Lie higher derivations on $B(X)$

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Abstract. Let $X$ be a Banach space of $\dim X > 2$ and $B(X)$ be the space of bounded linear operators on $X$. If $L : B(X) \to B(X)$ be a Lie higher derivation on $B(X)$, then there exists an additive higher derivation $D$ and a linear map $\tau : B(X) \to \mathbb{F}$ vanishing at commutators $[A,B]$ for all $A, B \in B(X)$ such that $L = D + \tau$.

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1. Introduction

Let $X$ be a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real number field $\mathbb{R}$ or the complex field $\mathbb{C}$. Recall that an additive map $L : B(X) \to B(X)$ is called a derivation if

$$L(AB) = L(A)B + AL(B) \quad \forall A, B \in B(X).$$

More generally, an additive map $L : B(X) \to B(X)$ is called a Lie derivation if $L([A,B]) = [L(A), B] + [A, L(B)]$ for all $A, B \in B(X)$. In recent years, Lie derivations has attracted the attentions of many researchers (see [1, 2, 8], and references therein). On the other hand, higher derivations were introduced and studied mainly in commutative rings and later, also in non-commutative rings and some operator algebras (see, for example, [3, 11] and the references therein). We first recall the concepts of higher derivations and Lie higher derivations.

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Definition 1.1 Let $L = (L_i)_{i \in \mathbb{N}}$ ($\mathbb{N}$ denotes the set of natural numbers including 0) be a sequence of additive maps of a ring $A$ such that $L_0 = \text{id}_A$. $L$ is said to be a higher derivation if for every $n \in \mathbb{N}$ we have $L_n(AB) = \sum_{i+j=n} L_i(A)L_j(B)$ for all $A, B \in A$; a Lie higher derivation if for every $n \in \mathbb{N}$ we have $L_n([A,B]) = \sum_{i+j=n}[L_i(A), L_j(B)]$ for all $A, B \in A$.

It is clear that all higher derivations are Lie higher derivations. However, the converse is not true in general. Assume that $D = (D_n)_{n \in \mathbb{N}}$ is a higher derivation on a ring $A$. For any $n \in \mathbb{N}$, let $L_n = D_n + h_n$, where $h_n$ is an additive map from $A$ into its center vanishing on every commutator. It is easily seen that $(L_n)_{n \in \mathbb{N}}$ is a Lie higher derivation, but not a higher derivation if $h_n \neq 0$ for some $n$. Then a natural question is to ask whether or not every Lie higher derivations have the above form?

In [10], Fei and Chen discussed the properties of Lie higher derivations on nest algebras. Generalized higher derivations, jordan higher derivations and higher derivations were studied by many authors (see [6, 7, 9, 11, 12]). The purpose of this paper is to show that every Lie higher derivations on $B(X)$ is proper.

2. Lie higher derivations on $B(X)$

In this section, we give a characterization of Lie higher derivations of $B(X)$. The following is our main result which generalizes the main result in [4].

Theorem 2.1 Let $X$ be a Banach space of dim$X > 2$ and $L : B(X) \to B(X)$ be a Lie higher derivation on $B(X)$. Then there exists an additive higher derivation $D$ and a linear map $\tau : B(X) \to \mathbb{F}I$ vanishing at commutators $[A,B]$ for all $A, B \in B(X)$ such that $L = D + \tau$.

Proof. In the following, we always assume that $L = (L_n)_{n \in \mathbb{N}}$ is an Lie higher derivation of $B(X)$. We proceed by induction on $n \in \mathbb{N}$.

If $n = 1$, then by the definition of Lie higher derivations, $L_1$ is a Lie derivation of $B(X)$. So by ([4, Theorem 1.1]), there exists an additive derivation $D$ and a linear map $\tau : B(X) \to \mathbb{F}I$ vanishing at commutators $[A,B]$ for all $A, B \in B(X)$ such that $L = D + \tau$.

For the convenience, in the sequel, take $x_0 \in X, f_0 \in X^*$ satisfying $f_0(x_0) = 1$. Let $P = x_0 \otimes f_0$ and $Q = I - P$ be idempotent of $B(X)$, it is obvious that $PQ = QP = 0$. Then $B(X) = B_{11} + B_{12} + B_{21} + B_{22}$, where $B_{11} = P B(X) P, B_{12} = P B(X) Q, B_{21} = Q B(X) P, B_{22} = Q B(X) Q$.

