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On duality of modular G-Riesz bases and G-Riesz bases in Hilbert C*-modules

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Abstract. In this paper, we investigate duality of modular g-Riesz bases and g-Riesz bases in Hilbert C*-modules. First we give some characterization of g-Riesz bases in Hilbert C*-modules, by using properties of operator theory. Next, we characterize the duals of a given g-Riesz basis in Hilbert C*-module. In addition, we obtain sufficient and necessary condition for a dual of a g-Riesz basis to be again a g-Riesz basis. We find a situation for a g-Riesz basis to have unique dual g-Riesz basis. Also, we show that every modular g-Riesz basis is a g-Riesz basis in Hilbert C*-module but the opposite implication is not true.

Keywords: Modular G-Riesz basis, G-Riesz basis, dual G-Riesz basis, Hilbert C^*-module.

2010 AMS Subject Classification: 42C15, 42C99, 42C40.

1. Introduction

Frames in Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [5] in the study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [4], and popularized from then on.

Let H be a Hilbert space, and J a set which is finite or countable. A sequence $\{f_j\}_{j\in J} \subseteq H$ is called a frame for H if there exist constants $C, D > 0$ such that

$$C\|f\|^2 \leqslant \sum_{j\in J} | \langle f, f_j\rangle |^2 \leqslant D\|f\|^2 \tag{1}$$

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for all $f \in H$. The constants C and D are called the frame bounds. We have a tight frame if $C = D$ and a Parseval frame if $C = D = 1$. We refer the reader to [2, 3] for more details.

In [15] Sun introduced a generalized notion of frames and suggested further generalizations, showing that many basic properties of frames can be derived within this more general framework.

Let U and V be two Hilbert spaces and $\{V_j : j \in J\}$ be a sequence of subspaces of V, where J is a subset of $\mathbb{Z}$. Let $L(U, V_j)$ be the collection of all bounded linear operators from U to V_j . We call a sequence $\{\Lambda_j \in L(U, V_j) : j \in J\}$ a generalized frame (or simply a g-frame) for U with respect to $\{V_j : j \in J\}$ if there are two positive constants C and D such that

$$C\|f\|^2 \leqslant \sum_{j\in J} \|\Lambda_j f\|^2 \leqslant D\|f\|^2 \tag{2}$$

for all $f \in U$. The constants C and D are called g-frame bounds. If $C = D$ we call have a tight g-frame and if $C = D = 1$ we have a Parseval g-frame.

The notions of frames and g-frames in Hilbert C^*-modules were introduced and investigated in [7, 10, 11, 16]. Frank and Larson [6, 7] defined the standard frames in Hilbert C^*-modules in 1998 and got a series of results for standard frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. Extending the results to this more general framework is not a routine generalization, as there are essential differences between Hilbert C^*-modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^*-module. Also there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^*-modules. We refer the readers to [13] for more details on Hilbert C^*-modules, and to [11] and [16], for a discussion of basic properties of g-frame in Hilbert C^*-modules.

Alijani and Dehghan in [1] studied dual g-frames in Hilbert C*-modules. They give some characterizations of dual g-frames for Hilbert spaces and Hilbert C*-modules. The main goal of this paper is to study duals of g-Riesz basis in Hilbert C*-modules.

This paper is organized as follows. In section 2 we review some basic properties of Hilbert C*-modules and g-Riesz bases in this space. In particular we characterize g-frames and g-Riesz bases in Hilbert C*-modules. In section 3 we study dual g-Riesz bases in Hilbert C^*-modules and characterize the duals of a given g-Riesz basis in Hilbert C*-module. We also obtain sufficient and necessary condition for a dual of a g-Riesz basis to be again a g-Riesz basis. We find a situation for a g-Riesz basis to have unique dual g-Riesz basis. Also, we show that every modular g-Riesz basis is a g-Riesz basis in Hilbert C*-module but the opposite implication is not true.

2. Preliminaries

In this section we review basic properties of g-frames in Hilbert C^*-modules. We also prove some results related to the notion of stability which is used in the next section. Our basic reference for Hilbert C^*-modules is [13]. For basic details on frames in Hilbert C^*-modules we refer the reader to [7].

