On fuzzy soft connected topological spaces

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Abstract. In this work, we introduce notion of connectedness on fuzzy soft topological spaces and present fundamentals properties. We also investigate effect to fuzzy soft connectedness. Moreover, $C_1$—connectedness which plays an important role in fuzzy topological space extend to fuzzy soft topological spaces.

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1. Introduction

There are several set theories dealing with uncertainties such as fuzzy set theory [24], intuitionistic fuzzy set theory, vague set theory, and rough set theory. But they have their own difficulties. For example, constructing membership function is a very important problem in fuzzy set theory. Due to these and similar reasons, soft set theory was defined by Molodtsov [17]. He introduced soft set theory as a new mathematical tool to model uncertainties. He presented the paper [17] about fundamental properties of the soft set theory and application areas such as Riemann-integration, Perron integration, game theory, operations research, probability theory, etc.

Recently, many researchers study on soft set theory. Not only applied mathematicians but also theoretic mathematicians are interested in this topic. For example, in algebra Aktaş [3] defined the concept of soft group. Then, Shabir [20], Aygünăoğlu [4], Zorlutuna [26], Roy [19] and Peyghan [18] wrote several articles about soft topological structures and Hussain [14] wrote a paper entitled a note on soft connectedness.

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After defining the soft set, fuzzy soft set theory was proposed by Maji [16] and its fundamental properties were investigated by Çağman [8]. Also, topological researches were done on this topic by Tanay [22], Varol [23], Gain [11], Şimşekler [21] and Gündüz [12].

2. Preliminary

In this section, we will give some fundamental definitions and theorems about fuzzy soft sets, fuzzy soft topology and fuzzy soft continuous mappings. Throughout this paper, \( \mathcal{P}(X) \), \( E \) and \( I^X \) denote power set of \( X \), set of parameter and set of all fuzzy sets over \( X \), respectively, where \( I = [0, 1] \).

**Definition 2.1** [24] A fuzzy set \( \mu \) on \( X \) is a mapping \( \mu : X \to I \). The value \( \mu(x) \) represents the degree of membership of \( x \) in the fuzzy set \( \mu \) for \( x \in X \).

**Definition 2.2** [24] Let \( \mu, \nu \in I^X \). Then, some basic set operations for fuzzy sets are defined as follows:

i. (Inclusion) \( \mu \subseteq \nu \) iff \( \mu(x) \leq \nu(x) \) for all \( x \in X \)

ii. (Equality) \( \mu = \nu \) iff \( \mu(x) = \nu(x) \) for all \( x \in X \)

iii. (Union) \( \eta = \mu \cup \nu \) iff \( \eta(x) = \mu(x) \vee \nu(x) \) for all \( x \in X \)

iv. (Intersection) \( \xi = \mu \cap \nu \) iff \( \xi(x) = \mu(x) \wedge \nu(x) \) for all \( x \in X \)

v. (Complement) \( \lambda = \mu^c \) iff \( \lambda(x) = 1 - \mu(x) \) for all \( x \in X \)

vi. (Empty set) If \( \mu(x) = 0 \) for all \( x \in X \), then \( \mu \) is called empty fuzzy set and denoted by \( \emptyset \)

vii. (Universal set) If \( \mu(x) = 1 \) for all \( x \in X \), then \( \mu \) is called universal fuzzy set and denoted by \( X \).

**Definition 2.3** [17] A pair \( (F, A) \) is called a soft set over \( X \), if \( F \) is a mapping defined by \( F : A \to \mathcal{P}(X) \), where \( A \subseteq E \).

Following definition was given by Çağman [9] and it is a new comment about soft sets.

**Definition 2.4** [9] A soft set \( F \) over \( X \) is a set valued function from \( E \) to \( \mathcal{P}(X) \). It can be written a set of ordered pairs

\[
F = \{(e, F(e)) : e \in E\}.
\]

Note that if \( F(e) = \emptyset \), then the element \( (e, F(e)) \) is not appeared in \( F \). Set of all soft sets over \( X \) is denoted by \( \mathcal{S} \).

According to above definition we will redefine fuzzy soft set and its set operations.

**Definition 2.5** A fuzzy soft set \( \tilde{f} \) over \( X \) is a set valued function from \( E \) to \( I^X \). It can be written a set of ordered pairs

\[
\tilde{f} = \{(e, \{(x, \mu_{\tilde{f}(e)}(x)) : x \in X\}) : e \in E\}
\]

where \( \mu_{\tilde{f}(e)} \in I^X \) for all \( e \in E \). Note that if \( \tilde{f}(e) = \emptyset \), then the element \( (e, \tilde{f}(e)) \) is not appeared in \( \tilde{f} \). Set of all fuzzy soft sets over \( X \) is denoted by \( \mathbb{F}_X \).

**Definition 2.6** Let \( \tilde{f}, \tilde{g} \in \mathbb{F}_X \). Then some basic set operations of fuzzy soft sets are defined as follows:
Let \( \varphi \) be the fuzzy soft mapping from \( X \) to \( Y \). Let \( \varphi \) be surjective, then \( \tilde{1}_X = \tilde{0}_Y \).

