Journal of
Linear and Topological Algebra
Vol. 04*, No.* 04*,* 2015*,* 275- 281

Quotient Arens regularity of $L^1(G)$

A. Zivari-Kazempour*

Department of Mathematics, University of Ayatollah Borujerdi,
Borujerd, Iran.

Received 9 December 2015; Revised 18 February 2016; Accepted 28 February 2016.

Abstract. Let $\mathcal{A}$ be a Banach algebra with BAI and E be an introverted subspace of $\mathcal{A}'$. In this paper we study the quotient Arens regularity of $\mathcal{A}$ with respect to E and prove that the group algebra $L^1(G)$ for a locally compact group G, is quotient Arens regular with respect to certain introverted subspace E of $L^\infty(G)$. Some related result are given as well.

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Keywords: Arens product, quotient Arens regular, introverted subspace, weakly almost periodic.

2010 AMS Subject Classification: Primary 46H20; Secondary 46H25.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. It is well-known, on the second dual space $\mathcal{A}''$ of $\mathcal{A}$, there are two multiplications, called the first and second Arens products which make $\mathcal{A}''$ into a Banach algebra, see [1] and [4]. By definition, the first Arens product $\Box$ on $\mathcal{A}''$ is induced by the left $\mathcal{A}$-module structure on $\mathcal{A}$. That is, for each $\Phi, \Psi \in \mathcal{A}''$, $f \in \mathcal{A}'$ and $a, b \in \mathcal{A}$, we have

$$\langle \Phi \Box \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle, \quad \langle \Psi \cdot f, a \rangle = \langle \Psi, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle.$$

Similarly, the second Arens product $\Diamond$ on $\mathcal{A}''$ is defined by considering $\mathcal{A}$ as a right $\mathcal{A}$-module. The Banach algebra $\mathcal{A}$ is said to be Arens regular if $\Box$ and $\Diamond$ coincide on $\mathcal{A}''$.

*Corresponding author.
E-mail address: zivari@abru.ac.ir (A. Zivari-Kazempour).

Print ISSN: 2252-0201
Online ISSN: 2345-5934
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For any fixed $\Phi \in \mathcal{A}''$, the map $\Psi \longmapsto \Psi\Box\Phi$ and $\Psi \longmapsto \Phi\Diamond\Psi$ are w^*-w^* continuous on $\mathcal{A}''$. Thus, with the w^*-topology, $(\mathcal{A}'', \Box)$ is a right topological semigroup and $(\mathcal{A}'', \Diamond)$ is a left topological semigroup. The following sets

$$Z_t^1(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Phi\Box\Psi \textit{ is } w^* - w^* \textit{ continuous on } \mathcal{A}''\},$$

$$Z_t^2(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Psi\Diamond\Phi \textit{ is } w^* - w^* \textit{ continuous on } \mathcal{A}''\},$$

are called the first and the second topological centres of $\mathcal{A}''$, respectively. One can verify that $\mathcal{A}$ is Arens regular if and only if $Z_t^1(\mathcal{A}'') = Z_t^2(\mathcal{A}'') = \mathcal{A}''$. For example, each C^*-algebra is Arens regular and for locally compact group G, the group algebra $L^1(G)$ is Arens regular if and only if G is finite. This was proved for abelian groups G by Civin and Yood [3] and Young [13] extend it for non-abelian case.

A linear functional $f \in \mathcal{A}'$ is said to be almost periodic (weakly almost periodic) if the map $a \longmapsto a \cdot f, \ \ \mathcal{A} \longrightarrow \mathcal{A}'$ is compact (weakly compact). The spaces of almost periodic and weakly almost periodic functionals on the Banach algebra $\mathcal{A}$ are denoted by AP($\mathcal{A}$) and WAP($\mathcal{A}$), respectively. Both AP($\mathcal{A}$) and WAP($\mathcal{A}$) are norm closed $\mathcal{A}$-submodule of $\mathcal{A}'$ and it was shown [5] that $\mathcal{A}$ is Arens regular if and only if WAP($\mathcal{A}$) $= \mathcal{A}'$.