Thus by ([4, Lemmas 2.2-2.12]) we have;

$$
\begin{align*}
PL_1(P)P + QL_1(P)Q & \in \mathbb{F}I; \\
PL_1(Q)P + QL_1(Q)Q & \in \mathbb{F}I; \\
\Delta_1(PAQ + QAP) = P\Delta_1(A)Q + Q\Delta_1(A)P & \quad \text{where} \\
\Delta_1(A) = L_1(A) - (AT - TA) & \quad \text{and} \quad T = PL_1(P)Q - QL_1(P)P; \\
\Delta_1(P) & \in \mathbb{F}I, \quad \Delta_1(Q) \in \mathbb{F}I; \\
\Delta_1(B_{ij}) & \subseteq B_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq 2; \\
\Delta_1(X_{ii}) & \in B_{ii} + \mathbb{F}I.
\end{align*}
$$

Assume that $L = (L_n)$ is a Lie higher derivation of $B(X)$. We proceed by induction on $n \in \mathbb{N}$. When $n = 1$, the conclusion is true by discussion. We now assume that $L_m(x) = D_m(x) + \tau_m(x)$ holds for all $x \in B(X)$ and for all $m < n \in \mathbb{N}$, where $\tau_m :$
\[ B(X) \to Z(B(X)) \text{ is such that } \tau_n([x, y]) = 0 \text{ for all } x, y \in B(X) \text{ and } D_m(xy) = \sum_{i+j=m} D_i(x) D_j(y) \text{ for all } x, y \in B(X). \]

Moreover, we have the following properties:

\[
P_m : \begin{cases} 
PL_m(P)P + QL_m(P)Q \in \mathbb{F}I; \\
PL_m(Q)P + QL_m(Q)Q \in \mathbb{F}I; \\
\Delta_m(PAQ + QAP) = P\Delta_m(A)Q + Q\Delta_m(A)P \quad \text{where} \\
\Delta_m(A) = L_m(A) - (AT - TA) \quad \text{and} \quad T = PL_m(P)Q - QL_m(P)P; \\
\Delta_m(P) \in \mathbb{F}I, \quad \Delta_m(Q) \in \mathbb{F}I; \\
\Delta_m(B_{ij}) \subseteq B_{ij}\text{ for } 1 \leq i \neq j \leq 2; \\
\Delta_m(X_{ii}) \in B_{ii} + \mathbb{F}I.
\end{cases}
\]

Our aim is to show that \( L_n \) also satisfies the similar properties and that \( L_n(x) = D_n(x) + \tau_n(x) \) holds for all \( x \in B(X) \) and \( \tau_n \) is linear map from \( B(X) \) into its center satisfying \( \tau_n([x, y]) = 0 \) for all \( x, y \in B(X) \). Therefore, by induction, the theorem is true.

We will prove it by several claims.

**Claim 1.** \( PL_n(P)P + QL_n(P)Q \in \mathbb{F}I \) and \( PL_n(Q)P + QL_n(Q)Q \in \mathbb{F}I \).

**proof.** Let \( x \in X, f \in X^* \). Then

\[
L_n(Px \otimes Q^* f) = L_n([P, Px \otimes Q^* f])
\]

\[
= \sum_{i+j=n} [L_i(P), L_j(Px \otimes Q^* f)]
\]

\[
= [L_n(P), Px \otimes Q^* f] + [P, L_n(Px \otimes Q^* f)] + \sum_{i+j=n; i \neq 0, n} [L_i(P), L_j(Px \otimes Q^* f)]
\]

\[
= L_n(P)Px \otimes Q^* f - Px \otimes Q^* f L_n(P) + PL_n(Px \otimes Q^* f)
\]

\[
- L_n(Px \otimes Q^* f)P + \sum_{i+j=n; i \neq 0, n} [L_i(P), L_j(Px \otimes Q^* f)]
\]

Multiplying this equation by \( P \) from the left and by \( Q \) from the right, we get, for all \( x \in X \) and \( f \in X^* \), that

\[
PL_n(P)(x \otimes f)Q = P(x \otimes f)QL_n(P)Q - \sum_{i+j=n; i \neq 0, n} [L_i(P), L_j(Px \otimes Q^* f)]Q.
\]

By \( P_m \), \( PL_m(P)P + QL_m(P)Q \in \mathbb{F}I \) and \( PL_m(Q)P + QL_m(Q)Q \in \mathbb{F}I \).