**Theorem 2.7** [23] Let \( \tilde{f}_i \in \mathcal{FS}_X \) and \( \tilde{g} \in \mathcal{FS}_X^E \). Then

i. \( \tilde{g} \cap \left( \bigcup_{i \in A} \tilde{f}_i \right) = \bigcup_{i \in A} (\tilde{g} \cap \tilde{f}_i) \)

ii. \( \tilde{g} \cup \left( \bigcap_{i \in A} \tilde{f}_i \right) = \bigcap_{i \in A} (\tilde{g} \cup \tilde{f}_i) \)

iii. \( \left( \bigcup_{i \in A} \tilde{f}_i \right)^c = \bigcap_{i \in A} \tilde{f}_i^c \)

iv. \( \left( \bigcap_{i \in A} \tilde{f}_i \right)^c = \bigcup_{i \in A} \tilde{f}_i^c \)

v. \( \tilde{0}_X \subseteq \tilde{f} \subseteq \tilde{1}_X, \tilde{1}_Y = \tilde{0}_X \) and \( \tilde{0}_Y = \tilde{1}_X \).

vi. \( \tilde{g} \cup \tilde{g}^c = \tilde{1}_X \) and \( (\tilde{g}^c)^c = \tilde{g} \).

**Definition 2.8** [15] Let \( \mathcal{FS}_X^E \) and \( \mathcal{FS}_Y^K \) be the families of all fuzzy soft sets over \( X \) and \( Y \), respectively. Let \( \varphi : X \to Y \) and \( \psi : E \to K \) be two functions. Then the \( \varphi \psi \) is called a fuzzy soft mapping from \( X \) to \( Y \), and denoted by \( \varphi \psi : \mathcal{FS}_X^E \to \mathcal{FS}_Y^K \).

i. Let \( \tilde{f} \in \mathcal{FS}_X^E \). Then the image of \( \tilde{f} \) under the fuzzy soft mapping \( \varphi \psi \) is the fuzzy soft set over \( Y \) defined by \( \varphi \psi (\tilde{f}) \), where

\[
\varphi \psi (\tilde{f})(k)(y) = \begin{cases} 
V_{x \in \varphi^{-1}(y)} \left( V_{e \in \psi^{-1}(k)} f(e) \right)(x), & \text{if } \varphi^{-1}(y) \neq \emptyset, \psi^{-1}(k) \neq \emptyset \\
0_Y, & \text{otherwise.}
\end{cases}
\]

ii. Let \( \tilde{g} \in \mathcal{FS}_X^E \). Then the pre-image of \( \tilde{g} \) under the fuzzy soft mapping \( \varphi \psi \) is a fuzzy soft set over \( X \) defined by \( \varphi^{-1} \psi (\tilde{g}) \), where

\[
\varphi^{-1} \psi (\tilde{g})(e)(x) = g(\psi(e))(\varphi(x)).
\]

If \( \varphi \) and \( \psi \) are injective (surjective) then the fuzzy soft mapping \( \varphi \psi \) is said to be injective (surjective).

**Theorem 2.9** [15] Let \( \varphi : \mathcal{FS}_X^E \to \mathcal{FS}_Y^K \) be a fuzzy soft mapping, \( \tilde{f} \in \mathcal{FS}_X^E \) and \( \{\tilde{f}_i : i \in I\} \subseteq \mathcal{FS}_X^E \). Then

i. If \( \tilde{f}_1 \subseteq \tilde{f}_2 \), then \( \varphi \psi (\tilde{f}_1) \subseteq \varphi \psi (\tilde{f}_2) \)

ii. \( \varphi \psi \left( \bigcup_{i \in A} \tilde{f}_i \right) = \bigcup_{i \in A} \varphi \psi (\tilde{f}_i) \)

iii. \( \varphi \psi \left( \bigcap_{i \in A} \tilde{f}_i \right) = \bigcap_{i \in A} \varphi \psi (\tilde{f}_i) \)

iv. \( (\varphi \psi (\tilde{f}))^c \subseteq \varphi \psi (\tilde{f}^c) \)

v. If \( \varphi \psi \) surjective, then \( \varphi \psi (\tilde{1}_X) = \tilde{1}_Y \)

vi. \( \varphi \psi (\tilde{0}_X) = \tilde{0}_Y \)

vii. \( \tilde{f} \subseteq \varphi^{-1} \psi (\varphi \psi (\tilde{f})) \), the equality holds if \( \varphi \psi \) is injective.
Theorem 2.10 [15] Let $\varphi_\psi : FS^{E}_X \rightarrow FS^{K}_Y$ be a fuzzy soft mapping, $\tilde{g} \in FS^{K}_Y$ and $\{\tilde{g}_j : j \in J\} \subseteq FS^{K}_Y$. Then

i. If $\tilde{g}_1 \subseteq \tilde{g}_2$, then $\varphi^{-1}_\psi(\tilde{g}_1) \subseteq \varphi^{-1}_\psi(\tilde{g}_2)$

ii. $\varphi^{-1}_\psi \left( \bigcup_{j \in J} \tilde{g}_j \right) = \bigcup_{j \in J} \varphi^{-1}_\psi(\tilde{g}_j)$

iii. $\varphi^{-1}_\psi \left( \bigcap_{j \in J} \tilde{g}_j \right) = \bigcap_{j \in J} \varphi^{-1}_\psi(\tilde{g}_j)$

iv. $(\varphi^{-1}_\psi(\tilde{g}))^c = \varphi^{-1}_\psi(\tilde{g}^c)$

v. $\varphi^{-1}_\psi(\tilde{I}_Y) = \tilde{I}_X$ and $\varphi^{-1}_\psi(\tilde{0}_Y) = \tilde{0}_X$

vi. $\varphi_\psi(\varphi^{-1}_\psi(\tilde{g})) \subseteq \tilde{g}$, the equality holds if $\varphi_\psi$ is surjective.