Definition 2.1 Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space H which is a left A-module with an inner product map $\langle ., . \rangle : H \times H \to A$ which satisfies the following properties:

1) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ for all $f, g, h \in H$ and $\alpha, \beta \in \mathbb{C}$;
2) $\langle af, g\rangle = a\langle f, g\rangle$ for all $f, g \in H$ and $a \in A$;
3) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g \in H$;
4) $\langle f, f\rangle \geqslant 0$ for all $f \in H$ and $\langle f, f\rangle = 0$ iff $f = 0$.
For $f \in H$, we define a norm on H by $\|f\|_H = \|\langle f, f\rangle\|_A^{1/2}$. If H is complete in this norm, it is called a (left) Hilbert C^*-module over A or a (left) Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^* = a$ and the spectrum of a is a subset of positive real number. In this case, we write $a \geqslant 0$. It is easy to see that $\langle f, f\rangle \geqslant 0$ for every $f \in H$, hence we define $|f| = \langle f, f\rangle^{1/2}$.

If H be a Hilbert C^*-module, and J a set which is finite or countable, a sequence $\{f_j\}_{j\in J} \subseteq H$ is called a frame for H if there exist constants $C, D > 0$ such that

$$C\langle f, f\rangle \leqslant \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leqslant D\langle f, f\rangle \tag{3}$$

for all $f \in H$. The constants C and D are called the frame bounds. The notion of (standard) frames in Hilbert C^*-modules is first defined by Frank and Larson [7]. Basic properties of frames in Hilbert C^*-modules are discussed in [8–10].

A. Khosravi and B. Khosravi [11] defined g-frame in Hilbert C^*-modules. Let U and V be two Hilbert C^*-modules over the same C^*-algebra A and $\{V_j : j \in J\}$ be a sequence of subspaces of V, where J is a subset of $\mathbb{Z}$. Let $End_A^*(U, V_j)$ be the collection of all adjointable A-linear maps from U into V_j, i.e. $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in H$ and $T \in End_A^*(U, V_j)$. We call a sequence $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ a generalized frame (or simply a g-frame) for Hilbert C^*-module U with respect to $\{V_j : j \in J\}$ if there are two positive constants C and D such that

$$C\langle f, f\rangle \leqslant \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leqslant D\langle f, f\rangle \tag{4}$$

for all $f \in U$. The constants C and D are called g-frame bounds. Those sequences which satisfy only the upper inequality in (2.2) are called g-Bessel sequences. A g-frame is tight, if $C = D$. If $C = D = 1$, it is called a Parseval g-frame.

Definition 2.2 [16] A g-frame $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ is called a g-Riesz basis if it satisfies:

(1) $\Lambda_j \neq 0$ for any $j \in J$;
(2) If an A-linear combination $\sum_{j\in K}\Lambda_j^* g_j$ is equal to zero, then every summand $\Lambda_j^* g_j$ is equal to zero, where $\{g_j\}_{j\in K} \in \bigoplus_{j\in K} V_j$ and $K \subseteq J$.

***Example* 2.3** Let H be an ordinary Hilbert space, then H is a Hilbert $\mathbb{C}$-module. Let $\{e_j : j \in J\}$ be an orthonormal basis for H, then $\{e_j : j \in J\}$ is a Parseval frame for Hilbert $\mathbb{C}$-module H.

***Example* 2.4** Let U be an ordinary Hilbert space, $J = N$ and $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for Hilbert $\mathbb{C}$-module U. For j=1,2,... we let $V_j = \overline{span}\{e_1, e_2, ..., e_j\}$, and $\Lambda_j : U \to V_j$, $\Lambda_j f = \sum_{k=1}^{j}\langle f, \frac{e_j}{\sqrt{j}}\rangle e_k$.

We have $\sum_{j=1}^{\infty}\langle \Lambda_j f, \Lambda_j f\rangle = \sum_{j=1}^{\infty}|\langle f, e_j\rangle|^2 = \langle f, f\rangle$, which implies that $\{\Lambda_j\}_{j=1}^{\infty}$ is a g-Parseval frame for U with respect to $\{V_j : j \in J\}$.

