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Expansion of Bessel and g-Bessel sequences to dual frames and dual g-frames

M. S. Asgari[a] and G. Kavian[a,*]

[a] *Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, PO. Code 14168-94351, Tehran, Iran*

Abstract. In this paper we study the duality of Bessel and g-Bessel sequences in Hilbert spaces. We show that a Bessel sequence is an inner summand of a frame and the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame. Next we develop this results to the g-frame situation.

Keywords: G-frames, dual frames, dual g-frames.

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1. Introduction

Let $\mathcal{H}$ denote a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a frame if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2 \qquad \forall f \in \mathcal{H}. \tag{1}$$

We call A and B the lower and upper frame bounds, respectively. The sequence $\{f_i\}_{i\in I}$ is a Bessel sequence if at least the upper bound in (1) is satisfied. For any frame $\{f_i\}_{i\in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i\in I}$ for which $f = \sum_{i\in I}\langle f, g_i\rangle f_i$ for all $f \in \mathcal{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence with bound $B < 1$, how can we find two

*Corresponding author.
E-mail addresses: msasgari@yahoo.com (M. S. Asgari), kaviangolsa@yahoo.com (G. Kavian).

sequences $\{g_i\}_{i\in I}$ and $\{p_i\}_{i\in I}$ such that $\{f_i+g_i\}_{i\in I}$ and $\{p_i\}_{i\in I}$ are dual frames, i.e., such that

$$f=\sum_{i\in I}\langle f,p_i\rangle(f_i+g_i)=\sum_{i\in I}\langle f,f_i+g_i\rangle p_i. \tag{2}$$

In this paper we aim at the more general results of the type in (2). For any Bessel sequence $\mathcal{F}=\{f_i\}_{i\in I}$ the synthesis operator is defined as follows:

$$T_{\mathcal{F}}:\ell^2(I)\to\mathcal{H},\qquad \text{with}\qquad T_{\mathcal{F}}(\{c_i\}_{i\in I})=\sum_{i\in I}c_if_i.$$

The analysis operator for $\mathcal{F}$ is $T_{\mathcal{F}}^*$ and is given by $T_{\mathcal{F}}^*f=\{\langle f,f_i\rangle\}_{i\in I}$. The frame operator is the positive self-adjoint invertible operator $S_{\mathcal{F}}=T_{\mathcal{F}}T_{\mathcal{F}}^*$ and satisfies $S_{\mathcal{F}}f=\sum_{i\in I}\langle f,f_i\rangle f_i$. The reconstruction formulas are as follows:

$$f=\sum_{i\in I}\langle f,f_i\rangle S_{\mathcal{F}}^{-1}f_i=\sum_{i\in I}\langle f,S_{\mathcal{F}}^{-1}f_i\rangle f_i\qquad\forall f\in\mathcal{H}. \tag{3}$$

Frames were first introduced in 1952 by Duffin and Schaeffer [4] in the study of nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossman and Meyer in [3]. G-frames for Hilbert spaces first formally were defined by Sun in [6].

Let $\mathcal{H},\mathcal{K}$ be two separable Hilbert spaces and $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{K}$, where I is a subset of $\mathbb{Z}$. Let $\mathcal{L}(\mathcal{H},W_i)$ be the collection of all bounded linear operators from $\mathcal{H}$ into W_i. Recall that a family of operators $\Lambda=\{\Lambda_i\in\mathcal{L}(\mathcal{H},W_i):\ \ i\in I\}$ is said to be a generalized frame, or simply a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ if there exist constants $0<C\leqslant D<\infty$ such that

$$C\|f\|^2\leqslant\sum_{i\in I}\|\Lambda_if\|^2\leqslant D\|f\|^2\qquad\forall f\in\mathcal{H}. \tag{4}$$