For a detailed account of Arens product and topological centres, we refer the reader to Memoire [5] and [6].

We denote by $LUC(G)$ ($RUC(G)$), the C^*-algebra of bounded left (right) uniformly continuous functions on G. It is well-known that if $\mathcal{A} = L^1(G)$, then $\mathcal{A}' \cdot \mathcal{A} = LUC(G)$ and $\mathcal{A} \cdot \mathcal{A}' = RUC(G)$, see [7] for example. If G is compact, then $LUC(G)$ coincide with $RUC(G)$.

A bounded net $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ is a bounded approximate identity (BAI for short) if, for each $a \in \mathcal{A}$, $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$. An element $\Phi_0 \in \mathcal{A}''$ is called mixed unit if it is a right unit for $(\mathcal{A}'', \Box)$ and a left unit for $(\mathcal{A}'', \Diamond)$. It is well-known that Φ_0 is a mixed unit if and only if it is a weak* cluster point of some BAI in $\mathcal{A}$, [3].

Let $\mathcal{A}$ be a Banach algebra with a BAI and let X be a Banach $\mathcal{A}$-module. Then by Cohen's factorization theorem [7], the set

$$X \cdot \mathcal{A} = \{x \cdot a : \quad x \in X, a \in \mathcal{A}\},$$

is a closed $\mathcal{A}$-submodule of X. Az in [9] we say that X factors in the left if the equality $X = X \cdot \mathcal{A}$ holds.

Throughout the paper we identify an element of a Banach algebra $\mathcal{A}$ with its canonical image in $\mathcal{A}''$.

2. Quotient Arens regularity

Let $\mathcal{A}$ be a Banach algebra and E be a closed $\mathcal{A}$-submodule of $\mathcal{A}'$. Then E is called left introverted (right introverted) if $\Phi \cdot f \in E$ ($f \cdot \Phi \in E$), for all $\Phi \in \mathcal{A}''$ and $f \in E$, and is introverted if it is both left and right introverted. It follows from the Hahn-Banach theorem that E is left introverted if and only if $\Phi \cdot f \in E$, for all $\Phi \in E'$ and $f \in E$, [5]. For example, $\mathcal{A}' \cdot \mathcal{A}$ is left introverted and $\mathcal{A} \cdot \mathcal{A}'$ is right introverted in $\mathcal{A}'$.

Let E be a left introverted Banach $\mathcal{A}$-submodule of $\mathcal{A}'$. Then E' is a Banach algebra by the following (first Arens type) product

$$\langle \Phi \Box \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle \qquad (\Phi, \Psi \in E', \quad f \in E).$$

The Banach algebra $\mathcal{A}$ is said to be left quotient Arens regular with respect to E, if $Z_t(E') = E'$, where

$$Z_t(E') = \{\Phi \in E' : \Psi \longmapsto \Phi \Box \Psi \textit{ is } w^* - w^* \textit{ continuous on } E'\}.$$

If $E = \mathcal{A}'$, then the space $Z_t(E')$ coincides with $Z_t^1(\mathcal{A}'')$. Similarly, if E is a right introverted, the second Arens product on $\mathcal{A}''$ induces naturally a Banach algebra product on E' which is denoted by $\Diamond$. The topological centre and right quotient Arens regularity can be defined analogously. The Banach algebra $\mathcal{A}$ is called quotient Arens regular(=QAR) with respect to E, if $\Phi \Box \Psi = \Phi \Diamond \Psi$ for all $\Phi, \Psi \in E'$. It is clear that if $\mathcal{A}$ is Arens regular, then $\mathcal{A}$ is quotient Arens regular for each introverted subspace E of $\mathcal{A}'$.

