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Analytical-approximate solution for nonlinear Volterra integro-differential equations

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Abstract. In this work, we conduct a comparative study among the combine Laplace transform and modified Adomian decomposition method (LMADM) and two traditional methods for an analytic and approximate treatment of special type of nonlinear Volterra integro-differential equations of the second kind. The nonlinear part of integro-differential is approximated by Adomian polynomials, and the equation is reduced to a simple equations. The proper implementation of combine Laplace transform and modified Adomian decomposition method can extremely minimize the size of work if compared to existing traditional techniques. Moreover, three particular examples are discussed to show the reliability and the performance of method.

Keywords: Nonlinear Volterra integro-differential equations, Laplace transform method, modified Adomian decomposition method.

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1. Introduction

This paper presents a comparative study between combine Laplace transform and modified Adomian decomposition method (MADM) [7, 8, 11] and two traditional methods, namely the Adomian decomposition method (ADM) [4, 5, 9], and the series solution method (SSM) [3], for solving an special type of nonlinear Volterra integro-differential equations of the second kind. The nonlinear Volterra integro-differential equation which

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we study here is given by the following:

$$u^{(n)}(x) = f(x) + \lambda \int_0^x k(x,t)[u'(t)]^p dt, \quad u^{(k)}(0) = b_k,\ 0 \leqslant k \leqslant (n-1), \tag{1}$$

where $u^{(n)}(x)$ is the nth derivative of the unknown function $u(x)$ that will be determined, $f(x)$ is an known analytic function, $k(x,t)$ is the kernel of the integral equation, u' represents the first order derivative of u with respect to t, p is a positive integer and λ is suitable constant.

The Volterra integro-differential equations (1) arise from the mathematical modeling of the spatiotemporal development of an epidemic model in addition to various physical and biological models [6], and also from many other scientific phenomena. Nonlinear phenomena, which appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by partial differential equations and by integral equations as well. The concepts of integral equations have motivated a large amount of research work in recent years. Several analytical and numerical methods were used such as the Adomian decomposition method and the direct computation method, the series solution method, the successive approximation method, the successive substitution method and the conversion to equivalent differential equations. However, these analytical solution methods are not easy to use and require tedious calculation. Our aim in this paper is to obtain the analytical solutions by using the LMADM for the integro-differential equations. The remainder of the paper is organized as follows: In Section 2, a brief discussion for the LMADM is presented and exact solutions for all examples are obtained. In Sections 3, we shall discuss the two traditional methods briefly. In Section 4 presented numerical examples that shows the efficiency and accuracy of proposed method in analogy to two traditional methods. We introduce the convergence analysis of the proposed method in section 5. Section 6 ends this paper with a brief conclusion.

2. The combined Laplace transform-modified Adomian decomposition method

In this work we will assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed by the difference $(x-t)$. The nonlinear Volterra integro-differential equation (1) can thus be expressed as following

$$u^{(n)}(x) = f(x) + \lambda \int_0^x k(x-t)[u'(t)]^p dt, \quad u^{(k)}(0) = b_k,\ 0 \leqslant k \leqslant (n-1). \tag{2}$$

Applying the Laplace transform to both sides of (2) gives

$$\mathcal{L}\{u^{(n)}(x)\} = \mathcal{L}\{f(x)\} + \lambda\mathcal{L}\{\int_0^x k(x-t)[u'(t)]^p dt\}. \tag{3}$$

On the other hand, using the differentiation property of Laplace transform, we have

$$\mathcal{L}\{u^{(n)}(x)\} = s^n\mathcal{L}\{u(x)\} - s^{n-1}u(0) - \cdots - su^{(n-2)}(0) - u^{(n-1)}(0). \tag{4}$$