The constants C and D are called g-frame bounds and $\sup_{i\in I}\Lambda_i$ is called the multiplicity of the g-frame. We call Λ a tight g-frame if $C=D$ and it is a Parseval g-frame if $C=D=1$. If the right-hand side of (4) holds, then Λ is said a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. The representation space associated with a g-Bessel sequence $\Lambda=\{\Lambda_i\}_{i\in I}$ is defined by

$$\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}=\Big\{\{g_i\}_{i\in I}|g_i\in W_i\ \text{ and }\ \sum_{i\in I}\|g_i\|^2<\infty\Big\}. \tag{5}$$

The synthesis operator of Λ given by

$$T_\Lambda:\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}\to\mathcal{H}\qquad T_\Lambda(\{g_i\}_{i\in I})=\sum_{i\in I}\Lambda_i^*g_i.$$

The adjoint operator of T_Λ, which is called the analysis operator also obtain as follows

$$T_\Lambda^*:\mathcal{H}\to\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}\qquad T_\Lambda^*f=\{\Lambda_if\}_{i\in I}.$$

By composing T_Λ with its adjoint T_Λ^*, we obtain the g-frame operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H} \qquad S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f,$$

which is a bounded, self-adjoint, positive and invertible operator and $CI_{\mathcal{H}} \leqslant S_\Lambda \leqslant DI_{\mathcal{H}}$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widetilde{\Lambda}_i\}_{i\in I}$ where $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for $\mathcal{H}$ with g-frame bounds $\frac{1}{D}$ and $\frac{1}{C}$, respectively. The reconstruction formulas are also as follows:

$$f = \sum_{i\in I} \Lambda_i^* \widetilde{\Lambda}_i f = \sum_{i\in I} \widetilde{\Lambda}_i^* \Lambda_i f \qquad \forall f \in \mathcal{H}.$$

For more details about the theory and applications of frames we refer the readers to [1, 5], about g-frames to [5, 6].

For using of the reconstruction formulas we need to invert the g-frame operator, which can be complicated. In the following similar to frame algorithm we use a g-frame algorithm to obtain approximations of linear operator $U \in B(\mathcal{H}, \mathcal{K})$.

Theorem 1.1 Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame with g-frame bounds C, D. For every $U \in B(\mathcal{H}, \mathcal{K})$, we define the sequence $\{U_n\}_{n\in\mathbb{N}}$ by

$$U_n = \begin{cases} 0 & n = 0 \\ U_{n-1} + \frac{2}{C+D}(U - U_{n-1})S_\Lambda & n \geqslant 1 \end{cases}$$

Then we have $U = \lim_{n\to\infty} U_n$ with the error estimate

$$\|U - U_n\| \leqslant \Big(\frac{D-C}{D+C}\Big)^n \|U\|.$$

Proof. By the g-frame condition for each $f \in \mathcal{H}$ we have

$$-\frac{D-C}{D+C}\|f\|^2 \leqslant \langle (I_{\mathcal{H}} - \frac{2}{C+D} S_\Lambda) f, f\rangle \leqslant \frac{D-C}{D+C}\|f\|^2.$$

Thus

$$\big\| Id_{\mathcal{H}} - \frac{2}{C+D} S_\Lambda \big\| \leqslant \frac{D-C}{D+C}.$$

Using the definition of $\{U_n\}_{n\in\mathbb{N}}$ we obtain

$$U - U_n = (U - U_0)\big(Id_{\mathcal{H}} - \frac{2}{C+D} S_\Lambda\big)^n.$$

Hence,

$$\|U - U_n\| \leqslant \Big(\frac{D-C}{D+C}\Big)^n \|U\|.$$

■

A family of operators $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : \ i \in I\}$ is called an orthonormal g-basis for $\mathcal{H}$ if it holds in the following conditions.

(1) $f = \sum_{i\in I} \Lambda_i^* \Lambda_i f \qquad \forall f \in \mathcal{H}.$
(2) $\langle \Lambda_i^* g, \Lambda_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle \qquad \forall g, g' \in W_i\ ,\ \forall i, j \in I.$

Lemma 1.2 Let $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : \ i \in I\}$ be a collection of partial isometries. Then the sequence $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$, if and only if the sequence $\{\Lambda_i^* \Lambda_i\}_{i\in I}$ be an orthonormal g-basis for $\mathcal{H}$.