Theorem 2.5 [16] Let $\Lambda_j \in End_A^*(U, V_j)$ for any $j \in J$ and $\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle$ converge in norm for $f \in U$. Then $\{\Lambda_j : j \in J\}$ is a g-frame for U with respect to $\{V_j : j \in J\}$ if and only if there exist constants $C, D > 0$ such that

$$C\|f\|^2 \leqslant \left\|\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\right\| \leqslant D\|f\|^2, \quad f \in U.$$

Definition 2.6 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ be a g-frame in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and $\{\Gamma_j \in End_A^*(U, V_j) : j \in J\}$ be a sequence of A-linear operators. Then $\{\Gamma_j : j \in J\}$ is called a dual sequence operator of $\{\Lambda_j : j \in J\}$ if

$$f = \sum_{j\in J}\Lambda_j^*\Gamma_j f$$

for all $f \in U$. The sequences $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are called a dual g-frame when moreover $\{\Gamma_j : j \in J\}$ is a g-frame.

In [11] the authors defined the g-frame operator S in Hilbert C^*-module as follow

$$Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j f, \ \ f \in U,$$

and showed that S is invertible, positive, and self-adjoint. Since

$$\langle Sf, f\rangle = \langle\sum_{j\in J}\Lambda_j^*\Lambda_j f, f\rangle = \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle,$$

it follows that

$$C\langle f, f\rangle \leqslant \langle Sf, f\rangle \leqslant D\langle f, f\rangle,$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j\in J}\Lambda_j^*\Lambda_j S^{-1}f = \sum_{j\in J}S^{-1}\Lambda_j^*\Lambda_j f,$$

for all $f \in U$. Let $\tilde{\Lambda}_j = \Lambda_j S^{-1}$, then

$$f = \sum_{j\in J}\Lambda_j^*\tilde{\Lambda}_j f = \sum_{j\in J}\tilde{\Lambda}_j\Lambda_j^* f.$$

The sequence $\{\tilde{\Lambda}_j : j \in J\}$ is also a g-frame for U with respect to $\{V_j : j \in J\}$(see [11]) which is called the canonical dual g-frame of $\{\Lambda_j : j \in J\}$.

Definition 2.7 Let $\{\Lambda_j : j \in J\}$ be a g-frame in Hilbert C*-module U with respect to $\{V_j : j \in J\}$, then the related analysis operator $T : U \to \bigoplus_{j\in J} V_j$ is defined by $Tf = \{\Lambda_j f : j \in J\}$, for all $f \in U$. We define the synthesis operator $F : \bigoplus_{j\in J} V_j \to U$

by $Ff = F(f_j) = \sum_{j\in J} \Lambda_j^* f_j$, for all $f = \{f_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$, where

$$\bigoplus_{j\in J} V_j = \left\{ f = \{f_j\} : f_j \in V_j, \ \left\| \sum_{j\in J} |f_j|^2 \right\| < \infty \right\}.$$

It has been showed in [16] that if for any $f = \{f_j\}_{j\in J}$ and $g = \{g_j\}_{j\in J}$ in V_j the A-valued inner product is defined by $\langle f, g\rangle = \sum_{j\in J} \langle f_j, g_j\rangle$ and the norm is defined by $\|f\| = \|\langle f, f\rangle\|^{1/2}$, then $\bigoplus_{j\in J} V_j$ is a Hilbert A-module. Hence the above operators are definable. Moreover, since for any $g = \{g_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$ and $f \in U$,

$$\begin{aligned} \langle Tf, g\rangle &= \sum_{j\in J} \langle \Lambda_j f, g_j\rangle = \sum_{j\in J} \langle f, \Lambda_j^* g_j\rangle \\ &= \langle f, \sum_{j\in J} \Lambda_j^* g_j\rangle = \langle f, Fg\rangle, \end{aligned}$$

it follows that T is adjointable and $T^* = F$. Also

$$T^*Tf = T^*(\Lambda_j f) = \sum_{j\in J} \Lambda_j^* \Lambda_j f = Sf,$$

for all $f \in U$. Let P_n be the projection on $\bigoplus_{j\in J} V_j$ that is $P_n : \bigoplus_{j\in J} V_j \to \bigoplus_{j\in J} V_j$ is defined by $P_n f = P_n(\{f_j\}_{j\in J}) = U_j$, for $f = \{f_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$, and $U_j = f_n$ when $j = n$ and $U_j = 0$ when $j \neq n$.