Definition 2.11 [22] A fuzzy soft topological space is a triplet $(X, \tau, E)$ where $X$ is a nonempty set and $\tau$ a family of fuzzy soft sets over $X$ satisfying the following properties:

i. $\tilde{0}_X, \tilde{I}_X \in \tau$,

ii. If $\tilde{f}, \tilde{g} \in \tau$, then $\tilde{f} \cap \tilde{g} \in \tau$,

iii. If $\{f_i : i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} f_i \in \tau$.

Then the family $\tau$ is called a fuzzy soft topology on $X$. Every member of $\tau$ is called fuzzy soft open. $\tilde{g}$ is called fuzzy soft closed in $(X, \tau, E)$ if $\tilde{g}^c \in \tau$.

Definition 2.12 Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{f} \in FS^{E}_X$. If $\tilde{f}$ is fuzzy soft open and fuzzy soft closed, then it is called fuzzy soft clopen set. In case $\tilde{f} \neq \tilde{1}_X$ and $\tilde{f} \neq \tilde{0}_X$, $\tilde{f}$ is called fuzzy soft proper set.

Example 2.13 $\tau = \{\tilde{1}_X, \tilde{0}_X\}$ and $\sigma = FS^{E}_X$ are fuzzy soft topologies on $X$.

Definition 2.14 [22] Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{f} \in FS^{E}_X$. Fuzzy soft interior of $\tilde{f}$ denoted by $int(\tilde{f})$ is the union of all fuzzy soft open subsets of $\tilde{f}$.

Definition 2.15 [22] Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{f} \in FS^{E}_X$. Fuzzy soft closure of $\tilde{f}$ denoted by $cl(\tilde{f})$ is the intersection of all fuzzy soft closed supersets of $\tilde{f}$.

It can be seen that $int(\tilde{f})$ and $cl(\tilde{f})$ are the largest fuzzy soft open set which contained in $\tilde{f}$ and the smallest fuzzy soft closed set which contains $\tilde{f}$ over $X$, respectively.

Theorem 2.16 [23] Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{f}, \tilde{g} \in FS^{E}_X$. Then,

i. If $\tilde{f} \subseteq \tilde{g}$, then $int(\tilde{f}) \subseteq int(\tilde{g})$ and $cl(\tilde{f}) \subseteq cl(\tilde{g})$

ii. $\tilde{f}$ is a soft open set iff $int(\tilde{f}) = \tilde{f}$

iii. $\tilde{f}$ is a soft closed set iff $cl(\tilde{f}) = \tilde{f}$

iv. $(cl(\tilde{f}))^c = int(\tilde{f}^c)$ and $cl(\tilde{f}^c) = (int(\tilde{f}))^c$

Definition 2.17 [23] Let $(X, \tau, E)$ and $(Y, \sigma, K)$ be two fuzzy soft topological spaces. A fuzzy soft mapping $\varphi_\psi : (X, \tau, E) \rightarrow (Y, \sigma, K)$ is called fuzzy soft continuous mapping if $\varphi^{-1}_\psi(\tilde{g}) \in \tau$ for all $\tilde{g} \in \sigma$.

Example 2.18 Let consider the fuzzy soft topology $\tau$ in Example 2.13 and $\sigma$ be a fuzzy soft topology on $Y$. If $\varphi_\psi : (X, \sigma, E) \rightarrow (Y, \sigma, K)$ is a fuzzy soft mapping, then the fuzzy soft mapping $\varphi_\psi$ is a fuzzy soft continuous mapping.
3. Fuzzy soft connectedness

In this section, we give definition of fuzzy soft connected spaces and their fundamental properties. Moreover, we will introduce SC$_i$-connectedness ($i = 1, 2, 3, 4$).

**Definition 3.1** Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{f} \in \mathcal{F}\mathcal{S}^E_X$. If there are two fuzzy soft proper open sets $\tilde{g}_1$ and $\tilde{g}_2$ such that $\tilde{f} \subseteq \tilde{g}_1 \cup \tilde{g}_2$ and $\tilde{g}_1 \cap \tilde{g}_2 = \emptyset_X$, then the fuzzy soft set $\tilde{f}$ is called fuzzy soft disconnected set. If there does not exist such two fuzzy soft proper open sets, then the fuzzy soft set $\tilde{f}$ is called fuzzy soft connected set.

In above definition, if we take $\tilde{1}_X$ instead of $\tilde{f}$, then the $(X, \tau, E)$ is called fuzzy soft disconnected (connected) space.

**Example 3.2** If we consider the fuzzy soft topological spaces $(X, \tau, E)$ and $(X, \sigma, E)$ in Example 2.13, $(X, \tau, E)$ is a fuzzy soft connected topological space, but $(X, \sigma, E)$ is a fuzzy soft disconnected topological space.

**Theorem 3.3** Let $(X, \tau, E)$ be a fuzzy soft topological space. $(X, \tau, E)$ fuzzy soft connected if and only if there does not exist a fuzzy soft proper clopen set $\tilde{f}$ in $(X, \tau, E)$.

**Proof.** ($\Rightarrow$): Let $(X, \tau, E)$ be a fuzzy soft connected space. Suppose that there exist a fuzzy soft proper clopen set $\tilde{f}$ in $(X, \tau, E)$ such that $\tilde{f} \cup \tilde{f}^c = 1_X$ and $\tilde{f} \cap \tilde{f}^c = 0_X$. It is a contradiction.