Proof. First note that Λ_i is a partial isometry, if and only if $\Lambda_i^* \Lambda_i$ is an orthogonal projection on $\mathcal{H}$. Now the claim follows from

$$\|f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2 = \sum_{i\in I} \langle \Lambda_i^* \Lambda_i f, f \rangle = \sum_{i\in I} \|\Lambda_i^* \Lambda_i f\|^2 \quad \forall f \in \mathcal{H}.$$

■

2. Duality of Bessel and *g*-Bessel sequences

Li and Sun in [5] expanded every Bessel sequence to a tight frame by adding some elements. In this section we show that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for $\mathcal{H}$ and we prove that a Bessel sequence is an inner summand of a frame.

Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be two Bessel sequences for $\mathcal{H}$ with synthesis operators $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ respectively. Then we say that $\mathcal{F}$ and $\mathcal{G}$ are dual frames for $\mathcal{H}$ if $T_{\mathcal{F}} T_{\mathcal{G}}^* = I_{\mathcal{H}}$ or $T_{\mathcal{G}} T_{\mathcal{F}}^* = I_{\mathcal{H}}$, i.e.,

$$f = \sum_{i\in I} \langle f, g_i \rangle f_i = \sum_{i\in I} \langle f, f_i \rangle g_i \qquad \forall f \in \mathcal{H}.$$

Notation 2.1 For every countable(or finite) index set I, we define the space $\ell(I, \mathcal{H})$ by

$$\ell(I, \mathcal{H}) = \Big\{ \{f_i\}_{i\in I} : \ \ f_i \in \mathcal{H}, \text{ and } \sup_{i\in I} \|f_i\| < \infty \Big\}.$$

It is easy to check that $\ell(I, \mathcal{H})$ with the pointwise operations and norm defined by

$$\|\{f_i\}_{i\in I}\| = \sup_{i\in I} \|f_i\|,$$

is a Banach space. Let $\mathcal{B}(I, \mathcal{H})$ be the set of all Bessel sequences, $\mathcal{F}(I, \mathcal{H})$ be the collection of all frames and $\mathcal{P}(I, \mathcal{H})$ denote the set of all Parseval frames indexed by I for $\mathcal{H}$ respectively. Then $\mathcal{B}(I, \mathcal{H})$ is a subspace of $\ell(I, \mathcal{H})$ and $\mathcal{P}(I, \mathcal{H}) \subset \mathcal{F}(I, \mathcal{H}) \subset \mathcal{B}(I, \mathcal{H}) \subset \ell(I, \mathcal{H})$.

The following theorem shows that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for $\mathcal{H}$.

Theorem 2.2 Let $\mathcal{F} = \{f_i\}_{i\in I} \in \mathcal{B}(I, \mathcal{H})$ with Bessel bound $B < 1$. Then

$$\mathcal{F} + \mathcal{P}(I, \mathcal{H}) \subset \mathcal{F}(I, \mathcal{H}).$$

Proof. Let $\mathcal{G} = \{g_i\}_{i\in I} \in \mathcal{P}(I, \mathcal{H})$. Then for all $f \in \mathcal{H}$ we have

$$\|T_{\mathcal{F}}T_{\mathcal{G}}^* f\|^2 = \sup_{\|g\|=1} \left|\langle T_{\mathcal{F}}T_{\mathcal{G}}^* f, g\rangle\right|^2 = \sup_{\|g\|=1} \left|\sum_{i\in I}\langle f, g_i\rangle\langle f_i, g\rangle\right|^2$$
$$\leqslant \sup_{\|g\|=1} \sum_{i\in I} |\langle f, g_i\rangle|^2 \sum_{i\in I} |\langle g, f_i\rangle|^2 \leqslant B\|f\|^2.$$