Substituting (4) into (3), and by using the convolution theorem we obtain

$$U(s) = F(s) + \lambda K(s)\mathcal{L}\{[u'(x)]^p\}, \tag{5}$$

where

$$\begin{aligned} U(s) &= \mathcal{L}\{u(x)\}, \\ F(s) &= \frac{b_0}{s} + ... + \frac{b_{n-2}}{s^{n-1}} + \frac{b_{n-1}}{s^n} + \frac{1}{s^n}\mathcal{L}\{f(x)\},\ K(s) = \frac{1}{s^n}\mathcal{L}\{k(x)\}. \end{aligned} \tag{6}$$

The modified Adomian decomposition method and the Adomian polynomials can be used to handle (5) and to address the nonlinear term $[u'(x)]^p$. We first represent the linear term $U(s)$ at the left side by an infinite series of components given by

$$U(s) = \sum_{n=o}^{\infty} U_n(s), \tag{7}$$

and similarly

$$u(x) = \sum_{n=o}^{\infty} u_n(x), \tag{8}$$

where the components $U_n(s)$, $n \geqslant 0$ will be determined recursively.
modified Adomian decomposition method suggest that the function $F(s)$ defined above in equation (5) can be decomposed into two parts namely, $F_0(s)$ and $F_1(s)$, i.e.

$$F(s) = F_0(s) + F_1(s). \tag{9}$$

In the above equation proper choice of $F_0(s)$ and $F_1(s)$ is essential and depends mainly on the trail basis. However, the nonlinear term $[u'(x)]^p$ at the right side of (5) will be represented by an infinite series of the Adomian polynomials A_n in the form

$$[u'(x)]^p = \sum_{n=o}^{\infty} A_n(x), \tag{10}$$

where A_n, $n \geqslant 0$ are defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]_{\lambda=0}, \qquad n = 0, 1, 2, ..., \tag{11}$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is $[u'(x)]^p$, therefore the Adomian polynomials are given by

$$A_0 = [u_0'(x)]^p,$$

$$A_1 = p[u_1'(x)][u_0'(x)]^{p-1},$$

$$A_2 = p[u_2'(x)][u_0'(x)]^{p-1} + \frac{p(p-1)}{2!}[u_1'(x)]^2[u_0'(x)]^{p-1}, \tag{12}$$

$$\cdots.$$

Substituting equations (7), (9) and (10) into equation (5), we get

$$\sum_{n=o}^{\infty} U_n(s) = F_0(s) + F_1(s) + \lambda K(s)\mathcal{L}\{\sum_{n=o}^{\infty} A_n(x)\}. \tag{13}$$

Thus the following recursive relations for the modified Adomian decomposition method are formulated as

$$U_0(s) = F_0(s),$$

$$U_1(s) = F_1(s) + \lambda K(s)\mathcal{L}\{A_0(x)\},$$

$$U_{n+1}(s) = \lambda K(s)\mathcal{L}\{A_n(x)\}, \quad n \geqslant 1. \tag{14}$$

Applying the inverse Laplace transform to the first part of (14) gives $u_0(x)$, that will define A_0. Using $A_0(x)$ will enable us to evaluate $u_1(x)$. The determination of $u_0(x)$ and $u_1(x)$ leads to the determination of $A_1(x)$ that will allows us to determine $u_2(x)$, and so on. This in turn will lead to the complete determination of the components of $u_n(s)$, $n \geqslant 0$ upon using the third part of (14). The series solution follows immediately after using (8). The series (8) is convergent to an exact solution for most of the cases, if such a solution exists, and also the rate of convergence depends on how we choose $F_0(s)$.

