Journal of
Linear and Topological Algebra
Vol. 02*, No.* 01*,* 2013, 9- 23

The extension of quadrupled fixed point results in K-metric spaces

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Abstract. Recently, Rahimi et al. [Comp. Appl. Math. 2013, In press] defined the concept of quadrupled fixed point in K-metric spaces and proved several quadrupled fixed point theorems for solid cones on K-metric spaces. In this paper some quadrupled fixed point results for T-contraction on K-metric spaces without normality condition are proved. Obtained results extend and generalize well-known comparable results in the literature.

Keywords: K-metric spaces; Quadrupled fixed point; T-contraction; Sequentially convergent.

2010 AMS Subject Classification: 47H10, 46J10.

1. Introduction

In 1922, Banach proved his famous fixed point theorem [2]. Suppose that (X, d) is a complete metric space and a self-map T of X satisfies $d(Tx, Ty) \leqslant \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$; that is, T is a contractive mapping. Then T has a unique fixed point. Afterward, many authors considered various definitions of contractive mappings and proved several fixed point theorems, which are extensions and generalizations of Banach's theorem (see [7, 19, 23] and the references contained therein). Fixed point theory in K-metric and K-normed spaces was developed by Perov et al. [12], Mukhamadijev and

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

Stetsenko [11] (also, see Zabrejko [25]). In 2007, Huang and Zhang [8] reintroduced K-metric spaces, replacing the set of real numbers by an ordered Banach space, and proved some fixed point theorems. Afterward, many authors proved several fixed and common fixed point results on cone metric spaces (see, for example, [1, 15, 16, 18, 21, 22] and the references contained therein).

On the other hand, Morales and Rajes [10] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric space and proved some fixed point theorems. Then, Filipović et al. [5] defined T-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. Very recently, Rahimi et al. [14, 16, 17] proved new fixed and periodic point theorems for T-contractions involving two mappings on cone metric spaces.

In 2006, Bhaskar and Lakshmikantham [4] considered the concept of coupled fixed point theorems in partially ordered metric spaces. Afterward, some other authors generalized this concept [16, 20, 24]. Finally, Berinde and Borcut [3] and Karapinar and Loung [9] introduced the notion of tripled and quadrupled fixed points and proved several n-tuple fixed point results.

In this paper, we consider the concept of T-contraction in quadrupled fixed point theory and obtain some quadrupled fixed point results on K-metric spaces without normality condition. Our theorems extend, unify and generalize the results of Rahimi et al. [13].

2. Preliminaries

Let us start by defining some important definitions.

Definition 2.1 [6, 8]. Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{\theta\}$,
(b) $a, b \in \mathbb{R}, a, b \geqslant 0, x, y \in P$ imply that $ax + by \in P$,
(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is interior of P). If $intP \neq \emptyset$, the cone P is called solid. The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leqslant K\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 2.2 [8]. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

($d1$) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
($d2$) $d(x, y) = d(y, x)$ for all $x, y \in X$,
($d3$) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric [8] or K-metric [25] on X and (X, d) is called a cone metric space [8] or K-metric space (or abstract metric spaces) [25].

The concept of a K-metric space is more general than that of a metric space, because each metric space is a K-metric space where $E = \mathbb{R}$ and $P = [0, \infty)$.

***Example* 2.3** [8]. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E | x, y \geqslant 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geqslant 0$ is a constant. Then (X, d) is a K-metric space.

Definition 2.4 [5]. Let (X, d) be a K-metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Also, a K-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.5 [5]. Let (X, d) be a K-metric space over an ordered real Banach space E. Then the following properties are often used.

(P_1) If $x \preceq y$ and $y \ll z$, then $x \ll z$.

(P_2) If $\theta \preceq x \ll c$ for each $c \in intP$, then $x = \theta$.

(P_3) If $x \preceq \lambda x$ where $x \in P$ and $0 \leqslant \lambda < 1$, then $x = \theta$.

(P_4) Let $x_n \to \theta$ in E and $\theta \ll c$. Then there exists a positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

Definition 2.6 [5]. Let (X, d) be a K-metric space, P a solid cone and $S : X \to X$. Then

(i) S is said to be sequentially convergent if we have for every sequence (x_n), if $S(x_n)$ is convergent, then (x_n) also is convergent.

(ii) S is said to be subsequentially convergent if we have for every sequence (x_n) that $S(x_n)$ is convergent, implies (x_n) has a convergent subsequence.

(iii) S is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} S(x_n) = S(x)$, for all (x_n) in X.

Definition 2.7 [5]. Let (X, d) be a K-metric space and $T, f : X \to X$ two mappings. A mapping f is said to be a T-Hardy-Rogers contraction, if there exist $\alpha_i \geqslant 0$, $i = 1, \cdots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,

$$d(Tfx, Tfy) \preceq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) + \alpha_5 d(Ty, Tfx).$$

In previous definition, suppose that $\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 \neq 0$ (resp. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 \neq 0$). Then we obtain T-Kannan (resp. T-Chatterjea) contraction from [10].

Theorem 2.8 [14, 16] Let (X,d) be a complete K-metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, $f : X \to X$ be a mapping satisfying

$$d(Tfx, Tfy) \preceq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tfy)] \\ +\gamma[d(Tx, Tfy) + d(Ty, Tfx)],$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ with $\alpha + 2\beta + 2\gamma < 1$. Then

(i) for each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$);
(ii) there exists a $z_{x_0} \in X$ such that $\lim_{n\to+\infty} Tf^n x_0 = z_{x_0}$;
(iii) if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence;
(iv) there exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point;
(v) if T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.

3. Main results

For simplicity, denote $X \times X \times X \times X$ by X^4, where X is a non-empty set.