Thus $\|T_{\mathcal{F}}T_{\mathcal{G}}^*\| \leqslant \sqrt{B} < 1$, and so $I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{G}}^*$ an invertible operator in $\mathcal{L}(\mathcal{H})$. If we set $\Theta = (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{G}}^*)^{-1}$ and $\mathcal{U} = \mathcal{F} + \mathcal{G}$ with $\mathcal{U} = \{u_i\}_{i\in I}$. Then we compute

$$\begin{aligned} f &= (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{G}}^*)\Theta f \\ &= \sum_{i\in I}\langle \Theta f, g_i\rangle f_i + \sum_{i\in I}\langle \Theta f, g_i\rangle g_i \\ &= \sum_{i\in I}\langle f, \Theta^* g_i\rangle u_i, \end{aligned}$$

for all $f \in \mathcal{H}$. This shows that $\mathcal{U} \in \mathcal{F}(I, \mathcal{H})$ with frame bounds $\|\Theta\|^{-2}$ and $(1 + \sqrt{B})^2$. ■

The next result shows that a Bessel sequence is an inner summand of a frame.

Corollary 2.3 Let $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$ be a Bessel sequence. Then there exists a tight frame $\mathcal{G} \in \mathcal{F}(I, \mathcal{H})$ such that $\mathcal{F} + \mathcal{G} \in \mathcal{F}(I, \mathcal{H})$.

Proof. Let B be the Bessel bound for $\mathcal{F} = \{f_i\}_{i\in I}$, then $\{\frac{1}{\sqrt{2B}} f_i\}_{i\in I}$ is a Bessel sequence with the Bessel bound less than one. By Theorem 2.2 $\{\frac{1}{\sqrt{2B}} f_i + e_i\}_{i\in I}$ is a frame for $\mathcal{H}$, where $\{e_i\}_{i\in I}$ is an arbitrary Parseval frame. Define $g_i = \sqrt{2B} e_i$ for all $i \in I$, then $\mathcal{G} = \{g_i\}_{i\in I}$ is a tight frame and $\mathcal{F} + \mathcal{G} = \left\{\sqrt{2B}(\frac{1}{\sqrt{2B}} f_i + e_i)\right\}_{i\in I}$ is also a frame for $\mathcal{H}$.■

The next theorem changes every Bessel sequence to a dual frame by summing it with any Parseval frame.

Theorem 2.4 Let $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$ with Bessel bound $B < 1$ and let $\mathcal{E} \in \mathcal{P}(I, \mathcal{H})$. Then there exists a $\mathcal{G} \in \mathcal{B}(I, \mathcal{H})$ such that $\mathcal{F} + \mathcal{E}$ and $\mathcal{G} + \mathcal{E}$ are dual frames.

Proof. Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_i\}_{i\in I}$. Since $B < 1$, hence $I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*$ is an invertible operator in $\mathcal{L}(\mathcal{H})$. If we define $\Theta = -(I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)^{-1}T_{\mathcal{F}}T_{\mathcal{E}}^*$ and $g_i = \Theta^* e_i$ for all $i \in I$. Then $\mathcal{G} = \{g_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$ and for all $f \in \mathcal{H}$ we have

$$\begin{aligned} f &= (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^* f + T_{\mathcal{F}}T_{\mathcal{E}}^* f \\ &= T_{\mathcal{E}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^* f + T_{\mathcal{F}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{F}}T_{\mathcal{E}}^* f \\ &= \sum_{i\in I}\langle \Theta f, e_i\rangle e_i + \sum_{i\in I}\langle f, e_i\rangle e_i + \sum_{i\in I}\langle \Theta f, e_i\rangle f_i + \sum_{i\in I}\langle f, e_i\rangle f_i \\ &= \sum_{i\in I}\langle f, g_i + e_i\rangle(f_i + e_i), \end{aligned}$$

which this finishes the proof. ■

Corollary 2.5 Let $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$ with Bessel bound $B < 1$ and let $\mathcal{E} \in \mathcal{P}(I, \mathcal{H})$. Then there exists a $\mathcal{G} \in \mathcal{B}(I, \mathcal{H})$ such that $\mathcal{F} + \mathcal{E}$ and $\mathcal{G}$ are dual frames.