So \( P \sum_{i+j=n; i \neq 0, n} [L_i(P), L_j(Px \otimes Q^* f)]Q = 0 \).

Thus \( PL_n(P)P = \mu P \) for some \( \mu \in \mathbb{F} \). Hence \( QL_n(P)Q = \mu Q \), which implies that \( PL_n(P)P + QL_n(P)Q = \mu I \). Similarly we can prove \( PL_n(Q)P + QL_n(Q)Q \in \mathbb{F}I \).

Now we put \( T = PL_n(P)Q - QL_n(P)P \). For \( A \in B(X) \), define \( \Delta_n(A) = L_n(A) - (AT - TA) \). Also we can easily checked that \( \Delta_n[A, B] = [\Delta_n(A), B] + [A, \Delta_n(B)] \) for all \( A, B \in B(X) \).

**Claim 2.** \( \Delta_n(P) \in \mathbb{F}I \).
proof. Using Claim 1, we have

$$\Delta_n(P) = L_n(P) - (PT - TP) = PL_n(P)P + QL_n(P)Q \in \mathbb{F}I.$$  

Claim 3. $\Delta_n(PAQ + QAP) = P\Delta_n(A)Q + Q\Delta_n(A)P$ for all $A \in B(X)$.

proof. Let $A \in B(X)$, then

$$\Delta_n(PAQ + QAP) = \Delta_n([P, [P, A]])$$

$$= \sum_{i+j=n} [\Delta_i(P), \Delta_j([P, A])]$$

$$= [\Delta_n(P), [P, A]] + [p, \Delta_n([P, A])] + \sum_{i+j=n, i\neq 0, n} [\Delta_i(P), \Delta_j([P, A])]$$

$$= [P, [P, \Delta_n(A)]] + \sum_{i+j=n, i\neq 0, n} [\Delta_i(P), \Delta_j([P, A])]$$

Since by Claim 2, $\Delta_i(P) \in \mathbb{F}I$, for $i < n$, so $\sum_{i+j=n, i\neq 0, n} [\Delta_i(P), \Delta_j([P, A])] = 0$.

So

$$\Delta_n(PAQ + QAP) = [P, [P, \Delta_n(A)]] = P\Delta_n(A)Q + Q\Delta_n(A)P.$$

Claim 4. $\Delta_n(Q) \in \mathbb{F}I$.

proof. Applying Claim 1, we have

$$P\Delta_n(Q)Q + Q\Delta_n(Q)P = \Delta_n(PQQ + QQP) = 0.$$

Thus $\Delta_n(Q) = PL_n(Q)Q + QL_n(Q)Q \in \mathbb{F}$.

Claim 5. $\Delta_n(B_{ij}) \subseteq B_{ij}$ for $1 \leq i \neq j \leq 2$.

proof. For any $X \in B_{12}$, we have

$$\Delta_n(X) = \Delta_n([P, X])$$

$$= \sum_{i+j=n} [\Delta_i(P), \Delta_j(X)]$$

$$= [P, \Delta_n(X)] + \sum_{i+j=n, i\neq 0, n} [\Delta_i(P), \Delta_j(X)]$$

$$= [P, \Delta_n(X)] + \sum_{j, j\neq 0, n} [P, \Delta_j(X)]$$

Since $\Delta_i(P) \in \mathbb{F}I$ is already shown, $\sum_{i+j=n, i\neq 0, n} [\Delta_i(P), \Delta_j(X)] = 0$ immediately follows.
So

\[ \Delta_n(X) = P \Delta_n(x)Q - Q \Delta_n(X)P \]

Multiplying above equation by \( P \) from the right, we get

\[ P \Delta_n(X)P = Q \Delta_n(X)P = Q \Delta_n(X)Q = 0 \]

Thus

\[ \Delta_n(X) = P \Delta_n(X)P \in B_{12} \]

Similarly we can prove that \( \Delta_n(y) \subseteq B_{21} \) for any \( y \in B_{21} \).

