Some results on higher numerical ranges and radii of quaternion matrices

Gh. Aghamollaei\textsuperscript{a}, N. Haj Aboutalebi\textsuperscript{b}

\textsuperscript{a}Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.
\textsuperscript{b}Department of Mathematics, Shahrood Branch, Islamic Azad University, Shahrood, Iran.

Received 20 December 2015; Revised 5 February 2016; Accepted 11 March 2016.

Abstract. Let $n$ and $k$ be two positive integers, $k \leq n$ and $A$ be an $n \times n$ square quaternion matrix. In this paper, some results on the $k-$numerical range of $A$ are investigated. Moreover, the notions of $k-$numerical radius, right $k-$spectral radius and $k-$norm of $A$ are introduced, and some of their algebraic properties are studied.

$\copyright$ 2015 IAUCTB. All rights reserved.

Keywords: $k-$Numerical radius; right $k-$spectral radius; $k-$norm, quaternion matrices.

2010 AMS Subject Classification: 15A60; 15B33; 15A18.

1. Introduction and preliminaries

As usual, let $\mathbb{R}$ and $\mathbb{C}$ denote the field of the real and complex numbers, respectively. Moreover, let $\mathbb{H}$ be the four-dimensional algebra of quaternions over $\mathbb{R}$ with the standard basis $\{1, i, j, k\}$ and multiplication rules:

\[ i^2 = j^2 = k^2 = -1, \]
\[ ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \quad \text{and} \]
\[ 1q = q1 = q \quad \text{for all} \ q \in \{1, i, j, k\}. \]

*Corresponding author.
E-mail address: aghamollaei@uk.ac.ir ; aghamollaei1976@gmail.com (Gh. Aghamollaei).
If $q \in \mathbb{H}$, then there are $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that
\[ q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k. \]

This representation of $q$ is called the canonical form of $q$. We define $Re \ q = \alpha_0$, the real part of $q$; $Co \ q = \alpha_0 + \alpha_1 i$, the complex part of $q$; $Im \ q = \alpha_1 i + \alpha_2 j + \alpha_3 k$, the imaginary part of $q$; $\bar{q} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$, the conjugate of $q$; $|q| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2} = (qq)^{\frac{1}{2}} = (\bar{q}q)^{\frac{1}{2}}$, the norm of $q$. Moreover, the set of all $q \in \mathbb{H}$ with $Re \ q = 0$ is denoted by $\mathbb{P}$, and $q \in \mathbb{H}$ is called a unit quaternion if $|q| = 1$.

Two quaternions $x$ and $y$ are said to be similar, denoted by $x \sim y$, if there exists a nonzero quaternion $q \in \mathbb{H}$ such that $x = q^{-1}yq$. It is known, e.g., see [4, Theorem 2.2], that $x \in \mathbb{H}$ is similar to $y \in \mathbb{H}$ if and only if $Re \ x = Re \ y$ and $|Im \ x| = |Im \ y|$. Obviously, $\sim$ is an equivalence relation on the quaternions. The equivalence class containing $x$ is denoted by $[x]$.

Let $\mathbb{H}^n$ be the collection of all $n$-column vectors with entries in $\mathbb{H}$, and $M_{m \times n}(\mathbb{H})$ (for the case $m = n$, $M_n(\mathbb{H})$) be the set of all $m \times n$ quaternion matrices. For any $m \times n$ quaternion matrix $A = (a_{ij}) \in M_{m \times n}(\mathbb{H})$, we define $\bar{A} = (\bar{a}_{ij}) \in M_{m \times n}(\mathbb{H})$, the conjugate of $A$; $A^T = (a_{ji}) \in M_{n \times m}(\mathbb{H})$, the transpose of $A$; $A^* = (\bar{a})^T \in M_{n \times m}(\mathbb{H})$, the conjugate transpose of $A$.

Let $A \in M_n(\mathbb{H})$. The matrix $A$ is said to be normal if $A^* A = AA^*$; Hermitian if $A^* = A$; skew-Hermitian if $A^* = -A$; and unitary if $A^* A = I_n$, where $I_n$ is the $n \times n$ identity matrix. A quaternion $\lambda$ is called a (right) eigenvalue of $A$ if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. The set of all right eigenvalues of $A$ is denoted by $\sigma_r(A)$; i.e., the right spectrum of $A$. Also, the right spectral radius of $A$ is defined as $\rho_r(A) = \max \{|z| : z \in \sigma_r(A)\}$.