Definition 3.1 [13]. An element $(x, y, z, u) \in X^4$ is called a quadrupled fixed point of a given mapping $F : X^4 \to X$ if $x = F(x, y, z, u)$, $y = F(y, z, u, x)$, $z = F(z, u, x, y)$ and $u = F(u, x, y, z)$.

Definition 3.2 [13]. Let (X, d) be a K-metric space and $T : X \to X$ be a mapping. A mapping $F : X^4 \to X$ is said to be a T-contraction, if there exist $\alpha, \beta, \gamma, \delta \geqslant 0$ with $\alpha + \beta + \gamma + \delta < 1$ such that for all $x, y, z, u, x^*, y^*, z^*, u^* \in X$,

$$d(TF(x, y, z, u), TF(x^*, y^*, z^*, u^*)) \preceq \alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*) \\ +\gamma d(Tz, Tz^*) + \delta d(Tu, Tu^*).$$

Now, we prove the main theorems of our work.

Theorem 3.3 Suppose that (X, d) is a complete K-metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let $F : X^4 \to X$ be a mapping satisfying

$$\begin{aligned}
&d(TF(x,y,z,t),TF(u,v,w,s))\\
&\preceq \alpha_1 d(TF(x,y,z,t),Tx)+\alpha_2 d(TF(y,z,t,x),Ty)+\alpha_3 d(TF(z,t,x,y),Tz)\\
&+\alpha_4 d(TF(t,x,y,z),Tt)+\alpha_5 d(T(u,v,w,s),Tu)+\alpha_6 d(TF(v,w,s,u),Tv)\\
&+\alpha_7 d(TF(w,s,u,v),Tw)+\alpha_8 d(TF(s,u,v,w),Ts)+\alpha_9 d(TF(u,v,w,s),Tx)\\
&+\alpha_{10} d(TF(v,w,s,u),Ty)+\alpha_{11} d(TF(w,s,u,v),Tz)+\alpha_{12} d(TF(s,u,v,w),Tt)\\
&+\alpha_{13} d(TF(x,y,z,t),Tu)+\alpha_{14} d(TF(y,z,t,x),Tv)+\alpha_{15} d(TF(z,t,x,y),Tw)\\
&+\alpha_{16} d(TF(t,x,y,z),Ts)+\alpha_{17} d(Tx,Tu)+\alpha_{18} d(Ty,Tv)+\alpha_{19} d(Tz,Tw)\\
&+\alpha_{20} d(Tt,Ts), \qquad (1)
\end{aligned}$$

for all $x,y,z,t,u,v,w,s \in X$, where α_i for $i = 1,2,\cdots,20$ are nonnegative constants with $\sum_{i=1}^{20}\alpha_i < 1$. Then

(t_1) For each $x_0,y_0,z_0,t_0 \in X$,

$$\{TF^n(x_0,y_0,z_0,t_0)\}\ ,\ \{TF^n(y_0,z_0,t_0,x_0)\},$$
$$\{TF^n(z_0,t_0,x_0,y_0)\}\ ,\ \{TF^n(t_0,x_0,y_0,z_0)\}$$

are Cauchy sequences.

(t_2) There exist $q_{x_0}, q_{y_0}, q_{z_0}, q_{t_0} \in X$ such that

$$\lim_{n\to\infty} TF^n(x_0,y_0,z_0,t_0) = q_{x_0}\ ,\ \lim_{n\to\infty} TF^n(y_0,z_0,t_0,x_0) = q_{y_0},$$
$$\lim_{n\to\infty} TF^n(z_0,t_0,x_0,y_0) = q_{z_0}\ ,\ \lim_{n\to\infty} TF^n(t_0,x_0,y_0,z_0) = q_{t_0}.$$

(t_3) If T is subsequentially convergent, then

$$\{TF^n(x_0,y_0,z_0,t_0)\}\ ,\ \{TF^n(y_0,z_0,t_0,x_0)\}$$
$$\{TF^n(z_0,t_0,x_0,y_0)\}\ ,\ \{TF^n(t_0,x_0,y_0,z_0)\}$$

have a convergent subsequence.

(t_4) There exist unique $r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0} \in X$ such that

$$F(r_{x_0},r_{y_0},r_{z_0},r_{t_0}) = r_{x_0}\ ,\ F(r_{y_0},r_{z_0},r_{t_0},r_{x_0}) = r_{y_0},$$
$$F(r_{z_0},r_{t_0},r_{x_0},r_{y_0}) = r_{z_0}\ ,\ F(r_{t_0},r_{x_0},r_{y_0},r_{z_0}) = r_{t_0};$$

that is, F has a unique quadruple fixed point.

(t_5) If T is sequentially convergent, then, for each $x_0,y_0,z_0,t_0 \in X$, the sequence $\{TF^n(x_0,y_0,z_0,t_0)\}$ converges to $r_{x_0} \in X$, the sequence $\{TF^n(y_0,z_0,t_0,x_0)\}$ converges to $r_{y_0} \in X$, the sequence $\{TF^n(z_0,t_0,x_0,y_0)\}$ converges to $r_{z_0} \in X$ and the sequence

$\{TF^n(t_0, x_0, y_0, z_0)\}$ converges to $r_{t_0} \in X$.