($\Leftarrow$): It is clear. ■

**Theorem 3.4** Let $(X, \tau, E)$ be a fuzzy soft connected topological space and $\sigma \subseteq \tau$. Then, $(X, \sigma, E)$ is a fuzzy soft connected topological space.

**Proof.** It is clear. ■

**Theorem 3.5** Let $(X, \tau, E)$ and $(Y, \sigma, K)$ be two fuzzy soft topological spaces, $\tilde{f} \in \mathcal{F}\mathcal{S}^E_X$ and $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be a fuzzy soft continuous function. If $\tilde{f}$ is a fuzzy soft connected set, then $\varphi_\psi(\tilde{f})$ is a fuzzy soft connected set.

**Proof.** Suppose that $\varphi_\psi(\tilde{f})$ is a fuzzy soft disconnected set. Then, there exist two fuzzy soft proper open sets $\tilde{g}$ and $\tilde{h}$ such that $\varphi_\psi(\tilde{f}) \subseteq \tilde{g} \cup \tilde{h}$ and $\tilde{g} \cap \tilde{h} = 0_Y$. Therefore, from Theorem 2.10

$$\tilde{f} \subseteq \varphi^{-1}_\psi(\varphi_\psi(\tilde{f})) \subseteq \varphi^{-1}_\psi(\tilde{g} \cup \tilde{h}) = \varphi^{-1}_\psi(\tilde{g}) \cup \varphi^{-1}_\psi(\tilde{h})$$

and

$$\varphi^{-1}_\psi(\tilde{g} \cap \tilde{h}) = \varphi^{-1}_\psi(\tilde{g}) \cap \varphi^{-1}_\psi(\tilde{h}) = \varphi^{-1}_\psi(0_Y) = 0_X.$$

So, we have a contradiction and this complete the proof. ■

**Theorem 3.6** Let $(X, \tau, E)$ and $(Y, \sigma, K)$ be two fuzzy soft topological spaces and $\varphi_\psi : (X, \tau, E) \to (Y, \sigma, K)$ be a fuzzy soft continuous and surjective function. If $(X, \tau, E)$ is a fuzzy soft connected space, then $(Y, \sigma, K)$ is also a fuzzy soft connected space.

**Proof.** Suppose that $(Y, \sigma, K)$ is a fuzzy soft disconnected space. Therefore, there exist two fuzzy soft proper open sets $\tilde{g}_1$ and $\tilde{g}_2$ such that $\tilde{g}_1 \cup \tilde{g}_2 = 1_Y$, $\tilde{g}_1 \cap \tilde{g}_2 = 0_Y$. By Theorem 2.10 $\varphi^{-1}_\psi(\tilde{g}_1) \cup \varphi^{-1}_\psi(\tilde{g}_2) = 1_X$ and $\varphi^{-1}_\psi(\tilde{g}_1) \cap \varphi^{-1}_\psi(\tilde{g}_2) = 0_X$. So, we have a contradiction and this complete the proof. ■
Definition 3.7 Let \((X, \tau, E)\) be a fuzzy soft topological space. If there exist \(\tilde{f}, \tilde{g} \in \mathbb{FS}_X^E\) which are proper, such that \(cl(\tilde{f}) \cap \tilde{g} = \tilde{0}_X\) and \(\tilde{f} \cap cl(\tilde{g}) = \tilde{0}_X\) then the fuzzy soft sets \(\tilde{f}\) and \(\tilde{g}\) are called fuzzy soft separated sets.

Theorem 3.8 Let \((X, \tau, E)\) be a fuzzy soft topological space, \(\tilde{f}\) and \(\tilde{g}\) be two fuzzy soft open sets. If \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\), then \(\tilde{f}\) and \(\tilde{g}\) are fuzzy soft separated sets.

Proof. Let \(\tilde{f}, \tilde{g} \in \tau\) and \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\). Then, \(\tilde{f}^c \cup \tilde{g}^c = \tilde{1}_X\). Therefore, \(\tilde{f} \subseteq \tilde{g}^c\) and \(\tilde{g} \subseteq \tilde{f}^c\), hence \(\tilde{f}^c\) and \(\tilde{g}^c\) are fuzzy soft closed sets. From Theorem 2.16, we have

\[
cl(\tilde{f}) \subseteq cl(\tilde{g}^c) = \tilde{g}^c \quad \text{and} \quad cl(\tilde{g}) \subseteq cl(\tilde{f}^c) = \tilde{f}^c.
\]

So, \(\tilde{f} \cap cl(\tilde{g}) = \tilde{0}_X\).

Theorem 3.9 Let \((X, \tau, E)\) be a fuzzy soft topological space, \(\tilde{f}\) and \(\tilde{g}\) be two fuzzy soft closed sets. If \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\), then \(\tilde{f}\) and \(\tilde{g}\) are fuzzy soft separated sets.

Proof. Let \(\tilde{f}^c, \tilde{g}^c \in \tau\) and \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\). By Theorem 2.16, \(cl(\tilde{f}) = \tilde{f}\) and \(cl(\tilde{g}) = \tilde{g}\). Therefore we have

\[
\tilde{f} \cap \tilde{g} = cl(\tilde{f}) \cap \tilde{g} = \tilde{0}_X
\]

and

\[
\tilde{f} \cap \tilde{g} = \tilde{f} \cap cl(\tilde{g}) = \tilde{0}_X
\]

So, \(\tilde{f}\) and \(\tilde{g}\) are fuzzy soft separated sets.

