Module-Amenability on Module Extension Banach Algebras

D. Ebrahimi bagha

Abstract. Let $A$ be a Banach algebra and $E$ be a Banach $A$-bimodule then $S = A \otimes E$, the $l^1$-direct sum of $A$ and $E$ becomes a module extension Banach algebra when equipped with the algebras product $(a,x)(a',x') = (aa',a.x' + x.a')$. In this paper, we investigate $\Delta$-amenability for these Banach algebras and we show that for discrete inverse semigroup $S$ with the set of idempotents $E_S$, the module extension Banach algebra $S = l^1(E_S) \oplus l^1(S)$ is $\Delta$-amenable as a $l^1(E_S)$-module if and only if $l^1(E_S)$ is amenable as Banach algebra.

Keywords: Module-amenability, module extension, Banach algebras

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in [8]. The main theorem in [8] asserts that the group algebra $L^1(G)$ of a locally compact group $G$ is amenable if and only if $G$ is amenable. This is far from true for semigroups. If $S$ is a discrete inverse semigroup, $l^1(S)$ is amenable if and only if $E_S$ is finite and all the maximal subgroups of $S$ are amenable [6]. This failure is due to the fact that $l^1(S)$, for a discrete inverse semigroup $S$ with the set of idempotents $E_S$, is equipped with two algebraic structures. It is a Banach algebra and a Banach module over $l^1(E_S)$.

The concept of module amenability for Banach algebras was introduced by M. Amini in [1]. The main theorem in [1] asserts that for an inverse semigroup $S$, with the set of idempotents $E_S$, $l^1(S)$ is module amenable as a Banach module over $l^1(E_S)$ if and only if $S$ is amenable. Also the second named author study the concept of weak module amenability in [2] and showed that for a commutative inverse semigroup $S$, $l^1(S)$ is always weak module amenable as a Banach module over $l^1(E_S)$. There are many examples of Banach modules which do not have any natural algebra structure. One example is $L^p(G)$ which is a left Banach $L^1(G)$-module, for a locally compact group $G$ [4]. The theory of amenability in [8] and module amenability developed in [1] does not cover these examples. There is one thing in common in these examples and that is the existence of a module homomorphism from the Banach module to the underlying Banach algebra. For instance if $G$ is a compact group and $f \in L^0(G)$, then on has the module homomorphism $\Delta_f : L^p(G) \rightarrow L^1(G)$ which sends $g$ to $f \ast g$. The concept of $\Delta$-amenability in [7] is defined for a Banach module $E$ over a Banach algebra $A$ with a given mod-

*Corresponding author. Email: dav.ebrahimibagha@iauctb.ac.ir

© 2013 IAUCTB
http://jlta.iauctb.ac.ir
ule homomorphism $\triangle : E \to A$. The authors in [7] give the basic properties of $\triangle$-amenability and in particular establishes the equivalence of this concept with the existence of module virtual (approximate) diagonals in an appropriate sense. Also the main example in [7] asserts that for a discrete abelian group $G$, $L^p(G)$ is $\triangle$-amenable as an $L^1(G)$-module if and only if $G$ is amenable. In this paper we shall focus on an especial kind of Banach algebras which are constructed from a Banach algebra $A$ and a Banach $A$-bimodule $E$, called module extension Banach algebras and we verify the concept of $\triangle$-amenability for these Banach algebras.

2. Preliminaries

Let $A$ be a Banach algebra and $E$ be a Banach space with a left $A$-module structure such that, for some $M > 0$, $\| a.x \| \leq M \| x \|$ ($a \in A, x \in E$). Then $E$ is called a left Banach $A$-module. Right and two-sided Banach $A$-modules are defined similarly. Throughout this section $E$ is a Banach $A$-bimodule and $\triangle : E \to A$ is a bounded Banach $A$-biomodule homomorphism.

**Definition 2.1** Let $X$ be a Banach $A$-Bimodule. A bounded linear map $D : A \to X$ is called a module derivation (or more specifically $\triangle$-derivation) if

$$D(\triangle(a.x)) = a.D(\triangle(x)) + D(a).\triangle(x)$$

$$D\triangle(x.a)) = D(\triangle(x)).a + \triangle(x).D(a)$$

For each $a \in A$ and $x \in E$. Also $D$ is called inner (or $\triangle$-inner) if there is $f \in X$ such that

$$D(\triangle(x)) = f.\triangle(x) - \triangle(x).f \quad (x \in E)$$

**Definition 2.2** A bimodule $E$ is called module amenable (or more specifically $\triangle$-amenable as a $A$-bimodule) if for each Banach $A$-bimodule $X$, all $\triangle$-derivation from $A$ to $X^*$ are $\triangle$-inner.