Proof. Let $x_0, y_0, z_0, t_0 \in X$ and set

$$\begin{cases} x_1 = TF(x_0, y_0, z_0, t_0) \\ y_1 = TF(y_0, z_0, t_0, x_0) \\ z_1 = TF(z_0, t_0, x_0, y_0) \\ t_1 = TF(t_0, x_0, y_0, z_0) \end{cases} \cdots \begin{cases} x_{n+1} = TF(x_n, y_n, z_n, t_n) \\ y_{n+1} = TF(y_n, z_n, t_n, x_n) \\ z_{n+1} = TF(z_n, t_n, x_n, y_n) \\ t_{n+1} = TF(t_n, x_n, y_n, z_n) \end{cases}$$

for $n = 0, 1, \cdots$. Now, according to (1), we have

$$\begin{aligned} & d(Tx_n, Tx_{n+1}) = d(TF(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), TF(x_n, y_n, z_n, t_n)) \\ \preceq\ & \alpha_1 d(Tx_n, Tx_{n-1}) + \alpha_2 d(Ty_n, Ty_{n-1}) + \alpha_3 d(Tz, Tz_{n-1}) + \alpha_4 d(Tt_n, Tt_{n-1}) \\ & +\alpha_5 d(Tx_{n+1}, Tx_n) + \alpha_6 d(Ty_{n+1}, Ty_n) + \alpha_7 d(Tz_{n+1}, Tz_n) + \alpha_8 d(Tt_{n+1}, Tt_n) \\ & +\alpha_9 d(Tx_{n+1}, Tx_{n-1}) + \alpha_{10} d(Ty_{n+1}, Ty_{n-1}) + \alpha_{11} d(Tz_{n+1}, Tz_{n-1}) \\ & +\alpha_{12} d(Tt_{n+1}, Tt_{n-1}) + \alpha_{17} d(Tx_{n-1}, Tx_n) + \alpha_{18} d(Ty_{n-1}, Ty_n) \\ & +\alpha_{19} d(Tz_{n-1}, Tz_n) + \alpha_{20} d(Tt_{n-1}, Tt_n) \end{aligned}$$

It follows

$$\begin{aligned} & (1 - \alpha_5 - \alpha_9) d(Tx_n, Tx_{n+1}) \\ \preceq\ & (\alpha_1 + \alpha_9 + \alpha_{17}) d(Tx_{n-1}, Tx_n) + (\alpha_2 + \alpha_{10} + \alpha_{18}) d(Ty_{n-1}, Ty_n) \\ & +(\alpha_3 + \alpha_{11} + \alpha_{19}) d(Tz_{n-1}, Tz_n) + (\alpha_4 + \alpha_{12} + \alpha_{20}) d(Tt_{n-1}, Tt_n) \\ & +(\alpha_6 + \alpha_{10}) d(Ty_{n+1}, Ty_n) + (\alpha_7 + \alpha_{11}) d(Tz_{n+1}, Tz_n) \\ & +(\alpha_8 + \alpha_{12}) d(Tt_{n+1}, Tt_n). \end{aligned} \tag{2}$$

Similarly, we obtain

$$\begin{aligned} & (1 - \alpha_5 - \alpha_9) d(Ty_n, Ty_{n+1}) \\ \preceq\ & (\alpha_1 + \alpha_9 + \alpha_{17}) d(Ty_{n-1}, Ty_n) + (\alpha_2 + \alpha_{10} + \alpha_{18}) d(Tz_{n-1}, Tz_n) \\ & +(\alpha_3 + \alpha_{11} + \alpha_{19}) d(Tt_{n-1}, Tt_n) + (\alpha_4 + \alpha_{12} + \alpha_{20}) d(Tx_{n-1}, Tx_n) \\ & +(\alpha_6 + \alpha_{10}) d(Tz_{n+1}, Tz_n) + (\alpha_7 + \alpha_{11}) d(Tt_{n+1}, Tt_n) \\ & +(\alpha_8 + \alpha_{12}) d(Tx_{n+1}, Tx_n), \end{aligned} \tag{3}$$

and

$$\begin{aligned}
&(1-\alpha_5-\alpha_9)d(Tz_n,Tz_{n+1})\\
\preceq\ &(\alpha_1+\alpha_9+\alpha_{17})d(Tz_{n-1},Tz_n)+(\alpha_2+\alpha_{10}+\alpha_{18})d(Tt_{n-1},Tt_n)\\
&+(\alpha_3+\alpha_{11}+\alpha_{19})d(Tx_{n-1},Tx_n)+(\alpha_4+\alpha_{12}+\alpha_{20})d(Ty_{n-1},Ty_n)\\
&+(\alpha_6+\alpha_{10})d(Tt_{n+1},Tt_n)+(\alpha_7+\alpha_{11})d(Tx_{n+1},Tx_n)\\
&+(\alpha_8+\alpha_{12})d(Ty_{n+1},Ty_n), \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha_5-\alpha_9)d(Tt_n,Tt_{n+1})\\
\preceq\ &(\alpha_1+\alpha_9+\alpha_{17})d(Tt_{n-1},Tt_n)+(\alpha_2+\alpha_{10}+\alpha_{18})d(Tx_{n-1},Tx_n)\\
&+(\alpha_3+\alpha_{11}+\alpha_{19})d(Ty_{n-1},Ty_n)+(\alpha_4+\alpha_{12}+\alpha_{20})d(Tz_{n-1},Tz_n)\\
&+(\alpha_6+\alpha_{10})d(Tx_{n+1},Tx_n)+(\alpha_7+\alpha_{11})d(Ty_{n+1},Ty_n)\\
&+(\alpha_8+\alpha_{12})d(Tz_{n+1},Tz_n). \qquad (5)
\end{aligned}$$