If $\lambda$ is an eigenvalue of $A$, then any element in $[\lambda]$ is also an eigenvalue of $A$. Moreover, it is known, e.g., see [4, Theorem 5.4], that $A$ has, counting multiplicities, exactly $n$ (right) eigenvalues which are complex numbers with nonnegative imaginary parts. These eigenvalues are called the standard right eigenvalues of $A$.

Throughout the paper, we assume that $k$ and $n$ are positive integers, and $k \leq n$. A matrix $X \in M_{n \times k}(\mathbb{H})$ is called an isometry if $X^*X = I_k$, and the set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n \times k}$. For the case $k = n$, $\mathcal{X}_{n \times n}$ is denoted by $\mathcal{U}_n$ which is the set of all $n \times n$ quaternionic unitary matrices. For $A \in M_n(\mathbb{H})$, the notion of $k$-numerical range of $A$ which was first introduced in [1], is defined and denoted by
\[
W^k(A) = \left\{ \frac{1}{k} tr(X^*AX) : X \in \mathcal{X}_{n \times k} \right\}. \tag{1}
\]

The sets $W^k(A)$, where $k \in \{1, 2, \ldots, n\}$, are generally called the higher numerical ranges of $A$. Let $A$ have the standard right eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicities. The right $k$–spectrum of $A$ is defined and denoted by
\[
\sigma_r^k(A) = \left\{ \frac{1}{k} \sum_{j=1}^{k} \alpha_{ij} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \ \alpha_{ij} \in [\lambda_{ij}] \right\}.
\]

Obviously, if $\alpha \in \sigma_r^k(A)$, then $[\alpha] \subseteq \sigma_r^k(A)$. Moreover, $\sigma_r^k(A) \subseteq W^k(A)$, $\sigma_r^k(A) = \sigma_r(A)$, and
\[
W^1(A) = W(A) := \{ x^*Ax : x \in \mathbb{H}^n, x^*x = 1 \}
\]
is the standard quaternionic numerical range of $A$, which was first studied in 1951 by Kippenhahn [2]. The numerical radius of $A$ is also defined as $r(A) = \max\{|z| : z \in W(A)\}$. Now, in the following theorem, we list some other properties of the $k$–numerical range of quaternion matrices which can be found in [1].

**Theorem 1.1** Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:

(a) $W^k(\alpha I + \beta A) = \alpha + \beta W^k(A)$, where $\alpha, \beta \in \mathbb{R}$;

(b) $W^k(A + B) \subseteq W^k(A) + W^k(B)$, where $B \in M_n(\mathbb{H})$;

(c) $W^k(U^*AU) = W^k(A)$, where $U \in U_n$;

(d) $aW^k(A) = W^k(A)$, where $a \in \mathbb{H}$ is such that $|a| = 1$;

(e) $W^k(A^*) = W^k(A)$;

(f) $W^{k+1}(A) \subseteq \text{conv}(W^k(A))$;

(g) $W^n(A) \subseteq \mathbb{R}$ if and only if $A$ is Hermitian;

(h) $W^n(A) = \{\frac{1}{n} \text{tr} A\}$ if and only if $A$ is Hermitian.

In this paper, we are going to study some properties of the $k$–numerical ranges and radii of quaternionic matrices. To this end, in the next section, we state some other properties of the $k$–numerical range of quaternion matrices. We also introduce and study, as in the complex case, the notions of right $k$–spectral, $k$–numerical radius and the $k$–norm of quaternion matrices. Moreover, we establish some relations among them.

2. **Main results**

We begin this section by a result about quaternion numbers which is important to study some properties of the $k$–numerical range of quaternion matrices.