3. The series solution method

This method is a traditional method that mainly depends on Taylor series and has been used in differential and integral equations as well. However, the method is mainly used for solving Volterra integral equations. In what follows, we present a brief idea about the method where details can be found in many references such as [2, 3]. Assuming that $u(x)$ is an analytic function, it can be represented by a series given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{15}$$

where a_k are constants that will be determined recursively. The first few coefficients can be determined by using the prescribed initial conditions where we may use

$$a_0 = u(0),$$

$$a_1 = u'(0),$$

$$a_2 = \frac{1}{2!}u''(0), \tag{16}$$

$$\ldots.$$

Substituting (16) into both sides of (1), and assuming that the kernel $k(x,t)$ is separable as $k(x,t) = p_1(x)p_2(t)$, we obtain

$$(\sum_{k=0}^{\infty} a_k x^k)^{(n)} = f(x) + \lambda p_1(x)\int_0^x p_2(t)[\sum_{k=1}^{\infty} k a_k t^{k-1}]^p dt. \tag{17}$$

Notice that by using this method, we put the procedure of the problem in a straightforward manner, although we can integrate both sides of (1) with respect to t. Using the Taylor series expansions for $f(x)$, $p_1(x)$ and $p_2(t)$, and equating the coefficients of like powers in both sides of the resulting equations, one can easily determine the coefficients a_k, $k \geqslant 0$. Substituting the results in equation (15) leads to the series solution of $u(x)$, as a result, we may obtain the closed form of the solution if the exact solution exists.

4. Numerical Examples

In this part three examples are provided. These examples are considered to illustrate ability and reliability of the new technique. The computations associated with the examples in this paper were performed using the package Matlab.

Example 4.1 Let us first consider the special case of nonlinear Volterra integro-differential equation of second kind [12]:

$$u'(x) = e^x + \frac{1}{2}(1 - e^{2x}) + \int_0^x [u'(t)]^2 dt, \qquad u(0) = 0. \tag{18}$$

The exact solution of this problem is $u(x) = e^x - 1$.

And we solve it by using combine Laplace transform and modified Adomian decomposition method, the Adomian decomposition method and the series solution method as follows:

4.1.1.Using LMADM: By taking Laplace transform of both sides equation (18), and by using the initial condition we obtain

$$U(s) = \frac{1}{s(s-1)} + \frac{1}{2s^2} - \frac{1}{2s(s-2)} + \frac{1}{s^2}\mathcal{L}\{[u'(x)]^2\}, \tag{19}$$

where

$$F(s) = \frac{1}{s(s-1)} + \frac{1}{2s^2} - \frac{1}{2s(s-2)},\ K(s) = \frac{1}{s^2},\ \lambda = 1. \tag{20}$$

Now splitting $F(s)$ into two part i.e.,

$$F_0(s) = \frac{1}{s(s-1)},\ F_1(s) = \frac{1}{2s^2} - \frac{1}{2s(s-2)}. \tag{21}$$

The modified Adomian decomposition method admits the use of

$$U_0(s) = \frac{1}{s(s-1)},$$

$$U_1(s) = \frac{1}{2s^2} - \frac{1}{2s(s-2)} + \frac{1}{s^2}\mathcal{L}\{A_0(x)\},$$

$$U_{n+1}(s) = \frac{1}{s^2}\mathcal{L}\{A_n(x)\}, \quad n \geqslant 1. \tag{22}$$

Using the recurrence relation (22), and by using the inverse Laplace transform of $U_0(s)$, we find

$$u_0(x) = e^x - 1. \tag{23}$$

Using this result in U_1 of (22), and by using the inverse Laplace transform we find $u_1(x)$. Proceeding in this manner we find the following components

$$u_0(x) = e^x - 1,$$

$$u_1(x) = 0,$$

$$u_{n+1}(x) = 0, \quad n \geqslant 1. \tag{24}$$

The series solution is therefore given by

$$u(x) = e^x - 1, \tag{25}$$

which is the exact solution. It is clear that this series solution satisfies the first condition $u(0) = 0$.