Proof. Suppose that $\mathcal{F}=\{f_i\}_{i\in I}$ and $\mathcal{E}=\{e_i\}_{i\in I}$. Since $I_{\mathcal{H}}+T_{\mathcal{F}}T_{\mathcal{E}}^*$ is invertible on $\mathcal{H}$. Thus if we set $\Theta=(I_{\mathcal{H}}+T_{\mathcal{F}}T_{\mathcal{E}}^*)^{-1}$ and $g_i=\Theta^*e_i$ for all $i\in I$. Then for all $f\in\mathcal{H}$ we have

$$\begin{aligned} f &= (I_{\mathcal{H}}+T_{\mathcal{F}}T_{\mathcal{E}}^*)\Theta f = T_{\mathcal{E}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{F}}T_{\mathcal{E}}^*\Theta f \\ &= \sum_{i\in I}\langle \Theta f, e_i\rangle e_i + \sum_{i\in I}\langle \Theta f, e_i\rangle f_i = \sum_{i\in I}\langle f, g_i\rangle (f_i+e_i). \end{aligned}$$

From this completes the proof. ■

Corollary 2.6 For every $\mathcal{F}\in\mathcal{B}(I,\mathcal{H})$ there exist $\mathcal{G}\in\mathcal{B}(I,\mathcal{H})$ and a tight frame $\mathcal{U}\in\mathcal{F}(I,\mathcal{H})$ such that $\mathcal{F}+\mathcal{U}$ and $\mathcal{G}$ are dual frames for $\mathcal{H}$.

Proof. Let B be the Bessel bound of $\mathcal{F}=\{f_i\}_{i\in I}$ and let $\{e_i\}_{i\in I}$ denote any Parseval frame for $\mathcal{H}$. By Theorem 2.4 there exists a Bessel sequence $\{v_i\}_{i\in I}$ for $\mathcal{H}$ such that $\{\frac{1}{\sqrt{2B}}f_i+e_i\}_{i\in I}$ and $\{v_i+e_i\}_{i\in I}$ are dual frames for $\mathcal{H}$. Put $\mathcal{G}=\{g_i\}_{i\in I}$,$\mathcal{U}=\{u_i\}_{i\in I}$ with $g_i=\frac{1}{\sqrt{2B}}v_i+\frac{1}{\sqrt{2B}}e_i$ and $u_i=\sqrt{2B}e_i$ for all $i\in I$. Then for all $f\in\mathcal{H}$ we have

$$\begin{aligned} \sum_{i\in I}\langle f, g_i\rangle (f_i+u_i) &= \sum_{i\in I}\langle f, \frac{1}{\sqrt{2B}}v_i+\frac{1}{\sqrt{2B}}e_i\rangle (f_i+\sqrt{2B}e_i) \\ &= \sum_{i\in I}\langle f, v_i+e_i\rangle (\frac{1}{\sqrt{2B}}f_i+e_i) = f. \end{aligned}$$

From this the claim follows immediately. ■

In the following theorem we show that every Bessel sequence can be expanded to a dual frame by adding it to a Parseval frame. Another form of this result can be found in [5] Corollary 3.2.

Theorem 2.7 Let $\mathcal{F}=\{f_i\}_{i\in I}$ be a Bessel sequence with Bessel bound B and $\mathcal{E}=\{e_i\}_{i\in I}$ be a Parseval frame for $\mathcal{H}$. Then for all $\alpha>B$, there exists a Bessel sequence $\{g_i\}_{i\in I}$ for $\mathcal{H}$ such that $\{f_i,e_i\}_{i\in I}$ and $\{\frac{1}{\alpha}f_i,g_i\}_{i\in I}$ are dual frames for $\mathcal{H}$.