**Claim 6.** There is a functional \( f_{ni} : B_{ii} \to \mathbb{F} \) such that \( \Delta_n(X_{ii}) - f_{ni}(X_{ii})I \in B_{ii} \) for all \( X_{ii} \in B_{ii} \), \( 1 \leq i \leq 2 \).

**proof.** Let \( X_{ii} \in B_{ii} \), Applying Claim 3, we obtain

\[ 0 = \Delta_n(0) = \Delta_n(PX_{ii}Q + QX_{ii}P) = P \Delta_n(X_{ii})Q + Q \Delta_n(X_{ii})P \]

Then we can assume that \( \Delta_n(X_{11}) = a_{11} + a_{22} \) and \( \Delta_n(X_{22}) = b_{11} + b_{22} \), where \( a_{11}, b_{11} \in B_{11}, a_{22}, b_{22} \in B_{22} \). Also

\[ 0 = \Delta_n([X_{11}, X_{22}]) \]

\[ = \sum_{i+j=n} [\Delta_i(X_{11}), \Delta_j(X_{22})] \]

\[ = [a_{22}, X_{22}] + [X_{11}, a_{11}] + \sum_{i+j=n, i \neq 0, n} [\Delta_i(X_{11}), \Delta_j(X_{22})]. \]

By the property \( \mathbb{P}_m \): \( \Delta_i(X_{11}) \in B_{11} + \mathbb{F} I \) and \( \Delta_j(X_{22}) \in B_{22} + \mathbb{F} I \), one gets \( \sum_{i+j=n, i \neq 0, n} [\Delta_i(X_{11}), \Delta_j(X_{22})] = 0 \) which implies that \( [a_{22}, X_{22}] = 0 \) for all \( X_{22} \in B_{22} \) and \( [X_{11}, a_{11}] = 0 \) for all \( X_{11} \in B_{11} \).

Therefore, there exist scalars \( f_{n1}(X_{11}) \) and \( f_{n2}(X_{22}) \) such that \( a_{22} = f_{n1}(X_{11})Q \) and \( b_{11} = f_{n2}(X_{22})P \). So \( \Delta_n(X_{11}) - f_{n1}(X_{11})I \in B_{11} \) and \( \Delta_n(X_{22}) - f_{n2}(X_{22})I \in B_{22} \).

**Claim 7.** \( \Delta_n \) is additive on \( B_{12} \) and \( B_{21} \).

**proof.** Let \( X_{22} \in B_{22} \), \( X_{12} \in B_{12} \) and \( Y_{21} \in B_{21} \), by applying Claim 5, 6 we
have

$$\Delta_n[X_{22} + X_{21}, Y_{21}] = \sum_{i+j=n} [\Delta_i(X_{22} + X_{21}), \Delta_j(Y_{21})]$$

$$= [\Delta_n(X_{22} + X_{21}), Y_{21}] + [X_{22} + X_{21}, \Delta_n(Y_{21})]$$

$$+ \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22} + X_{21}), \Delta_j(Y_{21})]$$

$$= [\Delta_n(X_{22} + X_{21}), Y_{21}] + [X_{22}, \Delta_n(Y_{21})]$$

$$+ \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22} + X_{21}), \Delta_j(Y_{21})]$$

$$\Delta_n[X_{22} + X_{21}, Y_{21}] = \Delta_n[X_{22}, Y_{21}]$$

$$= \sum_{i+j=n} [\Delta_i(X_{22}), \Delta_j(Y_{21})]$$

$$= [\Delta_n(X_{22}) + \Delta_n(X_{21}), Y_{21}] + [X_{22}, \Delta_n(Y_{21})]$$

$$+ \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22}) + \Delta_i(X_{21}), \Delta_j(Y_{21})]$$

which implies that

$$[\Delta_n(X_{22} + X_{21}) - \Delta_n(X_{22}) - \Delta_n(X_{21}), Y_{21}] + \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22} + X_{21}), \Delta_j(Y_{21})]$$

$$- \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22}), \Delta_j(Y_{21})] = 0$$

But by $P_m$, we have

$$\sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22} + X_{21}), \Delta_j(Y_{21})] = \sum_{i+j=n;i\neq 0,n} [\Delta_i(X_{22}) + \Delta_i(X_{22}), \Delta_j(Y_{21})]$$

Thus we conclude that $[\Delta_n(X_{22} + X_{21}) - \Delta_n(X_{22}) - \Delta_n(X_{21}), Y_{21}] = 0$. By a similar way to we can prove that $\Delta_n$ is additive on $B_{21}$ and similarly additivity of $\Delta_n$ on $B_{12}$ can be deduced easily.