The notion of the topological centre $Z_t(E')$ in the above sense was introduced in [8]. In the case where $E = \mathcal{A}' \cdot \mathcal{A}$, the space $Z_t(E')$ was denoted by $\widetilde{Z_1}$ in [9].

Proposition 2.1 Let $\mathcal{A}$ be a Banach algebra with closed subalgebra $\mathcal{B}$. Let E and F be introverted subspace of $\mathcal{A}'$ and $\mathcal{B}'$, respectively. If the restriction map $T : \mathcal{A}' \longrightarrow \mathcal{B}'$ maps E onto F, and $\mathcal{A}$ is QAR with respect to E, then $\mathcal{B}$ is QAR with respect to F.

Proof. This is immediate. ∎

Theorem 2.2 Let $\mathcal{A}$ be a Banach algebra with BAI. If $\mathcal{A}$ is a right ideal in $\mathcal{A}''$, then $\mathcal{A}$ is left QAR with respect to $\mathcal{A}' \cdot \mathcal{A}$.

Proof. Let $E = \mathcal{A}' \cdot \mathcal{A}$, $\Phi, \Psi \in E'$ and $\Psi_\alpha \longrightarrow \Psi$ in w^*-topology. Then for all $f \in \mathcal{A}'$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}
\langle \Phi \Box \Psi_\alpha, f \cdot a \rangle &= \lim_\alpha \langle \Phi, \Psi_\alpha \cdot (f \cdot a) \rangle \\
&= \lim_\alpha \langle \widehat{a} \cdot \Phi, \Psi_\alpha \cdot f \rangle \\
&= \lim_\alpha \langle \Psi_\alpha, f \cdot (a \cdot \Phi) \rangle = \langle \Psi, f \cdot (a \cdot \Phi) \rangle \\
&= \langle \widehat{a} \cdot \Phi, \Psi \cdot f \rangle \\
&= \langle \Phi, \Psi \cdot (f \cdot a) \rangle \\
&= \langle \Phi \Box \Psi, f \cdot a \rangle.
\end{aligned}$$

So $\Phi \Box \Psi_\alpha \longrightarrow \Phi \Box \Psi$ in w^*-topology of E', thus $\mathcal{A}$ is left QAR with respect to E. ∎

One can verify that if $\mathcal{A}$ is a left ideal in $\mathcal{A}''$, then $\mathcal{A}$ is right QAR with respect to $\mathcal{A} \cdot \mathcal{A}'$, hence $\mathcal{A}$ is QAR with respect to $\mathcal{A}' \cdot \mathcal{A} = \mathcal{A} \cdot \mathcal{A}'$, if $\mathcal{A}$ is an ideal in the second dual.

Theorem 2.3 The group algebra $L^1(G)$ for a locally compact group G is QAR with respect to $LUC(G)$ if and only if G is compact.

Proof. Suppose G is compact and $\mathcal{A} = L^1(G)$. Then $\mathcal{A}$ is an ideal in $\mathcal{A}''$ by Lemma 4.1 of [11]. Therefore by above Theorem $\mathcal{A}$ is QAR with respect to $LUC(G)$.

Conversely, let $\mathcal{A}$ be QAR with respect to $LUC(G)$. Then by Theorem 3.6 of [9] we have

$$WAP(\mathcal{A}) = \mathcal{A}' \cdot \mathcal{A} = LUC(G).$$

Thus, G is compact by Corollary 3.8 of [9]. ■

Let G be an infinite compact group and $\mathcal{A} = L^1(G)$. Then by above theorem $\mathcal{A}$ is QAR with respect to $LUC(G)$, but it is not Arens regular. Now let $\mathcal{U} = \mathcal{A}\widehat{\otimes}\mathcal{A}$, the projective tensor product of $\mathcal{A}$ and $\mathcal{A}$. Since $\mathcal{A}$ is not Arens regular, it follows from Corollary 3.5 of [10] that $\mathcal{U}$ is not Arens regular, but it is a right ideal in the second dual by Theorem 5.3 of [11]. Therefore $\mathcal{U}$ is left QAR with respect to $\mathcal{U}' \cdot \mathcal{U}$, by Theorem 2.2.