Theorem 2.8 ([14]) Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ be a g-frame in Hilbert C*-module U with respect to $\{V_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is a g-Riesz basis if and only if $\Lambda_n \neq 0$ and $P_n(RangT) \subseteq RangT$ for all $n \in J$, where T is the analysis operator of $\{\Lambda_j\}_{j\in J}$.

Corollary 2.9 A g-frame $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ is a g-Riesz basis if and only if

(1) $\Lambda_j \neq 0$ for any $j \in J$;
(2) If an A-linear combination $\sum_{j\in K} \Lambda_j^* g_j$is equal to zero, then every summand $\Lambda_j^* g_j$ is equal to zero, where $\{g_j\}_{j\in K} \in \bigoplus_{j\in K} V_j$ and $K \subseteq J$.

3. Dual of g-Riesz bases in Hilbert C*-modules

In this section, we study dual g-Riesz bases in Hilbert C^*-modules and characterize the duals of a given g-Riesz basis in Hilbert C*-module. We also obtain sufficient and necessary condition for a dual of a g-Riesz basis to be again a g-Riesz basis. We find a situation for a g-Riesz basis to have unique dual g-Riesz basis.

Proposition 3.1 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ and $\{\Gamma_j \in End_A^*(U, V_j) : j \in J\}$ be two g-Bessel sequences in Hilbert C*-module U with respect to $\{V_j : j \in J\}$. If $f = \sum_{j\in J} \Lambda_j^* \Gamma_j f$ holds for any $f \in U$, then both $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are g-frames in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and $f = \sum_{j\in J} \Gamma_j^* \Lambda_j f$.

Proof. Let us denote the g-Bessel bound of $\{\Gamma_j : j \in J\}$ by B_Γ. For all $f \in U$ we have

$$\begin{aligned}\|f\|^4 = \|\langle \sum_{j\in J} \Lambda_j^* \Gamma_j f, f\rangle\|^2 &= \|\sum_{j\in J} \langle \Gamma_j f, \Lambda_j f\rangle\|^2 \\ &\leqslant \|\sum_{j\in J} \langle \Gamma_j f, \Gamma_j f\rangle\|.\|\sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle\| \\ &\leqslant B_\Gamma \|f\|^2.\|\sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle\|.\end{aligned}$$

It follows that

$$B_\Gamma^{-1}\|f\|^2 \leqslant \|\sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle\|.$$

This implies that $\{\Lambda_j : j \in J\}$ is a g-frame in Hilbert C*-module. Similarly we can show that $\{\Gamma_j : j \in J\}$ is also a g-frame of U respect to $\{V_j : j \in J\}$. ■

Lemma 3.2 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ be a g-frame in Hilbert C*-module U with respect to $\{V_j : j \in J\}$. Suppose that $\{\Gamma_j \in End_A^*(U, V_j) : j \in J\}$ and $\{\Theta_j \in End_A^*(U, V_j) : j \in J\}$ are dual g-frames of $\{\Lambda_j : j \in J\}$ with the property that either $RangT_\Gamma \subseteq RangT_\Theta$ or $RangT_\Theta \subseteq RangT_\Gamma$. Then $\Gamma_j = \Theta_j \ \ \forall j \in J$.

Proof. Suppose that $RangT_\Theta \subseteq RangT_\Gamma$. Then for each $f \in U$ there exists $g_f \in U$ such that $T_\Theta g_f = T_\Gamma f$.
Applying T_Λ^* on both sides, we arrive at

$$g_f = \sum_{j\in J} \Lambda_j^* \Theta_j g_f = T_\Lambda^* T_\Theta g_f = T_\Lambda^* T_\Gamma f = \sum_{j\in J} \Lambda_j^* \Gamma_j f = f$$

and so $T_\Gamma f = T_\Theta f$, $\forall f \in U$.
Equivalently $\Gamma_j f = \Theta_j f$, $\forall j \in J$. ■

Theorem 3.3 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ be a g-frame in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ with analysis operator T_Λ, then the following are equivalence:
(1) $\{\Lambda_j : j \in J\}$ has a unique dual g-frame;
(2) $RangT_\Lambda = \oplus_{j\in J} V_j$;
(3) If $\sum_{j\in J} \Lambda_j^* f_j = 0$ for some sequence $\{f_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$, then $f_j = 0$ for each $j \in J$.
In case the equivalent conditions are satisfied, $\{\Lambda_j : j \in J\}$ is a g-Riesz basis.