Now we define $D_n(A) = \Delta_n(PAQ) + \Delta_n(QAP) + \Delta_n(PAP) + \Delta_n(QAQ) - (f_1(PAP) - f_2(QAQ))I$. Then by Claim 5, 6 we have

**Claim 8.** For $B_{ij} \in B_{ij}$, $1 \leq i, j \leq 2$ we have

1. $D_n(B_{ij}) \in B_{ij}$, $1 \leq i, j \leq 2$;
2. $D_n(B_{12}) = \Delta_n(B_{12})$ and $D_n(B_{21}) = \Delta_n(B_{21})$;
3. $D_n(B_{11} + B_{12} + B_{21} + B_{22}) = D_n(B_{11}) + D_n(B_{12}) + D_n(B_{21}) + D_n(B_{22})$.

The following claims immediately follows from Claim 7 and Claim 8.

**Claim 9.** $D_n$ is additive on $B_{12}$ and $B_{21}$.
Claim 10. $D_n$ is additive on $B_{11}$ and $B_{22}$.
It follows similarly to proof of Lemma 2.12 in [4].

Claim 11. $D_n$ is additive.
It follows similarly to proof of Lemma 2.13 in [4].

Claim 12. $D_n$ has the following properties:

1. $D_n(A_{ii}B_{ij}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij})$  \quad 1 \leq i \neq j \leq 2
2. $D_n(B_{ij}A_{jj}) = \sum_{t+k=n} D_t(B_{ij})D_k(A_{jj})$  \quad 1 \leq i \neq j \leq 2
3. $D_n(A_{ii}B_{ii}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ii})$

proof. Let $A_{ii} \in B_{ii}$ and $B_{ij} \in B_{ij}$, then

$$D_n(A_{ii}B_{ij}) = \Delta_n(A_{ii}B_{ij})$$
$$= \sum_{t+k=n} \left[ D_t(A_{ii}), D_k(B_{ij}) \right]$$
$$= \sum_{t+k=n} \left[ D_t(A_{ii}) + \tau_t(A_{ii}), D_k(B_{ij}) + \tau_t(B_{ij}) \right]$$
$$= \sum_{t+k=n} \left[ D_t(A_{ii}), D_k(B_{ij}) \right]$$

Since $D_t(A_{ii}) \in B_{ii} + FI$ and $D_k(B_{ij}) \in B_{ij}$, then

$$D_n(A_{ii}B_{ij}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij}).$$

Also

$$D_n(B_{ij}A_{jj}) = \Delta_n(B_{ij}A_{jj})$$
$$= \sum_{t+k=n} \left[ D_t(B_{ij}), D_k(A_{jj}) \right]$$
$$= \sum_{t+k=n} \left[ D_t(B_{ij}) + \tau_t(A_{jj}), D_k(B_{ij}) + \tau_t(A_{jj}) \right]$$
$$= \sum_{t+k=n} \left[ D_t(B_{ij}), D_k(A_{jj}) \right]$$

Since $D_k(A_{jj}) \in B_{jj} + FI$ and $D_t(B_{ij}) \in B_{ij}$, then

$$D_n(B_{ij}A_{jj}) = \sum_{t+k=n} D_t(B_{ij})D_k(A_{jj}).$$
Furthermore

\[ D_n(A_{ii}B_{ii}C_{ij}) = \sum_{t+k=n} D_t(A_{ii}B_{ii})D_k(C_{ij}) \]

\[ = \sum_{t+k=n,k\neq 0} D_t(A_{ii}B_{ii})D_k(C_{ij}) + D_n(A_{ii}B_{ii})C_{ij} \]