Theorem 3.10 Let \((X, \tau, E)\) be a fuzzy soft topological space and \(\tilde{f} \in \mathbb{FS}_X^E\). \(\tilde{f}\) is fuzzy soft connected if and only if it cannot be written as union of fuzzy soft separated sets.

Proof. (\(\Rightarrow\)) If \(\tilde{f} = \tilde{0}_X\), then it is fuzzy soft connected set. Thus, let \(\tilde{f} \neq \tilde{0}_X\). Assume that \(\tilde{f}\) can be written as union of fuzzy soft separated sets \(\tilde{g}\) and \(\tilde{h}\). But this situation contradicts by hypothesis.

(\(\Leftarrow\)) It is clear.

Theorem 3.11 A fuzzy soft topological space \((X, \tau, E)\) is connected if and only if \(\tilde{1}_X\) cannot be written as union of fuzzy soft separated sets.

Proof. (\(\Rightarrow\)) : Suppose that \(\tilde{1}_X\) can be written as union of fuzzy soft separated sets \(\tilde{f}\) and \(\tilde{g}\). Then, \(\tilde{1}_X = \tilde{f} \cup \tilde{g}\), \(cl(\tilde{f}) \cap \tilde{g} = \tilde{0}_X\) and \(\tilde{f} \cap cl(\tilde{g}) = \tilde{0}_X\). Therefore, we have \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\), \(\tilde{f} = \tilde{g}^c\) and \(\tilde{g} = \tilde{f}^c\). Thus,

\[
cl(\tilde{f}) = cl(\tilde{f}) \cap \tilde{1}_X
\]

\[
= cl(\tilde{f}) \cap (\tilde{f} \cup \tilde{g})
\]

\[
= (cl(\tilde{f}) \cap \tilde{f}) \cup (cl(\tilde{f}) \cap \tilde{g})
\]

\[
= \tilde{f}.
\]

So, \(\tilde{f}\) is a fuzzy soft closed set. Similarly, \(\tilde{g}\) is also a fuzzy soft closed set. From \(\tilde{f} = \tilde{g}^c\) and \(\tilde{g} = \tilde{f}^c\), \(\tilde{f}\) and \(\tilde{g}\) are fuzzy soft open sets. This is a contradiction.

(\(\Leftarrow\)) : Suppose that \((X, \tau, E)\) is not a fuzzy soft connected space. Then, there exist a fuzzy soft proper clopen set \(\tilde{f}\). But this contradicts by hypothesis.
**Theorem 3.12** Let \((X, \tau, E)\) be a fuzzy soft topological space and \(\tilde{f} \in \text{FS}_X^E\) be a fuzzy soft open connected set. If \(\tilde{g} \subseteq \tilde{f} \subseteq \text{cl}(\tilde{f})\), then \(\tilde{g}\) is a fuzzy soft connected set.

**Proof.** Suppose that \(\tilde{g}\) is a fuzzy soft disconnected set. Then, there exist two fuzzy soft open proper sets \(\tilde{h}_1\) and \(\tilde{h}_2\) such that
\[
\tilde{h}_1 \cap \tilde{h}_2 = \tilde{0}_X \text{ and } \tilde{g} \subseteq \tilde{h}_1 \cup \tilde{h}_2.
\]
Therefore
\[
\tilde{f} = [\tilde{f} \cap \tilde{h}_1] \cup [\tilde{f} \cap \tilde{h}_2]
\]
and
\[
[\tilde{f} \cap \tilde{h}_1] \cap [\tilde{f} \cap \tilde{h}_2] = \tilde{0}_X.
\]
But it is a contradiction. Thus \(\tilde{g}\) is a fuzzy soft connected set. \(\blacksquare\)

**Remark 1** Let \((X, \tau, E)\) be a fuzzy soft topological space and \(\tilde{f} \in \text{FS}_X^E\) be a fuzzy soft open set. If \(\tilde{f}\) is a fuzzy soft connected set, then \(\text{cl}(\tilde{f})\) is a fuzzy soft connected set.

In following definition, we will introduce fuzzy soft \(C_i\)-connected (or namely \(SC_i\)) spaces \((i = 1, 2, 3, 4)\) by helping of fuzzy \(C_i\)-connectedness [2]. Definition of \(SC_i\)-connected spaces can be seen as extension of fuzzy soft connected space.

**Definition 3.13** Let \((X, \tau, E)\) be a fuzzy soft topological space and \(\tilde{f} \in \text{FS}_X^E\). \(\tilde{f}\) is called

i. \(SC_1\)-connected iff does not exist two non null fuzzy soft open sets \(\tilde{g}\) and \(\tilde{h}\) such that \(\tilde{f} \subseteq \tilde{g} \cup \tilde{h}, \tilde{g} \cap \tilde{h} \subseteq \tilde{f}^c\), \(\tilde{f} \cap \tilde{g} = \tilde{0}_X\) and \(\tilde{f} \cap \tilde{h} = \tilde{0}_X\).

ii. \(SC_2\)-connected iff does not exist two non null fuzzy soft open sets \(\tilde{g}\) and \(\tilde{h}\) such that \(\tilde{f} \subseteq \tilde{g} \cup \tilde{h}, \tilde{f} \cap \tilde{g} \cap \tilde{h} = \tilde{0}_X\) and \(\tilde{f} \cap \tilde{h} = \tilde{0}_X\).