Proof. (2)⇒ (1) Let $\{\tilde{\Lambda}_j : j \in J\}$ be the canonical dual g-frame of $\{\Lambda_j : j \in J\}$ with analysis operator $T_{\tilde{\Lambda}}$. Then $\tilde{\Lambda}_j = \Lambda_j S^{-1}$, where S is g-frame operator.
Let $\{\Gamma_j : j \in J\}$ be any dual g-frame of $\{\Lambda_j : j \in J\}$ with analysis operator T_Γ.
Then

$$RangT_\Gamma \subseteq \oplus_{j\in J} V_j = RangT_\Lambda = RangT_{\tilde{\Lambda}}.$$

By Lemma 3.2, $\Gamma_j = \tilde{\Lambda}_j$ for all $j \in J$.

(1)⇒ (2) Assume on the contrary that $RangT_\Lambda \neq \oplus_{j\in J} V_j$. We have

$$\bigoplus_{j\in J} V_j = RangT_\Lambda \bigoplus KerT^*_\Lambda. \tag{5}$$

Let P_Λ be orthogonal projection from $\bigoplus_{j\in J} V_j$ onto $RangT_\Lambda$, then

$$\bigoplus_{j\in J} V_j = P_\Lambda(\bigoplus_{j\in J} V_j) \bigoplus P^\perp_\Lambda(\bigoplus_{j\in J} V_j).$$

Therefore $P^\perp_\Lambda = KerT^*_\Lambda \neq \{0\}$.
Choose $f_{j_0} \in \bigoplus_{j\in J} V_j$ such that $P^\perp_\Lambda f_{j_0} \neq 0$ where $f_{j_0} = 1_{j_0}$ if $j = j_0$ and $f_{j_0} = 0$ if $j \neq j_0$ and 1_{j_0} is unital element of V_{j_0}. Define an operator $W : \bigoplus_{j\in J} V_j \to U$ by $W\{g_j\} = \Lambda^*_{j_0} g_{j_0}$.
Now, let $\{\widetilde{\Lambda}_j : j \in J\}$ be the canonical dual of $\{\Lambda_j : j \in J\}$ with upper bound $D_{\tilde{\Lambda}}$ and $\Gamma_j = \widetilde{\Lambda}_j + \Pi_j \Pi W^*$ where $\Pi : \bigoplus_{j\in J} V_j \to KerT^*_\Lambda$ and $\Pi_j : \bigoplus_{j\in J} V_j \to V_j$ are projection operators.
We have

$$\begin{aligned}\sum_{j\in J}\langle \Gamma_j f, \Gamma_j f\rangle &\leqslant 2\left(\sum_{j\in J}\langle \widetilde{\Lambda}_j f, \widetilde{\Lambda}_j f\rangle + \sum_{j\in J}\langle \Pi_j\Pi W^* f, \Pi_j \Pi W^* f\rangle\right)\\ &\leqslant 2\left(D_{\tilde{\Lambda}}\langle f, f\rangle + \langle \Pi W^* f, \Pi W^* f\rangle\right)\\ &\leqslant (D_{\tilde{\Lambda}} + \|\Pi W^*\|^2)\langle f, f\rangle,\end{aligned}$$

which implies that $\{\Gamma_j : j \in J\}$ is a g-Bessel sequence.

Now for any $f \in U$,

$$\sum_{j\in J} \Lambda^*_j \Pi_j \Pi W^* f = T^*_\Lambda\{\Pi_j \Pi W^* f\} = 0.$$

This yields that $\sum_{j\in J} \Lambda^*_j \Gamma_j f = \sum_{j\in J} \Lambda^*_j \widetilde{\Lambda}_j f = f$ for all $f \in U$. By Proposition 3.1, $\{\Gamma_j : j \in J\}$ is a dual g-frame of $\{\Lambda_j : j \in J\}$ and is different from $\{\widetilde{\Lambda}_j : j \in J\}$, which contradicts with the uniqueness of dual g-frame of $\{\Lambda_j : j \in J\}$.
(2)⇔ (3) Obvious by (3.1). ■