Because of the symmetry in (1), we have

$$\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(Tx_{n+1},Tx_n)\\
\preceq\ &(\alpha_5+\alpha_{13}+\alpha_{17})d(Tx_n,Tx_{n-1})+(\alpha_6+\alpha_{14}+\alpha_{18})d(Ty_{n-1},Ty_n)\\
&+(\alpha_7+\alpha_{15}+\alpha_{19})d(Tz_{n-1},Tz_n)+(\alpha_8+\alpha_{16}+\alpha_{20})d(Tt_{n-1},Tt_n)\\
&+(\alpha_2+\alpha_{14})d(Ty_{n+1},Ty_n)+(\alpha_3+\alpha_{15})d(Tz_{n+1},Tz_n)\\
&+(\alpha_4+\alpha_{16})d(Tt_{n+1},Tt_n), \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(Ty_{n+1},Ty_n)\\
\preceq\ &(\alpha_5+\alpha_{13}+\alpha_{17})d(Ty_n,Ty_{n-1})+(\alpha_6+\alpha_{14}+\alpha_{18})d(Tz_{n-1},Tz_n)\\
&+(\alpha_7+\alpha_{15}+\alpha_{19})d(Tt_{n-1},Tt_n)+(\alpha_8+\alpha_{16}+\alpha_{20})d(Tx_{n-1},Tx_n)\\
&+(\alpha_2+\alpha_{14})d(Tz_{n+1},Tz_n)+(\alpha_3+\alpha_{15})d(Tt_{n+1},Tt_n)\\
&+(\alpha_4+\alpha_{16})d(Tx_{n+1},Tx_n), \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(Tz_{n+1},Tz_n)\\
&\preceq (\alpha_5+\alpha_{13}+\alpha_{17})d(Tz_n,Tz_{n-1})+(\alpha_6+\alpha_{14}+\alpha_{18})d(Tt_{n-1},Tt_n)\\
&\quad+(\alpha_7+\alpha_{15}+\alpha_{19})d(Tx_{n-1},Tx_n)+(\alpha_8+\alpha_{16}+\alpha_{20})d(Ty_{n-1},Ty_n)\\
&\quad+(\alpha_2+\alpha_{14})d(Tt_{n+1},Tt_n)+(\alpha_3+\alpha_{15})d(Tx_{n+1},Tx_n)\\
&\quad+(\alpha_4+\alpha_{16})d(Ty_{n+1},Ty_n), \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(Tt_{n+1},Tt_n)\\
&\preceq (\alpha_5+\alpha_{13}+\alpha_{17})d(Tt_n,Tt_{n-1})+(\alpha_6+\alpha_{14}+\alpha_{18})d(Tx_{n-1},Tx_n)\\
&\quad+(\alpha_7+\alpha_{15}+\alpha_{19})d(Ty_{n-1},Ty_n)+(\alpha_8+\alpha_{16}+\alpha_{20})d(Tz_{n-1},Tz_n)\\
&\quad+(\alpha_2+\alpha_{14})d(Tx_{n+1},Tx_n)+(\alpha_3+\alpha_{15})d(Ty_{n+1},Ty_n)\\
&\quad+(\alpha_4+\alpha_{16})d(Tz_{n+1},Tz_n). \qquad (9)
\end{aligned}$$

Let

$$\Delta_n = d(Tx_n,Tx_{n+1})+d(Ty_n,Ty_{n+1})+d(Tz_n,Tz_{n+1})+d(Tt_n,Tt_{n+1}).$$

Now, adding (2) to (5) and (6) to (9), we have

$$(1-\sum_{i=5}^{12}\alpha_i)\Delta_n \preceq (\sum_{i=1}^{4}\alpha_i+\sum_{i=9}^{12}\alpha_i+\sum_{i=17}^{20}\alpha_i)\Delta_{n-1}, \qquad (10)$$

and

$$(1-\sum_{i=1}^{4}\alpha_i-\sum_{i=13}^{16}\alpha_i)\Delta_n \preceq (\sum_{i=5}^{8}\alpha_i+\sum_{i=13}^{20}\alpha_i)\Delta_{n-1}. \qquad (11)$$

Ultimately, adding (10) and (11), we have

$$(2-\sum_{i=1}^{16}\alpha_i)\Delta_n \preceq (\sum_{i=1}^{16}\alpha_i+2\sum_{i=17}^{20}\alpha_i)\Delta_{n-1}.$$

Thus, for all n,

$$\theta \preceq \Delta_n \preceq \lambda\Delta_{n-1} \preceq \lambda^2\Delta_{n-2} \preceq \cdots \preceq \lambda^n\Delta_0, \qquad (12)$$

where

$$\lambda = \frac{\sum_{i=1}^{16} \alpha_i + 2\sum_{i=17}^{20} \alpha_i}{2 - \sum_{i=1}^{16} \alpha_i} < 1.$$

If $\Delta_0 = \theta$ then (x_0, y_0, z_0, t_0) is a quadrupled fixed point of F. Now, let $\theta \prec \Delta_0$. If $m > n$, we have

$$d(Tx_n, Tx_m) \preceq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m), \quad (13)$$

$$d(Ty_n, Ty_m) \preceq d(Ty_n, Ty_{n+1}) + d(Ty_{n+1}, Ty_{n+2}) + \cdots + d(Ty_{m-1}, Ty_m), \quad (14)$$

$$d(Tz_n, Tz_m) \preceq d(Tz_n, Tz_{n+1}) + d(Tz_{n+1}, Tz_{n+2}) + \cdots + d(Tz_{m-1}, Tz_m), \quad (15)$$

$$d(Tt_n, Tt_m) \preceq d(Tt_n, Tt_{n+1}) + d(Tt_{n+1}, Tt_{n+2}) + \cdots + d(Tt_{m-1}, Tt_m). \quad (16)$$