iii. \(SC_3\)-connected iff does not exist two non null fuzzy soft open sets \(\tilde{g}\) and \(\tilde{h}\) such that \(\tilde{f} \subseteq \tilde{g} \cup \tilde{h}, \tilde{g} \cap \tilde{h} \subseteq \tilde{f}^c, \tilde{g} \nsubseteq \tilde{f}^c\) and \(\tilde{h} \nsubseteq \tilde{f}^c\).

iv. \(SC_4\)-connected iff does not exist two non null fuzzy soft open sets \(\tilde{g}\) and \(\tilde{h}\) such that \(\tilde{f} \subseteq \tilde{g} \cup \tilde{h}, \tilde{f} \cap \tilde{g} \cap \tilde{h} = \tilde{0}_X, \tilde{g} \nsubseteq \tilde{f}^c\) and \(\tilde{h} \nsubseteq \tilde{f}^c\).

From Definition 3.13, relations between \(SC_i\)-connectedness \((i = 1, 2, 3, 4)\) can be described by the following diagram:

\[
\begin{array}{ccc}
SC_1 & \longrightarrow & SC_2 \\
\downarrow & & \downarrow \\
SC_3 & \longrightarrow & SC_4
\end{array}
\]

In the following examples, we illustrate all reverse implications.

**Example 3.14** Let \(X = [0, 1]\) and \(E = \{a, b\}\). Moreover, define soft sets \(\tilde{f}, \tilde{g}\) and \(\tilde{h}\) as
following:

\[
\tilde{f} = \left\{ (a, \{(x, \mu_{f(a)}(x)) : x \in X\}), (b, \{(x, \mu_{f(b)}(x)) : x \in X\}) \right\}
\]

\[
\tilde{g} = \left\{ (a, \{(x, \mu_{g(a)}(x)) : x \in X\}), (b, \{(x, \mu_{g(b)}(x)) : x \in X\}) \right\}
\]

\[
\tilde{h} = \left\{ (a, \{(x, \mu_{h(a)}(x)) : x \in X\}), (b, \{(x, \mu_{h(b)}(x)) : x \in X\}) \right\},
\]

where

\[
\mu_{g(a)}(x) = \begin{cases} 
\frac{4}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\
1, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

and

\[
\mu_{g(b)}(x) = \begin{cases} 
1, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{4}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\[
\mu_{h(a)}(x) = \begin{cases} 
1, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{4}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

and

\[
\mu_{h(b)}(x) = \begin{cases} 
1, & \text{if } \frac{1}{3} < x \leq 1 \\
0, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\[
\mu_{f(a)}(x) = \mu_{f(b)}(x) = \frac{3}{4}
\]

for all \(x \in [0, 1]\). \(\tau = \{\tilde{0}_X, \tilde{1}_X, \tilde{g}, \tilde{h}, \tilde{g} \sqcup \tilde{h}\}\) is a fuzzy soft topology on \(X\). It can be seen clearly that \(\tilde{f}\) is \(SC_4\)–connected but \(SC_3\)–disconnected.

**Example 3.15** Let \(X = [0, 1]\) and \(E = \{a, b\}\). Moreover, define soft sets \(\tilde{g}, \tilde{h}\) and \(\tilde{f}\) as following:

\[
\tilde{g} = \left\{ (a, \{(x, \mu_{g(a)}(x)) : x \in X\}), (b, \{(x, \mu_{g(b)}(x)) : x \in X\}) \right\}
\]

\[
\tilde{h} = \left\{ (a, \{(x, \mu_{h(a)}(x)) : x \in X\}), (b, \{(x, \mu_{h(b)}(x)) : x \in X\}) \right\},
\]

\[
\tilde{f} = \tilde{g} \sqcup \tilde{h}
\]

where

\[
\mu_{g(a)}(x) = \begin{cases} 
0, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{4}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

and

\[
\mu_{g(b)}(x) = \begin{cases} 
\frac{1}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\
0, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\[
\mu_{h(a)}(x) = \begin{cases} 
\frac{4}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\
0, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

and

\[
\mu_{h(b)}(x) = \begin{cases} 
0, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{4}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\(\tau = \{\tilde{0}_X, \tilde{1}_X, \tilde{g}, \tilde{h}, \tilde{g} \sqcup \tilde{h}\}\) is a fuzzy soft topology on \(X\). It can be seen clearly that \(\tilde{f}\) is \(SC_4\)–connected but \(SC_2\)–disconnected.

**Example 3.16** Let \(X = [0, 1]\) and \(E = \{a, b\}\). Moreover, define soft sets \(\tilde{f}, \tilde{g}\) and \(\tilde{h}\) as
following:

\[
\tilde{f} = \left\{ (a, \{(x, \mu_{f(a)}(x)) : x \in X\}), (b, \{(x, \mu_{f(b)}(x)) : x \in X\}) \right\} \\
\tilde{g} = \left\{ (a, \{(x, \mu_{g(a)}(x)) : x \in X\}), (b, \{(x, \mu_{g(b)}(x)) : x \in X\}) \right\} \\
\tilde{h} = \left\{ (a, \{(x, \mu_{h(a)}(x)) : x \in X\}), (b, \{(x, \mu_{h(b)}(x)) : x \in X\}) \right\},
\]