Theorem 3.4 Suppose that $\{\Lambda_j \in End^*_A(U, V_j) : j \in J\}$ is a g-Riesz basis in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and $\{\Gamma_j \in End^*_A(U, V_j) : j \in J\}$ is a sequence of A-linear operators. Then the following are equivalence:

(1) $\{\Gamma_j : j \in J\}$ is a dual g-frame of $\{\Lambda_j : j \in J\}$;
(2) $\{\Gamma_j : j \in J\}$ is a dual g-Bessel sequence of $\{\Lambda_j : j \in J\}$;
(3) For each $j \in J$, $\Gamma_j = \Lambda_j S^{-1} + \Theta_j$, where S is the g-frame operator of $\{\Lambda_j : j \in J\}$ and $\{\Theta_j : j \in J\}$ is a dual g-Bessel sequence of U with respect to $\{V_j : j \in J\}$ satisfying $\Lambda^*_j \Theta_j f = 0$ for all $f \in U$ and $j \in J$.

Theorem 3.5 Let $\{\Lambda_j \in End^*_A(U, V_j) : j \in J\}$ be a g-Riesz basis in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ a sequence of A-linear operators. Then $\{\Gamma_j : j \in J\}$ is a dual g-Riesz basis of $\{\Lambda_j : j \in J\}$ if and only if for each $j \in J$,

$\Gamma_j = \Lambda_j S^{-1} + \Theta_j$, where S is the g-frame operator of $\{\Lambda_j : j \in J\}$ and $\{\Theta_j : j \in J\}$ is a g-Bessel sequence of U with respect to $\{V_j : j \in J\}$ with the property that for each $j \in J$ there exists operator $F_j \in End_A^*(V_j, V_j)$ such that $\Theta_j = F_j \Lambda_j S^{-1}$ and $\Lambda_j^* F_j \Lambda_j f = 0$ holds for all $f \in U$.

Proof. $\Rightarrow$ Suppose that $\{\Gamma_j : j \in J\}$ is a dual g-Riesz basis of $\{\Lambda_j : j \in J\}$ and let $\Theta_j = \Gamma_j - \Lambda_j S^{-1}$. It is easy to see that $\{\Theta_j : j \in J\}$ is a g-Bessel sequence of U.
Now fix an $n \in J$. From $\sum_{j \in J} \Gamma_j \Lambda_j^*(\Gamma_n f) = \Gamma_n f$ we can infer that $\Gamma_n = \Gamma_n \Lambda_n^* \Gamma_n$, i.e.

$$\Lambda_n S^{-1} + \Theta_n = (\Lambda_n S^{-1} + \Theta_n)\Lambda_n^*(\Lambda_n S^{-1} + \Theta_n)$$

Consequently, we have

$$\begin{aligned}\Theta_n &= (\Lambda_n S^{-1}\Lambda_n^* + \Theta_n \Lambda_n^*)(\Lambda_n S^{-1} + \Theta_n) - \Lambda_n S^{-1} \\ &= \Lambda_n S^{-1}\Lambda_n^*\Lambda_n S^{-1} + \Lambda_n S^{-1}\Lambda_n^*\Theta_n + \Theta_n\Lambda_n^*\Lambda_n S^{-1} + \Theta_n \Lambda_n^* \Theta_n - \Lambda_n S^{-1} \\ &= \Lambda_n S^{-1}\Lambda_n^*\Theta_n + \Theta_n\Lambda_n^*\Lambda_n S^{-1} + \Theta_n\Lambda_n^*\Theta_n.\end{aligned}$$

We show that $\Lambda_n S^{-1}\Lambda_n^*\Theta_n + \Theta_n\Lambda_n^*\Lambda_n S^{-1} = 0$.
Note that

$$f = \sum_{j \in J} \Lambda_j^* \Gamma_j f = \sum_{j \in J} \Lambda_j^*(\Lambda_j S^{-1} + \Theta_j) f = \sum_{j \in J} \Lambda_j^* \Lambda_j S^{-1} f + \sum_{j \in J} \Lambda_j^* \Theta_j f = f + \sum_{j \in J} \Lambda_j^* \Theta_j f,$$