**Theorem 2.1** Let $S \subseteq \mathbb{H}$ be such that $\lambda \in S$ implies that $|\lambda| \subseteq S$. Then

$$\text{conv}(\mathbb{C} \cap S) = \mathbb{C} \cap \text{conv}(S).$$

**Proof.** It is clear that $\text{conv}(\mathbb{C} \cap S) \subseteq \mathbb{C} \cap \text{conv}(S)$. Conversely, let $\lambda = \sum_{l=1}^{m} \theta_l(a_l + b_l i + c_l j + d_l k) \in \mathbb{C} \cap \text{conv}(S)$, where $\theta_l \geq 0$, $\sum_{l=1}^{m} \theta_l = 1$, and $a_l + b_l i + c_l j + d_l k \in S$ for all $l = 1, \ldots, m$. Thus, we have

$$\lambda = \sum_{l=1}^{m} \theta_l(a_l + b_l i), \text{ and } \sum_{l=1}^{m} \theta_l(c_l j + d_l k) = 0.$$ 

Since $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in [a_l + b_l i + c_l j + d_l k]$, by our assumption, we have $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in \mathbb{C} \cap S$ for all $l = 1, \ldots, m$. So, for every $l \in \{1, \ldots, m\}$, we have

$$a_l + b_l i = t(a_l + i\sqrt{b_l^2 + c_l^2 + d_l^2}) + (1 - t)(a_l - i\sqrt{b_l^2 + c_l^2 + d_l^2}) \in \text{conv}(\mathbb{C} \cap S),$$

where

$$t = \frac{b_l + \sqrt{b_l^2 + c_l^2 + d_l^2}}{2\sqrt{b_l^2 + c_l^2 + d_l^2}}$$

for the case $\sqrt{b_l^2 + c_l^2 + d_l^2} \neq 0$, and for the case $b_l = c_l = d_l = 0$, $t \in [0, 1]$ is arbitrary. Therefore, $\lambda \in \text{conv}(\mathbb{C} \cap S)$. Hence, $\mathbb{C} \cap \text{conv}(S) \subseteq \text{conv}(\mathbb{C} \cap S)$. This completes the proof. 

By Theorem 2.1, we have the following results.
Corollary 2.2 Let $A \in M_n(\mathbb{H})$. Then

$$\text{conv}(C \bigcap W^k(A)) = C \bigcap \text{conv}(W^k(A)).$$

Corollary 2.3 (see also [1, Theorem 2.4(b)]); Let $A \in M_n(\mathbb{H})$. Then

$$\text{conv}(C \bigcap \sigma^k_r(A)) = C \bigcap \text{conv}(\sigma^k_r(A)).$$

Now, we introduce the notions of right $k$–spectral, $k$–numerical radius and the $k$–norm of quaternion matrices. To access more information about the similar results in the complex case, see [3].

Definition 2.4 Let $A \in M_n(\mathbb{H})$. The right $k$–spectral radius, the $k$–numerical radius, and the $k$–norm of $A$ are defined and denoted, respectively, by

$$\rho^{(k)}_r(A) = \max \{|z| : z \in \sigma^k_r(A)\},$$

$$r^{(k)}(A) = \max \{|z| : z \in W^k(A)\}, \text{ and}$$

$$\|A\|_{(k)} = \frac{1}{k}\max\{|\text{tr}(X^*AY)| : X, Y \in \mathcal{X}_{n\times k}\}.$$  

It is clear that $\rho^{(1)}_r(A) = \rho_r(A)$ and $r^{(1)}(A) = r(A)$. So, the notions of right $k$-spectral radius and $k$-numerical radius are generalizations of the classical spectral radius and numerical radius, respectively. In the following theorem, we state some basic properties of $r^{(k)}(.)$.

Theorem 2.5 Let $A, B \in M_n(\mathbb{H})$ and $c \in \mathbb{R}$. Then the following assertions are true:

(a) $r^{(k)}(A) \geq 0$;
(b) $r^{(k)}(cA) = |c|r^{(k)}(A)$;
(c) $r^{(k)}(U^*AU) = r^{(k)}(A)$, where $U \in \mathcal{U}_n$;
(d) $r^{(k)}(A) = r^{(k)}(A^*)$;
(e) Let $k < n$. Then $r^{(k)}(A) = 0$ if and only if $A = 0$. For the case $k = n$, $r^{(n)}(A) = 0$ if and only if $A$ is Hermitian and $\text{tr}A = 0$;
(f) $r^{(k)}(A + B) \leq r^{(k)}(A) + r^{(k)}(B)$;
(g) $r^{(n)}(A) \leq r^{(n-1)}(A) \leq \cdots \leq r^{(1)}(A) = r(A)$.