where

\[
\mu_{g(a)}(x) = \begin{cases} 
\frac{1}{2}, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{2}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\quad \text{and} \quad
\mu_{g(b)}(x) = \begin{cases} 
\frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\[
\mu_{h(a)}(x) = \begin{cases} 
\frac{2}{3}, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{1}{3}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\quad \text{and} \quad
\mu_{h(b)}(x) = \begin{cases} 
\frac{1}{2}, & \text{if } \frac{1}{3} < x \leq 1 \\
\frac{3}{4}, & \text{if } 0 \leq x \leq \frac{1}{3}
\end{cases}
\]

\[
\mu_{f(a)}(x) = \mu_{f(b)}(x) = \frac{1}{2}
\]

for all \( x \in [0, 1] \). \( \tau = \{0_X, \tilde{I}_X, \tilde{g}, \tilde{h}, \tilde{g} \cap \tilde{h}, \tilde{g} \cup \tilde{h}\} \) is a fuzzy soft topology on \( X \). It can be seen clearly that \( \tilde{f} \) is \( SC_3 \)-connected and \( SC_2 \)-connected but \( SC_1 \)-disconnected.

**Example 3.17** Let \( X = [0, 1] \) and \( E = \{a, b\} \). Moreover, define soft sets \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \) as following:

\[
\tilde{f} = \left\{ (a, \{(x, \mu_{f(a)}(x)) : x \in X\}), (b, \{(x, \mu_{f(b)}(x)) : x \in X\}) \right\} \\
\tilde{g} = \left\{ (a, \{(x, \mu_{g(a)}(x)) : x \in X\}), (b, \{(x, \mu_{g(b)}(x)) : x \in X\}) \right\} \\
\tilde{h} = \left\{ (a, \{(x, \mu_{h(a)}(x)) : x \in X\}), (b, \{(x, \mu_{h(b)}(x)) : x \in X\}) \right\},
\]

where

\[
\mu_{g(a)}(x) = \begin{cases} 
0, & \text{if } \frac{2}{3} < x \leq 1 \\
\frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\quad \text{and} \quad
\mu_{g(b)}(x) = \begin{cases} 
\frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\
0, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\]

\[
\mu_{h(a)}(x) = \begin{cases} 
\frac{1}{2}, & \text{if } \frac{2}{3} < x \leq 1 \\
0, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\quad \text{and} \quad
\mu_{h(b)}(x) = \begin{cases} 
\frac{2}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\
\frac{3}{4}, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\]

\[
\mu_{f(a)}(x) = \begin{cases} 
\frac{1}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\
\frac{2}{3}, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\quad \text{and} \quad
\mu_{f(b)}(x) = \begin{cases} 
\frac{2}{3}, & \text{if } \frac{2}{3} < x \leq 1 \\
\frac{3}{4}, & \text{if } 0 \leq x \leq \frac{2}{3}
\end{cases}
\]
\[ \tau = \{ \tilde{0}_X, \tilde{1}_X, \tilde{g}, \tilde{h}, \tilde{g} \sqcup \tilde{h} \} \] is a fuzzy soft topology on \( X \). It can be seen clearly that \( \tilde{f} \) is \( SC_3 \)-connected but \( SC_2 \)-disconnected and \( SC_1 \)-disconnected.

**Example 3.18** In the Example 3.16, if we take \( \mu_{f(a)}(x) = \mu_{f(b)}(x) = \frac{2}{3} \) for all \( x \in [0, 1] \), then \( \tilde{f} \) is \( SC_2 \)-connected but \( SC_3 \)-disconnected.

**Theorem 3.19** Let \( \varphi : (X, \tau, E) \to (Y, \sigma, K) \) be a fuzzy soft surjective continuous mapping and \( \tilde{f} \in FS_X^E \). If \( \tilde{f} \) is a \( SC_1 \)-connected, then \( \varphi(\tilde{f}) \) is \( SC_1 \)-connected.

**Proof.** Suppose that \( \varphi(\tilde{f}) \) is \( SC_1 \)-connected. Then, there exist two non null fuzzy soft open sets \( \tilde{g} \) and \( \tilde{h} \) in \( (Y, \sigma, K) \) such that

\[
\begin{align*}
\varphi(\tilde{f}) & \subseteq \tilde{g} \sqcup \tilde{h}, \\
\tilde{g} \cap \tilde{h} & \subseteq (\varphi(\tilde{f}))^c, \\
\varphi(\tilde{f}) & \nsubseteq \tilde{0}_Y, \\
\varphi(\tilde{f}) & \nsubseteq \tilde{0}_X.
\end{align*}
\]

Thus, by Theorem 2.10 we have

\[
\begin{align*}
\varphi(\tilde{f}) & \subseteq \tilde{g} \sqcup \tilde{h}, \\
\tilde{g} \cap \tilde{h} & \subseteq (\varphi(\tilde{f}))^c, \\
\varphi(\tilde{f}) & \nsubseteq \tilde{0}_X, \\
\varphi(\tilde{f}) & \nsubseteq \tilde{0}_X.
\end{align*}
\]

But this contradict by hypothesis. So, \( \varphi(\tilde{f}) \) is \( SC_1 \)-connected.

**Theorem 3.20** Let \( \varphi : (X, \tau, E) \to (Y, \sigma, K) \) be a fuzzy soft surjective continuous mapping and \( \tilde{f} \in FS_X^E \). If \( \tilde{f} \) is a \( SC_2 \)-connected, then \( \varphi(\tilde{f}) \) is \( SC_2 \)-connected.

