ in $\Gamma(G)$, is a contradiction.

By [10], we have $\mu(L_2(5^3)) = \{5, 62, 63\}$. Therefore, if $P \cong L_2(5^3)$, then, we have $2 \sim 31$ in $\Gamma(G)$, a contradiction.

By [**?**], we have $\mu(L_2(31)) = \{15, 16, 31\}$. Therefore, if $P \cong L_2(31)$, then, $7 \in \pi(N)$. By Lemma 2.7, P has a Frobenius subgroup $31 : 15$, then, by Lemma 2.6, G has an element of order $5 \cdot 7$, a contradiction.

By [**?**], we have $\mu(L_3(5)) = \{20, 24, 31\}$. Therefore, if $P \cong L_3(5)$, then, $7 \in \pi(N)$. By Lemma 2.7, P has a Frobenius subgroup $25 : 24$, then, by Lemma 2.6, G has an element of order $2 \cdot 7$, a contradiction. Therefore $P \cong G_2(5)$.
$G/N \cong G_2(5)$
So far we proved that $P \leqslant G/N \leqslant Aut(P)$ where $P \cong G_2(5)$. But $Aut(G_2(5)) = G_2(5)$, therefore, $G/N \cong G_2(5)$.
$\pi(N) \subseteq \{2, 3, 5\}$
We Know that N is a nilpotent normal $\{2, 3, 5, 7\}$-subgroup of G. Regarding Figure 1 we obtain:

If $2, 5 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5\}$
If $3 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5, 7\}$
If $7 \mid |N|$, then $\pi(N) \subseteq \{3, 7\}$

Now we observe that the group $G_2(5)$ contains Frobenius subgroup $31 : 5$. We may assume N is elementary abelian p-group for $p \in \{2, 3, 5, 7\}$. Now if $7 \mid |N|$, then by Lemma 2.6, G has an element of order $5 \cdot 7$, a contradiction. Therefore, $\pi(N) \subseteq \{2, 3, 5\}$.

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