Adding (13) to (16) and using (12). Since $\lambda < 1$, we have

$$\begin{aligned} d' &\preceq d_n + d_{n+1} + \cdots + d_{m-1} \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d_0 \\ &\preceq \frac{\lambda^n}{1-\lambda} d_0 \to \theta \quad as \quad n \to \infty, \end{aligned}$$

where

$$d' = d(Tx_n, Tx_m) + d(Ty_n, Ty_m) + d(Tz_n, Tz_m) + d(Tt_n, Tt_m).$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exist positive integer N such that $d' \ll c$ for every $m > n > N$. Thus $\{Tx_n\}$, $\{Ty_n\}$, $\{Tz_n\}$ and $\{Tt_n\}$ are Cauchy sequences in X. Since X is a complete K-metric space, there exist $q_{x_0}, q_{y_0}, q_{z_0}, q_{t_0} \in X$ such that

$$\begin{aligned} &\lim_{n\to\infty} TF^n(x_0, y_0, z_0, t_0) = q_{x_0} \,,\, \lim_{n\to\infty} TF^n(y_0, z_0, t_0, x_0) = q_{y_0}, \\ &\lim_{n\to\infty} TF^n(z_0, t_0, x_0, y_0) = q_{z_0} \,,\, \lim_{n\to\infty} TF^n(t_0, x_0, y_0, z_0) = q_{t_0}. \end{aligned} \quad (17)$$

Now, if T is subsequentially convergent, then $F^n(x_0, y_0, z_0, t_0)$, $F^n(y_0, z_0, t_0, x_0)$, $F^n(z_0, t_0, x_0, y_0)$ and $F^n(t_0, x_0, y_0, z_0)$ have convergent subsequences. Thus, there exist $r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0} \in X$ and the sequences $\{x_{n_i}\}$, $\{y_{n_i}\}$, $\{z_{n_i}\}$ and $\{t_{n_i}\}$ such that

$$\begin{aligned} &\lim_{i\to\infty} F^{n_i}(x_0, y_0, z_0, t_0) = r_{x_0} \,,\, \lim_{i\to\infty} F^{n_i}(y_0, z_0, t_0, x_0) = r_{y_0}, \\ &\lim_{i\to\infty} F^{n_i}(z_0, t_0, x_0, y_0) = r_{z_0} \,,\, \lim_{i\to\infty} F^{n_i}(t_0, x_0, y_0, z_0) = r_{t_0}. \end{aligned}$$

Because of the continuity of T, we have

$$\begin{aligned}&\lim_{i\to\infty} TF^{n_i}(x_0,y_0,z_0,t_0)=Tr_{x_0}\ ,\ \lim_{i\to\infty} TF^{n_i}(y_0,z_0,t_0,x_0)=Tr_{y_0},\\&\lim_{i\to\infty} TF^{n_i}(z_0,t_0,x_0,y_0)=Tr_{z_0}\ ,\ \lim_{i\to\infty} TF^{n_i}(t_0,x_0,y_0,z_0)=Tr_{t_0}.\end{aligned}\tag{18}$$

Now, by (17) and (18), we conclude that

$$Tr_{x_0}=q_{x_0}\quad,\quad Tr_{y_0}=q_{y_0}\quad,\quad Tr_{z_0}=q_{z_0}\quad,\quad Tr_{t_0}=q_{t_0}.$$

On the other hand,

$$\begin{aligned}d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})&\preceq d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),TF(x_{n_i},y_{n_i},z_{n_i},t_{n_i}))\\&\quad+d(Tx_{n_i+1},Tr_{x_0})\end{aligned}$$

Using (1), we obtain

$$\begin{aligned}&(1-\alpha_1-\alpha_{13})d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})\\&-(\alpha_2+\alpha_{14})d(TF(r_{y_0},r_{z_0},r_{t_0},r_{x_0}),Tr_{y_0})\\&-(\alpha_3+\alpha_{15})d(TF(r_{z_0},r_{t_0},r_{x_0},r_{y_0}),Tr_{z_0})-(\alpha_4+\alpha_{16})d(TF(r_{t_0},r_{x_0},r_{y_0},r_{z_0}),Tr_{t_0})\\&\preceq \alpha_5 d(Tx_{n_i+1},Tx_{n_i})+\alpha_6 d(Ty_{n_i+1},Ty_{n_i})+\alpha_7 d(Tz_{n_i+1},Tz_{n_i})\\&+\alpha_8 d(Tt_{n_i+1},Tt_{n_i})+(1+\alpha_9)d(Tx_{n_i+1},Tr_{x_0})+\alpha_{10}d(Ty_{n_i+1},Tr_{y_0})\\&+\alpha_{11}d(Tz_{n_i+1},Tr_{z_0})+\alpha_{12}d(Tt_{n_i+1},Tr_{t_0})+(\alpha_{13}+\alpha_{17})d(Tr_{x_0},Tx_{n_i})\\&+(\alpha_{14}+\alpha_{18})d(Tr_{y_0},Ty_{n_i})+(\alpha_{15}+\alpha_{19})d(Tr_{z_0},Tz_{n_i})\\&+(\alpha_{16}+\alpha_{20})d(Tr_{t_0},Tt_{n_i}).\end{aligned}\tag{19}$$

Similarly, we obtain

$$\begin{aligned}&(1-\alpha_1-\alpha_{13})d(TF(r_{y_0},r_{z_0},r_{t_0},r_{x_0}),Tr_{y_0})\\&-(\alpha_2+\alpha_{14})d(TF(r_{z_0},r_{t_0},r_{x_0},r_{y_0}),Tr_{z_0})\\&-(\alpha_3+\alpha_{15})d(TF(r_{t_0},r_{x_0},r_{y_0},r_{z_0}),Tr_{t_0})-(\alpha_4+\alpha_{16})d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})\\&\preceq \alpha_5 d(Ty_{n_i+1},Ty_{n_i})+\alpha_6 d(Tz_{n_i+1},Tz_{n_i})+\alpha_7 d(Tt_{n_i+1},Tt_{n_i})\\&+\alpha_8 d(Tx_{n_i+1},Tx_{n_i})+(1+\alpha_9)d(Ty_{n_i+1},Tr_{y_0})+\alpha_{10}d(Tz_{n_i+1},Tr_{z_0})\\&+\alpha_{11}d(Tt_{n_i+1},Tr_{t_0})+\alpha_{12}d(Tx_{n_i+1},Tr_{x_0})+(\alpha_{13}+\alpha_{17})d(Tr_{y_0},Ty_{n_i})\\&+(\alpha_{14}+\alpha_{18})d(Tr_{z_0},Tz_{n_i})+(\alpha_{15}+\alpha_{19})d(Tr_{t_0},Tt_{n_i})\\&+(\alpha_{16}+\alpha_{20})d(Tr_{x_0},Tx_{n_i}),\end{aligned}\tag{20}$$