which implies that $\sum_{j \in J} \Lambda_j^* \Theta_j f = 0$ and $\Lambda_j^* \Theta_j f = 0$ for all $f \in U$ and $j \in J$.
Particularly, we have $\Lambda_n^* \Theta_n f = 0$ for all $f \in U$. This yields that $\Lambda_n S^{-1}\Lambda_n^* \Theta_n = 0$ and $\Theta_n \Lambda_n^* \Theta_n = 0$.
Therefore $\Theta_n = \Theta_n \Lambda_n^* \Lambda_n S^{-1}$. Suppose $F_n = \Theta_n \Lambda_n^*$, then $\Theta_n = F_n \Lambda_n S^{-1}$.
From $\Lambda_n^* \Theta_n = 0$, we have $\Lambda_n^* \Theta_n \Lambda_n^* \Lambda_n f = 0$ i.e.$F_n \Lambda_n^* \Lambda_n f = 0$.
$\Leftarrow$ Suppose that for each $j \in J$ there exists operator $F_j \in End_A^*(V_j, V_j)$ such that $\Theta_j = F_j \Lambda_j S^{-1}$ and $\Lambda_j^* F_j \Lambda_j f = 0$ holds for all $f \in U$. Then for all $f \in U$ we have

$$\sum_{j \in J} \Lambda_j^* \Gamma_j f = \sum_{j \in J} \Lambda_j^* \Lambda_j S^{-1} f = \sum_{j \in J} \Lambda_j^* \Theta_j f = f + \sum_{j \in J} \Lambda_j^* F_j \Lambda_j S^{-1} f = f.$$

Therefore $\{\Gamma_j : j \in J\}$ is a dual sequence of $\{\Lambda_j : j \in J\}$.
With similar proof of Theorem 3.3, $\{\Gamma_j : j \in J\}$ is a g-Bessel sequence and by Proposition 3.1, $\{\Gamma_j : j \in J\}$ is dual g-frame of $\{\Lambda_j : j \in J\}$.
To complete the proof, we need to show that $\{\Gamma_j : j \in J\}$ is a g-Riesz basis of U with respect to $\{V_j : j \in J\}$.
Let $\sum_{j \in J} \Gamma_j^* f_j = 0$, then we have

$$0 = \sum_{j \in J} (S^{-1}\Lambda_j^* + \Theta_j^*) f_j = \sum_{j \in J} (S^{-1}\Lambda_j^* + S^{-1}\Lambda_j^* F_j^*) f_j = \sum_{j \in J} S^{-1}\Lambda_j^*(I_j + F_j^*) f_j.$$

Since $\{S^{-1}\Lambda_j^* : j \in J\}$ is a g-Riesz basis then $S^{-1}\Lambda_j^*(I_j + F_j^*) f_j = 0$, i.e. $\Gamma_j^* f_j = 0$ for all $j \in J$.
We now show that $\Gamma_j \neq 0$ for all $j \in J$.
Assume on the contrary that $\Gamma_n = 0$ for some $n \in J$. Then $\Theta_n = -\Lambda_n S^{-1}$. It follows

that

$$0 = \Lambda_n^* F_n \Lambda_n f = \Lambda_n^* S \Theta_n f = -\Lambda_n^* \Lambda_n f$$

holds for all $f \in U$.
In particular, letting $f = S^{-1}\Lambda_n^* g$ for some $g \in U$, we have $-\Lambda_n^* \Lambda_n S^{-1} \Lambda_n^* g = -\Lambda_n^* g = 0$, therefore $\Lambda_n = 0$, a contradiction. This completes the proof. ■

Corollary 3.6 Suppose that $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ is a g-Riesz basis in Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and Λ_j is surjective for any $j \in J$. Then $\{\Lambda_j : j \in J\}$ has a unique dual g-Riesz basis.

Proof. Let $f_j \in V_j$ for some $j \in J$, then there exists $f \in U$ such that $\Lambda_j f = f_j$. Therefore, we have

$$\Theta_j^* f_j = S^{-1} \Lambda_j^* F_j^* \Lambda_j f = S^{-1} 0 = 0.$$

■

Corollary 3.7 Suppose that $\{f_j : j \in J\}$ is a Riesz basis in Hilbert A-module H and operator $T_j : H \to A$ defined by $T_j f =< f, f_j >$ is surjective for any $j \in J$. Then $\{f_j : j \in J\}$ has a unique dual Riesz basis.