Theorem 2.4 Let G be a locally compact group and $\mathcal{A} = L^1(G)$. Suppose E is a closed $\mathcal{A}$-submodule of $\mathcal{A}'$ and $E \subseteq Wap(G)$. Then $\mathcal{A}$ is QAR with respect to E.

Proof. See Theorem 8.13 of [6]. ■

As an consequence of above Theorem we have the next result.

Corollary 2.5 Let E denotes one of the space $C_0(G)$, $AP(\mathcal{A})$ or $WAP(\mathcal{A})$, where $\mathcal{A} = L^1(G)$. Then $\mathcal{A}$ is QAR with respect to E.

Proposition 2.6 Suppose $\mathcal{A}$ is a Banach algebra with BAI and E is an introverted subspace of $\mathcal{A}'$. If $\mathcal{A}$ is a right ideal in E' and $E = E \cdot \mathcal{A}$, then $\mathcal{A}$ is QAR with respect to E.

Proof. Let $\Phi, \Psi \in E'$ and $\Psi_\alpha \longrightarrow \Psi$ in w^*-topology of E'. Let $f \in E$, since $E = E \cdot \mathcal{A}$, there exist $g \in E$ and $a \in \mathcal{A}$ such that $f = g \cdot a$. Then we get

$$\begin{aligned}
\langle \Phi \Box \Psi_\alpha, f \rangle &= \lim_\alpha \ \langle \Phi \Box \Psi_\alpha, g \cdot a \rangle \\
&= \lim_\alpha \langle \widehat{a} \cdot \Phi, \Psi_\alpha \cdot g \rangle \\
&= \lim_\alpha \langle \Psi_\alpha, g \cdot (a \cdot \Phi) \rangle = \langle \Psi, g \cdot (a \cdot \Phi) \rangle \\
&= \langle \widehat{a} \cdot \Phi, \Psi \cdot g \rangle \\
&= \ \langle \Phi, \Psi \cdot (g \cdot a) \rangle \\
&= \ \langle \Phi \Box \Psi, f \rangle.
\end{aligned}$$

Thus $\mathcal{A}$ is QAR with respect to E. ■

Let $\mathcal{A}$ be a Banach algebra. We recall that a bounded linear operator $T : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a right multiplier if, for all $a, b \in \mathcal{A}$, $T(ab) = aT(b)$. We denote by $RM(\mathcal{A})$ the set of all right multipliers of $\mathcal{A}$. In [12], Wong proved that $M(\mathcal{A})$, the multiplier algebra of $\mathcal{A}$, is isometrically isometric with $(\mathcal{A}'', \Box)$ if and only if $\mathcal{A}$ is Arens regular, have a BAI and is an ideal in the second dual. Now let $\mathcal{A} = L^1(G)$ for an infinite compact group G. Since $\mathcal{A}$ is not regular, thus $M(\mathcal{A})$ dose not isometrically isometric with $\mathcal{A}''$.

The following result generalized Wong's Theorem on introverted subspace.

Theorem 2.7 Let $\mathcal{A}$ be a Banach algebra with BAI bounded by 1 and let E be an introverted subspace of $\mathcal{A}'$. Then $RM(\mathcal{A})$ is isometrically isometric with $(E', \Box)$ if and only if E factors on the left and $\mathcal{A} \cdot E' \subset \mathcal{A}$.

Proof. Suppose $RM(\mathcal{A})$ is isometrically isometric with $(E', \Box)$. Then $(E', \Box)$ is unital and so E factors on the left. Since $\mathcal{A}$ is an ideal in $RM(\mathcal{A})$, the inclusion $\mathcal{A} \cdot E' \subset \mathcal{A}$ follows.