and

$$
\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(TF(r_{z_0},r_{t_0},r_{x_0},r_{y_0}),Tr_{z_0})\\
&-(\alpha_2+\alpha_{14})d(TF(r_{t_0},r_{x_0},r_{y_0},r_{z_0}),Tr_{t_0})\\
&-(\alpha_3+\alpha_{15})d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})-(\alpha_4+\alpha_{16})d(TF(r_{y_0},r_{z_0},r_{t_0},r_{x_0}),Tr_{y_0})\\
&\preceq \alpha_5 d(Tz_{n_i+1},Tz_{n_i})+\alpha_6 d(Tt_{n_i+1},Tt_{n_i})+\alpha_7 d(Tx_{n_i+1},Tx_{n_i})\\
&+\alpha_8 d(Ty_{n_i+1},Ty_{n_i})+(1+\alpha_9)d(Tz_{n_i+1},Tr_{z_0})+\alpha_{10}d(Tt_{n_i+1},Tr_{t_0})\\
&+\alpha_{11}d(Tx_{n_i+1},Tr_{x_0})+\alpha_{12}d(Ty_{n_i+1},Tr_{y_0})+(\alpha_{13}+\alpha_{17})d(Tr_{z_0},Tz_{n_i})\\
&+(\alpha_{14}+\alpha_{18})d(Tr_{t_0},Tt_{n_i})+(\alpha_{15}+\alpha_{19})d(Tr_{x_0},Tx_{n_i})\\
&+(\alpha_{16}+\alpha_{20})d(Tr_{y_0},Ty_{n_i}), \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-\alpha_1-\alpha_{13})d(TF(r_{t_0},r_{x_0},r_{y_0},r_{z_0}),Tr_{t_0})\\
&-(\alpha_2+\alpha_{14})d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})\\
&-(\alpha_3+\alpha_{15})d(TF(r_{y_0},r_{z_0},r_{t_0},r_{x_0}),Tr_{y_0})-(\alpha_4+\alpha_{16})d(TF(r_{z_0},r_{t_0},r_{x_0},r_{y_0}),Tr_{z_0})\\
&\preceq \alpha_5 d(Tt_{n_i+1},Tt_{n_i})+\alpha_6 d(Tx_{n_i+1},Tx_{n_i})+\alpha_7 d(Ty_{n_i+1},Ty_{n_i})\\
&+\alpha_8 d(Tz_{n_i+1},Tz_{n_i})+(1+\alpha_9)d(Tt_{n_i+1},Tr_{t_0})+\alpha_{10}d(Tx_{n_i+1},Tr_{x_0})\\
&+\alpha_{11}d(Ty_{n_i+1},Tr_{y_0})+\alpha_{12}d(Tz_{n_i+1},Tr_{z_0})+(\alpha_{13}+\alpha_{17})d(Tr_{t_0},Tt_{n_i})\\
&+(\alpha_{14}+\alpha_{18})d(Tr_{x_0},Tx_{n_i})+(\alpha_{15}+\alpha_{19})d(Tr_{y_0},Ty_{n_i})\\
&+(\alpha_{16}+\alpha_{20})d(Tr_{z_0},Tz_{n_i}). \qquad (22)
\end{aligned}
$$

Now, set

$$
\begin{aligned}
\mu_1 &= d(TF(r_{x_0},r_{y_0},r_{z_0},r_{t_0}),Tr_{x_0})+d(TF(r_{y_0},r_{z_0},r_{t_0},r_{x_0}),Tr_{y_0})\\
&\quad +d(TF(r_{z_0},r_{t_0},r_{x_0},r_{y_0}),Tr_{z_0})+d(TF(r_{t_0},r_{x_0},r_{y_0},r_{z_0}),Tr_{t_0}),\\
\mu_2 &= d(Tx_{n_i+1},Tr_{x_0})+d(Ty_{n_i+1},Tr_{y_0})+d(Tz_{n_i+1},Tr_{z_0})+d(Tt_{n_i+1},Tr_{t_0}),\\
\mu_3 &= d(Tx_{n_i},Tr_{x_0})+d(Ty_{n_i},Tr_{y_0})+d(Tz_{n_i},Tr_{z_0})+d(Tt_{n_i},Tr_{t_0}).
\end{aligned}
$$

Adding (19) to (22). We have

$$
\begin{aligned}
(1-\sum_{i=1}^{4}\alpha_i-\sum_{i=13}^{16}\alpha_i)\mu_1 &\preceq \sum_{i=5}^{8}\alpha_i\Delta_n+(1+\sum_{i=9}^{12}\alpha_i)\mu_2+\sum_{i=13}^{20}\alpha_i\mu_3\\
&\preceq \Delta_n+2\mu_2+\mu_3.
\end{aligned}
$$