But

\[ D_n(A_{ii}B_{ii}C_{ij}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ii}C_{ij}) \]

\[ = \sum_{t+k+l=n} D_t(A_{ii})D_k(B_{ij})D_l(C_{ij}) \]

\[ = \sum_{t+k+l=n, L\neq 0} D_t(A_{ii})D_k(B_{ij})D_l(C_{ij}) + \sum_{t+k=n, l\neq 0} D_t(A_{ii})D_k(B_{ij})C_{ij} \]

Thus

\[ D_n(A_{ii}B_{ii})C_{ij} = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij})C_{ij} \]

Which implies that

\[ D_n(A_{ii}B_{ii}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij}). \]

**Claim 14.** \( \Delta_n(B_{11} + C_{22}) - \Delta_n(B_{11}) - \Delta_n(C_{22}) \in \mathbb{F}I. \)

**Claim 15.** \( \Delta_n(A_{ij}B_{ji}) = \sum_{i+j=n} [\Delta_i(A_{ij}, \Delta_j(B_{ji})] \quad 1 \leq i \neq j \leq 2 \)

**proof.** Note that \([A_{12}, B_{21}] = [[A_{12}, P_2], B_{21}] \) holds true for all \( A_{12} \in B_{12} \) and \( B_{21} \in B_{21}. \) So we have

\[ \Delta_n([B_{21}, C_{12}]) - D_n([B_{21}, C_{12}]) = \sum_{i+j=n} [\Delta_i(B_{21}, \Delta_j(C_{12})] - D_n(B_{21})C_{12} - D_n(C_{12}B_{21}) \]

\[ = [\Delta_n(B_{21}), C_{12}] + [B_{21}, \Delta_n(C_{12})] \]

\[ + \sum_{i+j=n; i\neq 0, n} [\Delta_i(B_{21}), \Delta_j(D_{12})] - D_n(B_{21})C_{12} - D_n(C_{12}B_{21}) \]

\[ = [D_n(B_{21}), C_{12}] + [B_{21}, D_n(C_{12})] \]

\[ + D_n(B_{21}C_{12}) - D_n(B_{21}C_{12}) \]

\[ = D_n(A_{12})B_{21} - P_2D_n(A_{12})B_{21} - B_{21}D_n(A_{12})P_2 + B_{21}D_n(A_{12}) \]

\[ + A_{12}D_n(B_{21}) - D_n(B_{21})A_{12} - D_n(A_{12}B_{21}) + D_n(B_{21}A_{12}) \]
Which implies that
\[(D_n(A_{12})B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21})) + (D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12})) = K \in \mathbb{F}I.\]

Using Claim 12 and multiplying $B_{22}$ from the left in the above equality, We obtain
\[B_{22}D_n(B_{21}A_{12}) - B_{22}D_n(B_{21})A_{12} - B_{22}B_{21}D_n(A_{12}) = B_{22}K.\]

So
\[K = D_n(A_{12}B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21})) + (D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12})) = 0.\]

Therefore
\[D_n(A_{12})B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21}) = -(D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12}))\]
\[\in B_{11}\bigcap B_{22}\]

So we have
\[D_n(A_{12})B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21}) = 0 \text{ and } D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12}) = 0\]
So $D_n$ is a derivation.

**Claim 16.** For all $A \in B(X)$, $\Delta_n(A) - \Delta_n(PAQ) - \Delta_n(QAP) \in \mathbb{F}I$.

For any $A \in B(X)$ and Claim it is easy to checked that $\Delta_n(A) - \Delta_n(PAQ) - \Delta_n(QAP) \in \mathbb{F}I$.

**Claim 17.** $\tau_n([X,Y]) = 0$ for all $X, Y \in B(X)$.

**proof.** For any $X, Y \in B(X)$, we have
\[\tau_n([X,Y]) = \Delta_n([X,Y]) - D_n([X,Y])\]
\[= \sum_{i+j=n} [\Delta_i(X), \Delta_j(Y)] - D_n(XY) - D_n(YX)\]
\[= \sum_{i+j=n} [D_i(X) + \tau_i(X), D_j(Y) + \tau_j(Y)] - \sum_{i+j=n} D_i(X)D_j(Y) - D_j(Y)D_i(X)\]
\[= \sum_{i+j=n} [D_i(X), D_j(Y)] - \sum_{i+j=n} D_i(X)D_j(Y) - D_j(Y)D_i(X)\]
\[= 0\]

The proof of theorem is completed. ___

**References**