For the converse let $(e_\alpha)_{\alpha \in I}$ be a BAI in $\mathcal{A}$ bounded by one and let Φ_0 be a corresponding mixed unit of it in E' such that $\|\Phi_0\| = 1$. Define

$$\theta : RM(\mathcal{A}) \longrightarrow E', \quad \theta(T) = T''(\Phi_0).$$

Then θ is a continuous homomorphism. Since E factors on the left, we have

$$\|\theta(T)\| \leqslant \|T\|, \qquad (T \in RM(\mathcal{A})).$$

On the other hand for all $a \in \mathcal{A}$ and $f \in E$, we have

$$\begin{aligned}
\langle \widehat{a} \cdot \theta(T), f \rangle &= \langle \theta(T), f \cdot a \rangle \\
&= \langle T''(\Phi_0), f \cdot a \rangle \\
&= \langle \Phi_0, (f \cdot a)T \rangle \\
&= \lim_{\alpha} \langle f, T(ae_\alpha) \rangle = \langle f, T(a) \rangle \\
&= \langle \widehat{T(a)}, f \rangle.
\end{aligned}$$

Thus, $\widehat{a} \cdot \theta(T) = \widehat{T(a)}$. So

$$\|T(a)\| = \|\widehat{a} \cdot \theta(T)\| \leqslant \|a\| \|\theta(T)\|,$$

hence $\|T\| \leqslant \|\theta(T)\|$ and θ is isometry. Now for all $\Phi \in E'$, define

$$T(a) = \widehat{a} \cdot \Phi, \qquad (a \in \mathcal{A}).$$

Then $T \in RM(\mathcal{A})$ and we deduce

$$\begin{aligned}
\langle \theta(T), f \cdot a \rangle &= \langle T''(\Phi_0), f \cdot a \rangle \\
&= \lim_{\alpha} \langle f \cdot a, T(e_\alpha) \rangle \\
&= \lim_{\alpha} \langle f, \widehat{ae_\alpha} \cdot \Phi \rangle \\
&= \langle \widehat{a} \cdot \Phi, f \rangle \\
&= \langle \Phi, f \cdot a \rangle.
\end{aligned}$$

Thus, $\theta(T) = \Phi$ and θ is onto. This complete the proof. ■

Let $\mathcal{A} = L^1(G)$ and $E = C_0(G)$. Then all conditions of Theorem 2.7 are valid, so we deduce the following corollary which is due to J. Wendel [4].

Corollary 2.8 Let G be a locally compact group. Then

$$RM(L^1(G)) = M(G).$$

Theorem 2.9 Let $\mathcal{A}$ be a Banach algebra and E be an introverted subspace of $\mathcal{A}'$. Then $\mathcal{A}$ is QAR with respect to E if and only if the map $T_f : \mathcal{A} \longrightarrow E$, $a \longmapsto f \cdot a$ is weakly compact.

Proof. Suppose $\mathcal{A}$ is QAR with respect to E, $\Phi \in E'$ and $a_\alpha \longrightarrow \Phi$ in w^*-topology of E'. Then for all $\Psi \in E'$ and $f \in E$ we get

$$\langle \Psi, f \cdot a_\alpha \rangle = \langle \widehat{a_\alpha}, \Psi \cdot f \rangle \longrightarrow \langle \Phi, \Psi \cdot f \rangle = \langle \Phi \Box \Psi, f \rangle = \langle \Phi \Diamond \Psi, f \rangle = \langle \Psi, f \cdot \Phi \rangle.$$

Thus, $f \cdot a_\alpha \longrightarrow f \cdot \Phi$ in w-topology of E', that is T_f is weakly compact.