Therefore,

$$\mu_1 \preceq \frac{1}{A}\Delta_n + \frac{2}{A}\mu_2 + \frac{1}{A}\mu_3,$$

where

$$A = 1 - \sum_{i=1}^{4} \alpha_i - \sum_{i=13}^{16} \alpha_i.$$

By applying Lemma 2.5, we can obtain

$$\begin{aligned} &d(TF(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0}), Tr_{x_0}) = \theta \ , \ d(TF(r_{y_0}, r_{z_0}, r_{t_0}, r_{x_0}), Tr_{y_0}) = \theta \\ &d(TF(r_{z_0}, r_{t_0}, r_{x_0}, r_{y_0}), Tr_{z_0}) = \theta \ , \ d(TF(r_{t_0}, r_{x_0}, r_{y_0}, r_{z_0}), Tr_{t_0}) = \theta. \end{aligned}$$

which implies that

$$\begin{aligned} &TF(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0}) = Tr_{x_0} \ , \ TF(r_{y_0}, r_{z_0}, r_{t_0}, r_{x_0}) = Tr_{y_0} \\ &TF(r_{z_0}, r_{t_0}, r_{x_0}, r_{y_0}) = Tr_{z_0} \ , \ TF(r_{t_0}, r_{x_0}, r_{y_0}, r_{z_0}) = Tr_{t_0}. \end{aligned}$$

Since T is one to one, then

$$\begin{aligned} &F(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0}) = r_{x_0} \ , \ F(r_{y_0}, r_{z_0}, r_{t_0}, r_{x_0}) = r_{y_0} \\ &F(r_{z_0}, r_{t_0}, r_{x_0}, r_{y_0}) = r_{z_0} \ , \ F(r_{t_0}, r_{x_0}, r_{y_0}, r_{z_0}) = r_{t_0}. \end{aligned}$$

Thus, $(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0})$ is quadrupled fixed point of the mapping F. Now, if $(r'_{x_0}, r'_{y_0}, r'_{z_0}, r'_{t_0})$ is another quadrupled fixed point of F, then we obtain

$$\begin{aligned} d(Tr_{x_0}, Tr'_{x_0}) &= d(TF(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0}), TF(r'_{x_0}, r'_{y_0}, r'_{z_0}, r'_{t_0})) \\ &\preceq (\alpha_9 + \alpha_{13} + \alpha_{17}) d(Tr'_{x_0}, Tr_{x_0}) + (\alpha_{10} + \alpha_{14} + \alpha_{18}) d(Tr'_{y_0}, Tr_{y_0}) \\ &\quad + (\alpha_{11} + \alpha_{15} + \alpha_{19}) d(Tr'_{z_0}, Tr_{z_0}) + (\alpha_{12} + \alpha_{16} + \alpha_{20}) d(Tr'_{t_0}, Tr_{t_0}), \end{aligned} \tag{23}$$

and

$$\begin{aligned} d(Tr_{y_0}, Tr'_{y_0}) &= d(TF(r_{y_0}, r_{z_0}, r_{t_0}, r_{x_0}), TF(r'_{y_0}, r'_{z_0}, r'_{t_0}, r'_{x_0})) \\ &\preceq (\alpha_9 + \alpha_{13} + \alpha_{17}) d(Tr'_{y_0}, Tr_{y_0}) + (\alpha_{10} + \alpha_{14} + \alpha_{18}) d(Tr'_{z_0}, Tr_{z_0}) \\ &\quad + (\alpha_{11} + \alpha_{15} + \alpha_{19}) d(Tr'_{t_0}, Tr_{t_0}) + (\alpha_{12} + \alpha_{16} + \alpha_{20}) d(Tr'_{x_0}, Tr_{x_0}), \end{aligned} \tag{24}$$

and

$$\begin{aligned} d(Tr_{z_0}, Tr'_{z_0}) &= d(TF(r_{z_0}, r_{t_0}, r_{x_0}, r_{y_0}), TF(r'_{z_0}, r'_{t_0}, r'_{x_0}, r'_{y_0})) \\ &\preceq (\alpha_9 + \alpha_{13} + \alpha_{17}) d(Tr'_{z_0}, Tr_{z_0}) + (\alpha_{10} + \alpha_{14} + \alpha_{18}) d(Tr'_{t_0}, Tr_{t_0}) \\ &\quad + (\alpha_{11} + \alpha_{15} + \alpha_{19}) d(Tr'_{x_0}, Tr_{x_0}) + (\alpha_{12} + \alpha_{16} + \alpha_{20}) d(Tr'_{y_0}, Tr_{y_0}), \end{aligned} \tag{25}$$

and

$$\begin{aligned} d(Tr_{t_0}, Tr'_{t_0}) &= d(TF(r_{t_0}, r_{x_0}, r_{y_0}, r_{z_0}), TF(r'_{t_0}, r'_{x_0}, r'_{y_0}, r'_{z_0})) \\ &\preceq (\alpha_9 + \alpha_{13} + \alpha_{17}) d(Tr'_{t_0}, Tr_{t_0}) + (\alpha_{10} + \alpha_{14} + \alpha_{18}) d(Tx', Tr_{x_0}) \\ &\quad + (\alpha_{11} + \alpha_{15} + \alpha_{19}) d(Tr'_{y_0}, Tr_{y_0}) + (\alpha_{12} + \alpha_{16} + \alpha_{20}) d(Tr'_{z_0}, Tr_{z_0}). \end{aligned} \tag{26}$$

Adding (23) to (26), we have

$$\begin{aligned} & d(Tr_{x_0}, Tr'_{x_0}) + d(Tr_{y_0}, Tr'_{y_0}) + d(Tr_{z_0}, Tr'_{z_0}) + d(Tr_{t_0}, Tr'_{t_0}) \\ \preceq & \sum_{i=9}^{20} \alpha_i [d(Tr_{x_0}, Tr'_{x_0}) + d(Tr_{y_0}, Tr'_{y_0}) + d(Tr_{z_0}, Tr'_{z_0}) + d(Tr_{t_0}, Tr'_{t_0})]. \end{aligned}$$

which implies that

$$d(Tr_{x_0}, Tr'_{x_0}) + d(Tr_{y_0}, Tr'_{y_0}) + d(Tr_{z_0}, Tr'_{z_0}) + d(Tr_{t_0}, Tr'_{t_0}) = \theta$$