4.1.2.Using ADM: For using this method, we substitute the decomposition series (8) for $u(x)$ into both sides of (18) which gives

$$\sum_{n=o}^{\infty} u'_n(x) = e^x + \frac{1}{2}(1 - e^{2x}) + \int_0^x [\sum_{n=o}^{\infty} A_n(t)]dt. \tag{26}$$

Identifying the zero component $u'_0(x)$ in (26) by $e^x + \frac{1}{2}(1 - e^{2x})$, the remaining components $u'_n(x)$, $n \geqslant 1$ can be determined by using the following recurrence relation

$$u'_0(x) = e^x + \frac{1}{2}(1 - e^{2x}),$$

$$u'_n(x) = \int_0^x \sum_{n=0}^{\infty} A_n(t)dt, \quad n \geqslant 1,$$

where A_n are the so-called Adomian polynomials. Therefore, we obtain

$$u_0(x) = \frac{1}{2}x - \frac{1}{4}e^{2x} + e^x - \frac{3}{4},$$

$$u_1(x) = \frac{1}{8}e^{2x} - \frac{47}{48}x - \frac{1}{9}e^{3x} + \frac{1}{64}e^{4x} + e^x + \frac{1}{8}x^2 - \frac{593}{576},$$

$$u_2(x) = \frac{551}{2880}x + \frac{167}{192}e^{2x} - \frac{5}{54}e^{3x} - \frac{41}{768}e^{4x} + \frac{11}{600}e^{5x}$$

$$- \frac{1}{576}e^{6x} - \frac{47}{24}e^x - \frac{1}{16}xe^{2x} + \frac{1}{2}xe^x - \frac{47}{96}x^2 + \frac{1}{24}x^3 + \frac{210457}{172800}, \tag{27}$$

$$\ldots .$$

In view of the above equations, the solution $u(x)$ is readily obtained in a series form by

$$u(x) = -\frac{1422665425256885}{2533274790395904} + \frac{1}{24}x^3 - \frac{35}{96}x^2 + \frac{1}{2}xe^x - \frac{1}{16}te^{2x} + \frac{1}{24}e^x - \frac{1}{576}e^{6x}$$

$$+\frac{11}{600}e^{5x} - \frac{29}{768}e^{4x} - \frac{11}{54}e^{3x} - \frac{829}{2880}x + \frac{143}{192}e^{2x} + \cdots . \tag{28}$$

The last expression shows that using the Adomian decomposition method cannot lead to the closed form of the solution.

4.1.3.Using SSM: Evidently, from (18), we get $u(0) = 0$. Now, we consider the series expansion (15):

$$u(x) = \sum_{k=1}^{\infty} a_k x^k. \tag{29}$$

By substituting (29) into both sides of (18), and using the Taylor series of the functions involved, and finally integrating the right-hand side of the result equation, we have

$$a_1x + 2a_2x^2 + \cdots = 1 + x - (\frac{1}{2} - 2a_1a_2)x^2 + (-\frac{1}{2} + 4a_2^2 + 6a_1a_3)x^3 + \cdots . \tag{30}$$

Now, we equate the coefficients of like powers of x from the both sides of (30), yields

$$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{6},\ a_4 = \frac{1}{24}, \tag{31}$$

and so on. Substituting these results into (29) gives the following series solution:

$$u(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots , \tag{32}$$

and in a closed form, $u(x) = e^x - 1$.
Example 4.2 Second we can study the following nonlinear Volterra integro-differential equation [1]:

$$u''(x) = sinh(x) + \frac{1}{2}x + \frac{1}{2}cosh(x)sinh(x) - \int_0^x [u'(t)]^2 dt, \ u(0) = 0, \ u'(0) = 1 \quad (33)$$

with the exact solution $u(x) = sinh(x)$.
4.2.1.Using LMADM: Applying the Laplace transform of both sides equation (33), and by using the initial conditions we have

$$U(s) = \frac{1}{s^2-1} + \frac{1}{2s^2(s^2-4)} + \frac{1}{2s^4} - \frac{1}{s^3}\mathcal{L}\{[u'(x)]^2\}, \quad (34)$$

where

$$F(s) = \frac{1}{s^2-1} + \frac{1}{2s^2(s^2-4)} + \frac{1}{2s^4}, \ K(s) = \frac{1}{s^3}, \ \lambda = -1. \quad (35)$$