Proof. Since $\alpha>B$, hence $\Theta=I_{\mathcal{H}}-\frac{1}{\alpha}T_{\mathcal{F}}T_{\mathcal{F}}^*$ is a linear bounded and positive operator on $\mathcal{H}$. Thus if we define $g_i=\Theta^*e_i$ for all $i\in I$. Then $\{g_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$ and for all $f\in\mathcal{H}$ we have

$$\begin{aligned} \sum_{i\in I}\langle f, \frac{1}{\alpha}f_i\rangle f_i + \sum_{i\in I}\langle f, g_i\rangle e_i &= \sum_{i\in I}\langle f, \frac{1}{\alpha}f_i\rangle f_i + \sum_{i\in I}\langle \Theta f, e_i\rangle e_i \\ &= \frac{1}{\alpha}T_{\mathcal{F}}T_{\mathcal{F}}^*f+\Theta f = f \end{aligned}$$

which this finishes the proof. ■

Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Gamma=\{\Gamma_i\}_{i\in I}$ be g-Bessel sequences for $\mathcal{H}$ with synthesis operators T_Λ and T_Λ respectively. Then we say that Λ and Γ are dual g-frames for $\mathcal{H}$ if $T_\Lambda T_\Gamma^*=I_{\mathcal{H}}$ or $T_\Gamma T_\Lambda^*=I_{\mathcal{H}}$. In the following we show that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [2] to the situation of g-frames.

Theorem 2.8 Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be two g-Bessel sequences for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then there exist g-Bessel sequences $\{\Xi_j\}_{j\in J}$ and $\{\Omega_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{V_j\}_{j\in J}$, such that $\{\Lambda_i\}_{i\in I} \cup \{\Xi_j\}_{j\in J}$ and $\{\Gamma_i\}_{i\in I} \cup \{\Omega_j\}_{j\in J}$ form a pair of dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I} \cup \{V_j\}_{j\in J}$.

Proof. Assume that $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are any pair of dual g-frames for $\mathcal{H}$ respect to $\{V_j\}_{j\in J}$ and let $\Theta = I_{\mathcal{H}} - T_\Gamma T_\Lambda^*$. Then for any $f \in \mathcal{H}$ we have

$$f = \Theta f + T_\Gamma T_\Lambda^* f = \sum_{j\in J} \Psi_j^* \Phi_j \Theta f + \sum_{i\in I} \Gamma_i^* \Lambda_i f.$$

If we set $\Xi_j = \Phi_j \Theta$ and $\Omega_j = \Psi_j$ for all $j \in J$. Then $\{\Lambda_i\}_{i\in I} \cup \{\Xi_j\}_{j\in J}$ and $\{\Gamma_i\}_{i\in I} \cup \{\Omega_j\}_{j\in J}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I} \cup \{V_j\}_{j\in J}$. ■

The following corollaries are generalizations of the above results to the g-frames situation. We leave the proofs to interested readers.

Corollary 2.9 Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-Bessel bound $B < 1$. Then there exists g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, such that $\{\Xi_i + \Lambda_i\}_{i\in I}$ and $\{\Xi_i + \Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, where $\{\Xi_i\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

Corollary 2.10 For every g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ with Bessel bound $B < 1$ and each Parseval g-frame $\Xi = \{\Xi_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, there exists g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ such that $\{\Lambda_i + \Xi_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

Corollary 2.11 For every g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ there exist g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ and a tight g-frame $\{\Xi_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ such that $\{\Lambda_i + \Xi_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

3. Conclusions

In this paper, using of g-frame algorithm we obtain an approximation of a bounded linear operator $U(\mathcal{H}, \mathcal{K})$. We also show that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for $\mathcal{H}$ and we prove that a Bessel sequence is an inner summand of a frame. The important result of this paper changing every Bessel sequence to a dual frame by summing it with any Parseval frame.

References

[1] O. Christensen, An Introduction to Frames and Riesz Bases. *Birkhauser, Boston* (2003)

[2] O. Christensions, H. O. Kim, R. Y. Kim, Extensions of Bessel sequences to dual pairs of frames. *Appl. Comput. Harmon. Anal.* **34**, (2013), 224-233.

[3] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions. *J. Math. Phys.* **27**, (1986), 1271-1283.

[4] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.* **72**, **(2)**, (1952), 341-366.

[5] D. F. Li, W. Sun, Expansion of frames to tight frames. *Acta Math. Sin. (Engl. Ser.)* **25** (2), (2009), 287292.

[6] W. Sun, G-frames and G-Riesz bases. *J. Math. Anal. Appl.* **322**, (2006), 437-452.