Definition 3.8 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$

(i) If the A-linear hull of $\bigcup_{j \in J} \Lambda_j^*(V_j)$ is dense in U, then $\{\Lambda_j : j \in J\}$ is g-complete.

(ii) If $\{\Lambda_j : j \in J\}$ is g-complete and there exist real constant A, B such that for any finite subset $S \subseteq J$ and $g_j \in V_j, j \in S$

$$A \Big\| \sum_{j \in S} |g_j|^2 \Big\| \leqslant \Big\| \sum_{j \in S} \Lambda_j^* g_j \Big\|^2 \leqslant B \Big\| \sum_{j \in S} |g_j|^2 \Big\|,$$

then $\{\Lambda_j : j \in J\}$ is a modular g-Riesz basis for U with respect to $\{V_j : j \in J\}$. A and B are called bounds of $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$.

Theorem 3.9 ([12]) A sequence $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ is a modular g-Riesz basis if and only if the synthesis operator F is a homeomorphism.

Theorem 3.10 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ then the following two statements are equivalent:

(1) The sequence $\{\Lambda_j : j \in J\}$ is a modular g-Riesz basis for Hilbert C*-module U with respect to $\{V_j : j \in J\}$ with bounds A and B;

(2)The sequence $\{\Lambda_j : j \in J\}$ is a g-frame for Hilbert C*-module U with respect to $\{V_j : j \in J\}$ with bounds A and B, and if an A-linear combination $\sum_{j \in S} \Lambda_j^* g_j = 0$ for $\{g_j\}_{j \in J} \in \oplus_{j \in J} V_j$, then $g_j = 0$ for all $j \in J$.

Proof. (1)$\to$ (2) By Theorem 3.9 the operator $F : \oplus V_j \to U$ is a linear homeomorphism. Hence the operator F is onto and therefore by Theorem 3.2 in [16] $\{\Lambda_j : j \in J\}$ is a g-frame. Also, since F is injective

$$KerF = \Big\{ \{g_j\}_{j \in J} \in \oplus_{j \in J} V_j : F(\{g_j\}_{j \in J}) = \sum_{j \in S} \Lambda_j^* g_j = 0 \Big\} = \{0\}. \qquad (6)$$

This implies the statement (2).

(2)→ (1)By Theorem 3.2 in [16] the operator F is injective and by (3.2) F is injective. Therefore F is homeomorphism and by Theorem ? $\{\Lambda_j : j \in J\}$ is a modular g-Riesz basis. ■

Corollary 3.11 Every modular g-Riesz basis is a g-Riesz basis.

Proof. By Definition 3.8 and Theorem 3.9. ■

Corollary 3.12 Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ then the following two statements are equivalent:

(1) The sequence $\{\Lambda_j : j \in J\}$ is a modular g-Riesz basis for Hilbert C*-module U with respect to $\{V_j : j \in J\}$.

(2)The sequence $\{\Lambda_j : j \in J\}$ has a unique dual modular g-Riesz basis.

Proof. (1)→ (2) Every modular g-Riesz basis is a g-Riesz basis and every g-Riesz basis is a g-frame. So every modular g-Riesz basis is a g-frame. Now by Theorem 3.3 and Theorem 3.10 $\{\Lambda_j : j \in J\}$ has a unique dual modular g-Riesz basis.

(2)→ (1)The proof by Theorem 3.3 and Theorem 3.10 is straightforward. ■

Next example shows in Hilbert C*-module setting, every Riesz basis is not a modular Riesz basis, so every g-Riesz basis is not a modular g-Riesz basis.

***Example* 3.13** Let $A = M_{2\times 2}(C)$ be the C*-algebra of all 2×2 complex matrices. Let $H = A$ and for any $A, B \in H$ define $\langle A, B\rangle = AB^*$. Then H is a Hilbert A-module.

Let $E_{i,j}$ be the matrix with 1 in the (i, j)th entry and 0 elsewhere, where $1 \leqslant i, j \leqslant 2$. Then $\Phi = \{E_{1,1}, E_{2,2}\}$ is a Riesz basis of H but is not a modular Riesz basis.

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