Now splitting $F(s)$ into two part i.e.,

$$F_0(s) = \frac{1}{s^2-1}, \ F_1(s) = \frac{1}{2s^2(s^2-4)} + \frac{1}{2s^4}. \quad (36)$$

Applying the same procedure as in the previous example we arrive the modified recursive relation given below

$$u_0(x) = sinh(x),$$

$$u_1(x) = 0,$$

$$u_{n+1}(x) = 0, \quad n \geqslant 1. \quad (37)$$

The series solution is therefore given by

$$u(x) = sinh(x), \quad (38)$$

which is the exact solution. It is clear that this series solution satisfies the first conditions $u(0) = 0$, $u'(0) = 1$.
4.2.2.Using ADM: By employing the Adomian decomposition method, we obtain

$$u_0(x) = -\frac{9}{8}x + \frac{1}{12}x^3 + sinh(x) + \frac{1}{16}sinh(2x),$$

$$u_1(x) = \frac{17833}{18432}x - \frac{13}{256}sinh(2x) - \frac{1}{8192}sinh(4x) - \frac{35}{9}sinh(x)$$

$$+\frac{3}{128}xcosh(2x)-\frac{1}{2}x^2sinh(x)-\frac{1}{54}sinh^3(x)-\frac{1}{128}x^2sinh(2x)$$

$$+3xcosh(x)-\frac{227}{768}x^3+\frac{3}{320}x^5-\frac{1}{3360}x^7, \quad (39)$$

$$\cdots.$$

In view of (39), the solution $u(x)$ is readily obtained in a series form by

$$u(x)=-\frac{2903}{18432}x+\frac{3}{256}sinh(2x)-\frac{1}{8192}sinh(4x)-\frac{26}{9}sinh(x)$$

$$+\frac{3}{128}xcosh(2x)-\frac{1}{2}x^2sinh(x)-\frac{1}{54}sinh^3(x)-\frac{1}{128}x^2sinh(2x)$$

$$+3xcosh(x)-\frac{163}{768}x^3+\frac{3}{320}x^5-\frac{1}{3360}x^7+\cdots. \quad (40)$$

This solution also cannot be written in the closed form.
4.2.3.Using SSM: We use the series (15), namely

$$u(x)=\sum_{k=0}^{\infty}a_kx^k, \quad (41)$$

and substitute (41) into both sides of (33), and next we use the Taylor series expansion of the functions involved, integrating the right-hand side of the result equation, and we finally equate the coefficients of like powers of x from both sides, as a result, we have the following:

$$a_0=0,\ a_1=1,\ a_2=0,\ a_3=\frac{1}{3!},\ a_4=0,\ a_5=\frac{1}{5!}, \quad (42)$$

and so on. Substituting the results given in (42) into (41), gives the following series solution

$$u(x)=x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\cdots, \quad (43)$$

and in a closed form we have, $u(x)=sinh(x)$.
Example 4.3 In the final example we consider the nonlinear Volterra integro-differential equation:

$$u'(x)=xe^x+\frac{1}{3}x^3e^{2x}-\int_0^x e^{2(x-t)}[u'(t)]^2dt, \qquad u(0)=0 \quad (44)$$

with the exact solution $u(x)=e^x(x-1)+1$.
4.3.1.Using LMADM: First, we apply the Laplace transform and by using the initial

condition we obtain

$$U(s)=\frac{1}{s(s-1)^2}+\frac{2}{s(s-2)^4}-\frac{1}{s(s-2)}\mathcal{L}\{[u'(x)]^2\}, \tag{45}$$

where

$$F(s)=\frac{1}{s(s-1)^2}+\frac{2}{s(s-2)^4},\ K(s)=\frac{1}{s(s-2)},\ \lambda=-1. \tag{46}$$