Since T is one to one, we have $(r_{x_0}, r_{y_0}, r_{z_0}, r_{t_0}) = (r'_{x_0}, r'_{y_0}, r'_{z_0}, r'_{t_0})$. Ultimately, if T is sequentially convergent, then we can replace n by n_i. Thus, we have

$$\begin{aligned} \lim_{n\to\infty} TF^n(x_0, y_0, z_0, t_0) = r_{x_0} \ , \ \lim_{n\to\infty} TF^n(y_0, z_0, t_0, x_0) = r_{y_0} \\ \lim_{n\to\infty} TF^n(z_0, t_0, x_0, y_0) = r_{z_0} \ , \ \lim_{n\to\infty} TF^n(t_0, x_0, y_0, z_0) = r_{t_0}. \end{aligned}$$

This completes the proof of Theorem 3.3. ■

The results of Rahimi et al.'s work can be obtained from Theorem 3.3.

Corollary 3.4 [13]. Suppose that (X, d) is a complete K-metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let $F : X^4 \to X$ be a mapping satisfying

$$\begin{aligned} d(TF(x, y, z, u), TF(x^*, y^*, z^*, u^*)) &\preceq \alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*) \\ &\quad + \gamma d(Tz, Tz^*) + \delta d(Tu, Tu^*) \end{aligned}$$

for all $x, y, z, u, x^*, y^*, z^*, u^* \in X$, where $\alpha, \beta, \gamma, \delta \geqslant 0$ with

$$\alpha + \beta + \gamma + \delta < 1.$$

Then, the results of Theorem 3.3 are hold.

Corollary 3.5 [13]. Suppose that (X, d) is a complete K-metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let $F : X^4 \to X$ be a mapping satisfying

$$d(TF(x,y,z,u), TF(x^*,y^*,z^*,u^*)) \preceq \frac{k}{4}[d(Tx,Tx^*) + d(Ty,Ty^*) \\ +d(Tz,Tz^*) + d(Tu,Tu^*)]$$

for all $x, y, z, u, x^*, y^*, z^*, u^* \in X$, where $k \in [0, 1)$. Then, the results of Theorem 3.3 hold.

References

[1] M. Abbas, B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. 22 (2009), pp. 511-515.

[2] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math. J. 3 (1922), pp. 133-181.

[3] V. Berinde, M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. 74 (2011), pp. 4889-4897.

[4] T. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006), pp. 1379-1393.

[5] M. Filipović, L. Paunović, S. Radenović, M. Rajović, *Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings"*, Math. Comput. Modelling. 54 (2011), pp. 1467-1472.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[7] G.E. Hardy, T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. 1 (6) (1973), pp. 201–206.

[8] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), pp. 1467–1475.

[9] E. Karapinar, N.V. Luong, *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl. 64 (6) (2012) pp. 1839–1848.

[10] J.R. Morales, E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, Int. J. Math. Anal. 4 (4) (2010), pp. 175–184.

[11] E.M. Mukhamadiev, V.J. Stetsenko, *Fixed point principle in generalized metric space*, Izvestija AN Tadzh. SSR, fiz.-mat.igeol.-chem.nauki. 10 (4) (1969), pp. 8–19 (in Russian).

[12] A.I. Perov, *The Cauchy problem for systems of ordinary differential equations*, Approximate Methods of Solving Differential Equations. Kiev. Naukova Dumka. (1964), pp. 115–134 (in Russian).

[13] H. Rahimi, S. Radenović, G. Soleimani Rad, P. Kumam, *Quadrupled fixed point results in abstract metric spaces*, Comp. Appl. Math. (2013), DOI 10.1007/s40314-013-0088-5.

[14] H. Rahimi, B.E. Rhoades, S. Radenović, G. Soleimani Rad, *Fixed and periodic point theorems for T-contractions on cone metric spaces*, Filomat. 27 (5) (2013), pp. 881–888 (DOI 10.2298/FIL1305881R).

[15] H. Rahimi, G. Soleimani Rad, *Common fixed point theorems and c-distance in ordered cone metric spaces*, Ukrainian Mathematical Journal, (2013), to appear.

[16] H. Rahimi, G. Soleimani Rad, *Fixed point theory in various spaces*, Lambert Academic Publishing, Germany, 2013.

[17] H. Rahimi, G. Soleimani Rad, *New fixed and periodic point results on cone metric spaces*, Journal of Linear and Topological Algebra 1 (1) (2012), pp. 33–40.

[18] H. Rahimi, G. Soleimani Rad, *Note on "Common fixed point results for noncommuting mappings without continuity in cone metric spaces"*, Thai. J. Math. 11 (3) (2013), pp. 589–599.

[19] H. Rahimi, G. Soleimani Rad, *Some fixed point results in metric type space*, J. Basic Appl. Sci. Res. 2 (9) (2012), pp. 9301–9308.

[20] H. Rahimi, G. Soleimani Rad, P. Kumam, *Coupled common fixed point theorems under weak contractions in cone metric type spaces* , Thai. J. Math, In press.

[21] H. Rahimi, P. Vetro, G. Soleimani Rad, *Some common fixed point results for weakly compatible mappings in cone metric type space*, Miskolc Math. Notes. 14 (1) (2013), pp. 233–243.

[22] S. Rezapour, R. Hamlbarani, *Some note on the paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 345 (2008), pp. 719-724.

[23] B.E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. 266 (1977), pp. 257-290.

[24] B. Samet, C. Vetro, *Coupled fixed point, f-invariant set and fixed point of N-order*, Ann. Funct. Anal. 1 (2) (2010), pp. 46-56.

[25] P. P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math. 48 (4-06)(1997), pp. 825-859.