**Proof.** it can be proved similar way to Theorem 3.19.

**Theorem 3.21** Let \( \varphi : (X, \tau) \to (Y, \sigma) \) be fuzzy soft continuous surjective mapping and \( \tilde{f} \in FS_X^E \). If \( \tilde{f} \) is a \( SC_3 \)-connected, then \( \varphi(\tilde{f}) \) is a \( SC_3 \)-connected.

**Proof.** Assume that, \( \varphi(\tilde{f}) \) is not \( SC_3 \)-connected. Then, there exist two non null fuzzy soft open sets \( \tilde{g} \) and \( \tilde{h} \) in \( (Y, \sigma, K) \) such that

\[
\begin{align*}
\varphi(\tilde{f}) & \subseteq \tilde{g} \sqcup \tilde{h}, \\
\tilde{g} \cap \tilde{h} & \subseteq (\varphi(\tilde{f}))^c, \\
\tilde{g} & \nsubseteq (\varphi(\tilde{f}))^c, \\
\tilde{h} & \nsubseteq (\varphi(\tilde{f}))^c.
\end{align*}
\]

By Theorem 2.10,

\[
\tilde{f} \subseteq \varphi^{-1}(\varphi(\tilde{f})) \subseteq \varphi^{-1}(\tilde{g} \sqcup \varphi^{-1}(\tilde{h})) = \varphi^{-1}(\tilde{g}) \sqcup \varphi^{-1}(\tilde{h})
\]
and

$$\varphi^{-1}_\psi (\tilde{g} \cap \varphi^{-1}_\psi (\tilde{h})) = \varphi^{-1}_\psi (\tilde{g}) \cap \varphi^{-1}_\psi (\tilde{h}) \subseteq \tilde{f}^c.$$  

Since, $\tilde{f} \subseteq \varphi^{-1}_\psi (\varphi_\psi (\tilde{f}))$ implies $(\varphi^{-1}_\psi (\varphi_\psi (\tilde{f})))^c \subseteq \tilde{f}^c$ and $\varphi_\psi$ is a fuzzy soft continuous function, so $\varphi^{-1}_\psi (\tilde{g}), \varphi^{-1}_\psi (\tilde{h}) \in \sigma$. Moreover, from $\tilde{g} \nsubseteq (\varphi_\psi (\tilde{f}))^c$ and $\tilde{h} \nsubseteq (\varphi_\psi (\tilde{f}))^c$, there exist $y_1, y_2 \in Y$ such that

$$g_c(y_1) \geq 1 - \varphi_\psi (\tilde{f})(k)(y_1)$$

(1)

$$h_c(y_2) \geq 1 - \varphi_\psi (\tilde{f})(k)(y_2)$$

(2)

We claim that $\varphi^{-1}_\psi (\tilde{g}) \nsubseteq \tilde{f}^c$ and $\varphi^{-1}_\psi (\tilde{h}) \nsubseteq \tilde{f}^c$. To prove the claim, we suppose $\varphi^{-1}_\psi (\tilde{g}) \subseteq \tilde{f}^c$. Clearly, this claim contradicts (1). Similarly, $\varphi^{-1}_\psi (\tilde{h}) \subseteq \tilde{f}^c$ contradicts (2). So, $\varphi_\psi (\tilde{f})$ is $SC_3$–connected.

**Theorem 3.22** Let $\varphi_\psi : (X, \tau) \rightarrow (Y, \sigma)$ be fuzzy soft continuous surjective mapping and $\tilde{f} \in \mathcal{FS}_X$. If $\tilde{f}$ is a $SC_4$–connected, then $\varphi_\psi (\tilde{f})$ is $SC_4$ connected.

**Proof.** It can be proved similarly way in Theorem 3.21.

**Theorem 3.23** Let $(X, \tau, E)$ be a fuzzy soft topological space, $\tilde{f}_1$ and $\tilde{f}_2$ be two $SC_1$–connected fuzzy soft sets such that $\tilde{f}_1 \cap \tilde{f}_2 \neq \emptyset_X$. Then, $\tilde{f}_1 \cup \tilde{f}_2$ is $SC_1$–connected.

**Proof.** It is easy.

**Remark 2** From Theorem 3.23, we can say easily that if $\tilde{f}_1$ and $\tilde{f}_2$ be two $SC_2$–connected fuzzy soft sets such that $\tilde{f}_1 \cap \tilde{f}_2 \neq \emptyset_X$, then $\tilde{f}_1 \cup \tilde{f}_2$ is $SC_2$–connected.

**Theorem 3.24** Let $(X, \tau, E)$ be a fuzzy soft topological space and $\{\tilde{f}_k : k \in \Lambda\} \subseteq \mathcal{FS}_X$ be family of $SC_1$–connected fuzzy soft sets such that $\tilde{f}_i \cap \tilde{f}_j \neq \emptyset_X$ for $i, j \in \Lambda (i \neq j)$. Then, $\bigcup_{k \in \Lambda} \tilde{f}_k$ is is a $SC_1$–connected fuzzy soft set.

**Proof.** It can be proved by using Theorem 3.23.

4. Conclusion

In this study we introduced fuzzy soft connectedness and presented fundamentals properties such as protecting image of connected fuzzy soft set under the fuzzy soft continuous mapping and $SC_1$–connectedness. For future works, we consider to study on intuitionistic fuzzy soft connected sets.

References