Now splitting $F(s)$ into two part i.e.,

$$F_0(s)=\frac{1}{s(s-1)^2},\ F_1(s)=\frac{2}{s(s-2)^4}. \tag{47}$$

Applying the same procedure as in the previous examples we arrive the modified recursive relation given as

$$u_0(x)=e^x(x-1)+1,$$

$$u_1(x)=0,$$

$$u_{n+1}(x)=0, \quad n\geqslant 1. \tag{48}$$

The series solution is therefore given by

$$u(x)=e^x(x-1)+1, \tag{49}$$

which is the exact solution. It is clear that $u(0)=0$.

4.3.2.Using ADM: Proceeding with the previous steps, we obtain

$$u_0(x)=e^x(x-1)+\frac{e^{2x}(8x^3-12x^2+12x-6)}{48}+\frac{9}{8},$$

$$u_1(x)=\frac{135}{16}e^{2x}-\frac{1936}{243}e^{3x}-\frac{635}{2048}e^{4x}-\frac{1}{4}xe^{2x}+\frac{640}{81}xe^{3x}+\frac{315}{512}xe^{4x}$$

$$+\frac{1}{4}x^2e^{2x}-\frac{104}{27}x^2e^{3x}-\frac{1}{6}x^3e^{2x}-\frac{155}{256}x^2e^{4x}+\frac{32}{27}x^3e^{3x}+\frac{25}{64}x^3e^{4x}$$

$$-\frac{2}{9}x^4e^{3x}-\frac{35}{192}x^4e^{4x}+\frac{1}{16}x^5e^{4x}-\frac{1}{72}x^6e^{4x}-\frac{79807}{497664}, \tag{50}$$

$$\vdots$$

In view of the above equations, the solution $u(x)$ can be obtained in a following series form

$$u(x) = \frac{135}{16}e^{2x} - \frac{1936}{243}e^{3x} - \frac{635}{2048}e^{4x} - \frac{1}{4}xe^{2x} + \frac{640}{81}xe^{3x} + \frac{315}{512}xe^{4x}$$

$$+ e^{x}(x-1) + \frac{e^{2x}(8x^3 - 12x^2 + 12x - 6)}{48} + \frac{1}{4}x^2e^{2x} + \frac{104}{27}x^2e^{3x}$$

$$-\frac{1}{6}x^3e^{2x} - \frac{155}{256}x^2e^{4x} + \frac{32}{27}x^3e^{3x} + \frac{25}{64}x^3e^{4x} - \frac{2}{9}x^4e^{3x}$$

$$-\frac{35}{192}x^4e^{4x} + \frac{1}{16}x^5e^{4x} - \frac{1}{72}x^6e^{4x} + \frac{480065}{497664} - \cdots . \quad (51)$$

This solution also cannot be written in the closed form.
4.3.3.Using SSM: By employing the series solution method, we obtain

$$a_0 = 0,\ a_1 = 0,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{8},\ a_5 = \frac{1}{30}, \quad (52)$$

and so on. Substituting the results given in (52) into (15), gives the following series solution

$$u(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots , \quad (53)$$

and in a closed form we have, $u(x) = e^x(x-1) + 1$.

5. Convergence Analysis

Here, we will study the convergence analysis as same manner in [7, 10] of the LMADM applied to the nonlinear Volterra integro-differential equations.

6. Conclusion

In this paper, coupling Laplace transform and modified Adomian decomposition method, for solving special type of nonlinear Volterra integro-differential equations of the second kind, is presented successfully. Moreover we conducted a comparative study between LMADM and the traditional methods, i.e. the ADM, and the SSM. The LMADM is implemented in a straightforward manner, and it accelerates the rapid convergence of the series solution. The two traditional methods suffer from the tedious work of calculations. However, the traditional methods were capable of providing more than one solution which is consistent with the theory of nonlinear equations. Generally speaking, LMADM is reliable and more efficient compared to other techniques.

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