It is clear that $A$ is $A$-module amenable (which $\triangle$ =id) if and only if it is amenable as a Banach algebra. A right bounded approximate identity of $E$ is a bounded net $a_\alpha$ in $A$ such that for each $x \in E$, $(\triangle(x).a_\alpha - \triangle(x)) \to 0$ as $\alpha \to 0$. The left and two sided approximate identities are defined similarly.

**Proposition 2.3** If $E$ is module amenable, Then $E$ has a bounded approximate identity.

**Proposition 2.4** If $I$ is a closed ideal of $A$ which contains a bounded approximate identity, $E$ is a Banach $A$-bimodule with module homomorphism $\triangle : E \to A$, and $X$ is a essential Banach I-module, then $X$ is a Banach $A$-module and each $\triangle_I$-derivation $D : I \to X$ uniquely extends to a $\triangle$-derivation $D : A \to X$ which is continuous with respect to the strict topology of $A$ (induced by $I$ ) and $W$-topology of $X^*$.

**Proposition 2.5** If $\triangle : E \to A$ has a dense range, then $\triangle$-amenability of $E$ is equivalent to amenability of $A$.

**Definition 2.6** let $\triangle : A \hat{\otimes} A \to A$ be the continuous lift of the multiplication map of $A$ to the projective tensor product $A \hat{\otimes} A$. A module approximate diagonal of $E$ is
a bounded net $e_{\alpha}$ in $A \hat{\otimes} A$ such that
\[ \| e_{\alpha} \delta(x) - \delta(x) e_{\alpha} \| \to 0 \]
\[ \| \pi(e_{\alpha} \delta(x) - \delta(x)) \| \to 0, \quad (x \in E) \]

As $\alpha \to \infty$. A module virtual diagonal of $E$ is an element $M$ in $(A \hat{\otimes} A)^{**}$ such that
\[ M \delta(x) - \delta(x) M = 0 \]
\[ \pi^{**}(M) \delta(x) - \delta(x) = 0, \quad (x \in E) \]

It is clear that if $E$ has a module virtual diagonal, then $A$ contains a bounded approximate identity.

**Theorem 2.7** Consider the following assertions
i) $E$ is module amenable,
ii) $E$ has a module virtual diagonal,
iii) $E$ has a module approximate diagonal.

We have (i) $\Rightarrow$ (ii)$\Rightarrow$(iii). If moreover $\delta$ has a dense range, all the assertions are equivalent.

**Example 2.8** let $1 \leq P < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $l^1$ is a Banach algebra and $l^p$ is a Banach $l^1$ module, with respect to pointwise multiplication. Also each $f \in l^q$ defines a module homomorphism $\delta_f : l^p \to l^1$ by $\delta_f(g) = g^* f$. If $f = \sum_{k=-\infty}^{\infty} \delta_k$, then $\delta_f$ has dense range and $l^p$ is $\delta_f$-amenable.

3. **$\Delta$-amenability of Module extension Banach algebras**

The module extension Banach algebra corresponding to $A$ and $E$ is $S = A \oplus E$, the $l^1$-direct sum of $A$ and $E$, with the algebra product defined a follows:

\[ (a, x) . (a', x') = (aa', a.x' + x.a') \quad (a, a' \in A, x, x' \in E). \]

Some aspects of algebras of this form have been discussed in [3] and [5] also the amenability and $n$-weak amenability of module extension Banach algebras investigated by Zhang in [7]. In this section we show that the amenability of Banach algebra $A$ is equivalent to $\delta$-amenability $A \oplus E$ as a Banach $A$-module.

By the following module actions the module extension Banach algebra $A \oplus E$ is a Banach $A$-module

\[ a.(b, x) = (ab, x), \quad (b, x).a = (ba, x) \quad (a, b \in A, x \in E). \]

Also $\delta : A \oplus E \to A$ by $(a, x) \to a(a \in A, x \in E)$ is a surjective A-module homomorphism ,so we have:

**Proposition 3.1** The Banach algebra $A$ is amenable if and only if the module extension Banach algebra $A \oplus E$ is $\delta$-amenable as a $A$-module.

**Example 3.2** Let $S$ is a discrete inverse semigroup with the set of idempotents $E_S$ and $E = l^1(S) , A = l^1(E_S)$ and $l^1(E_S)$ act on $l^1(S)$ by multiplication in this case; the module extension Banach algebra $S = l^1(E_S) \oplus l^1(S)$ is $\delta$-amenable as a $l^1(E_S)$-module if and only if $l^1(E_S)$ is amenable.
The authors wishes to thank the islamic Azad university central tehran branch for their kind support.

References