Conversely, let $\Phi \in E'$ and $a_\alpha \longrightarrow \Phi$ in w^*-topology of E'. Then $f \cdot a_\alpha$ tend to $f \cdot \Phi$ in w-topology, and so for all $\Psi \in E'$ we have

$$\langle \Psi, f \cdot a_\alpha \rangle \longrightarrow \langle \Psi, f \cdot \Phi \rangle.$$

Therefore $\widehat{a_\alpha} \Box \Psi \longrightarrow \Phi \Diamond \Psi$. It follows that $\Phi \Box \Psi = \Phi \Diamond \Psi$ and $\mathcal{A}$ is QAR with respect to E. ■

As an consequence of above Theorem we have the next result.

Corollary 2.10 Let G be a locally compact group. Then G is compact if and only if $T_f : L^1(G) \longrightarrow LUC(G)$, $g \longmapsto f \star g$ is weakly compact.

Theorem 2.11 Let $\mathcal{A}$ be a Banach algebra and E be a closed $\mathcal{A}$-submodule of $\mathcal{A}'$. Then $E \subseteq WAP(\mathcal{A})$ if and only if $\mathcal{A}$ is QAR with respect to E.

Proof. Suppose $E \subseteq WAP(\mathcal{A})$, then E is introverted by Proposition 5.7 of [5]. Let $f \in E$, take

$$K = \{f \cdot a : \quad \|a\| \leqslant 1\}.$$

Then the w-closure K in $\mathcal{A}'$ is weakly compact, because f is weakly almost periodic. Since K is Hausdorff in w^*-topology, the weak and w^*-topologies agree on K and both of them coincide with the norm topology on K, by the Mazur's Theorem. Now let $\Psi_\alpha \longrightarrow \Psi$ in w^*-topology of E', then

$$\langle \Psi_\alpha \cdot f, a \rangle = \langle \Psi_\alpha, f \cdot a \rangle \longrightarrow \langle \Psi, f \cdot a \rangle = \langle \Psi \cdot f, a \rangle.$$

Since $\Psi_\alpha \cdot f \longrightarrow \Psi \cdot f$, for all $a \in \mathcal{A}$, thus $\Psi_\alpha \cdot f \longrightarrow \Psi \cdot f$ in w-topology. So $\Phi \Box \Psi_\alpha \longrightarrow \Phi \Box \Psi$ for each $\Phi \in E'$ and $\mathcal{A}$ is QAR with respect to E.

Conversely, assume that $\mathcal{A}$ is QAR with respect to E and let $f \in E$, then the map

$$T : E' \longrightarrow E, \quad \Phi \longmapsto f \cdot \Phi$$

is w^*-w continuous. Thus, the set $S = \{f \cdot \Phi : \ \|\Phi\| \leqslant 1\}$ is relatively weakly compact in E. Since $K \subset S$, it follows that K is relatively weakly compact and hence f is almost periodic, as required. ■

The proof of the next result is immediate and we omit it.

Proposition 2.12 Let $\mathcal{A}$ be a Banach $*$-algebra and E be a introverted subspace of $\mathcal{A}'$. If the involution of $\mathcal{A}$ can be extend to $(E', \Box)$, then $\mathcal{A}$ is QAR with respect to E.

A locally compact group G is said an SIN-group if the identity e of G has a basis consisting of compact sets invariant under inner automorphisms. It was shown that G is SIN-group if and only if $LUC(G) = RUC(G)$.

Remark 1 *1) Let G be a locally compact SIN-group, and $\mathcal{A} = L^1(G)$. Let $(LUC(G)', \Box)$ has an involution extending the natural involution of $\mathcal{A}$, then by proposition* 2.12*, we have $Z_t(LUC(G)', \Box) = LUC(G)'$. Therefore G is compact by Theorem 2.3.*
2) Let G be any totally bounded topological group. Then by Corollary 4.11 *of [2] we have $LUC(G) = Wap(G)$. Since the multiplication on $Wap(G)'$ is w^*-w^* continuous, hence $Z_t(LUC(G)', \Box) = LUC(G)'$.*

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