Proof. The part (a) follows from Definition 2.4. The parts (b), (c), (d) and (f) follow easily from Theorem 1.1. 

To prove (e), at first, we assume that $r^{(k)}(A) = 0$ and $k < n$. We will show that $A = 0$. Since $r^{(k)}(A) = 0$, for any $z \in W^k(A)$, $|z| = 0$. Therefore, $W^k(A) = \{0\}$, and hence, by Theorem 1.1(g), $A$ is Hermitian. Now, since $k < n$, by a simple calculation we see that $A = 0$. The converse is trivial. For the case $k = n$, let $A$ have the standard right eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicities, and $r^{(n)}(A) = 0$. Then $W^n(A) = \{0\}$. Since $\frac{1}{n}\text{tr}A \in \sigma^n(A)$, by [1, Theorem 2.5(e)], $W^n(A) = \{0\} = \left\{\frac{1}{n}\text{tr}A\right\}$. So, $\text{tr}A = 0$ and also by Theorem 1.1(h), $A$ is Hermitian. The converse is trivial.

To prove (g), let $1 < k \leq n$ be given. Then by Theorem 1.1(f), we have $W^k(A) \subseteq \text{conv}(W^{k-1}(A))$. Now, let $r^{(k)}(A) = |\mu|$ for some $\mu \in W^k(A)$. Hence, $\mu \in \text{conv}(W^{k-1}(A))$. 

Then there are nonnegative real numbers \( t_1, \ldots, t_n \in \mathbb{R} \) summing to 1, and \( \alpha_1, \ldots, \alpha_n \in W^{k-1}(A) \) such that \( \mu = \sum_{i=1}^n t_i \alpha_i \). Therefore,

\[
\rho_r^{(k)}(A) = |\mu| \leq \sum_{i=1}^n t_i |\alpha_i| \leq \sum_{i=1}^n t_i r^{(k-1)}(A) = r^{(k-1)}(A).
\]

This completes the proof. \( \square \)

Using Definition 2.4 and this fact that \( \sigma_r^{(k)}(A) \subseteq W^k(A) \), we have the following result which states the relation between \( \rho_r^{(k)}(.) \), \( r^{(k)}(.) \) and \( \|\cdot\|_{(k)} \).

**Proposition 2.6** Let \( A \in M_n(\mathbb{H}) \). Then

\[
\rho_r^{(k)}(A) \leq r^{(k)}(A) \leq \|A\|_{(k)}.
\]

The following example shows that in Proposition 2.6, the equality \( \rho_r^{(k)}(A) = r^{(k)}(A) \) does not hold in general.

**Example 2.7** Let \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{H}) \). Then \( A \) is a matrix with eigenvalues 1, 0, 0.

Therefore, \( \rho_r^{(2)}(A) = \frac{1}{2} \). By a simple calculation, we have \( r^{(2)}(A) = 1 \). So, \( \rho_r^{(2)}(A) = \frac{1}{2} < 1 = r^{(2)}(A) \).

In the following proposition, we show that the left inequality in Proposition 2.6 is sharp. It follows easily from [1, Theorem 2.13].

**Proposition 2.8** Let \( A \in M_n(\mathbb{H}) \) be a Hermitian matrix. Then

\[
\rho_r^{(k)}(A) = r^{(k)}(A).
\]

In the following theorem, we state some basic properties of \( \|A\|_{(k)} \).

**Theorem 2.9** Let \( A, B \in M_n(\mathbb{H}) \) and \( c \in \mathbb{R} \). Then the following assertions are true:

(a) \( \|A\|_{(k)} \geq 0 \);
(b) \( \|cA\|_{(k)} = |c|\|A\|_{(k)} \);
(c) Let \( k < n \). Then \( \|A\|_{(k)} = 0 \) if and only if \( A = 0 \);
(d) \( \|A + B\|_{(k)} \leq \|A\|_{(k)} + \|B\|_{(k)} \);
(e) \( \|A\|_{(n)} \leq \|A\|_{(n-1)} \leq \ldots \leq \|A\|_{(1)} \).

**Proof.** The assertions in (a), (b), and (d) follow easily from Definition 2.4.

To prove (c), at first, we assume that \( \|A\|_{(k)} = 0 \) and \( k < n \). Then by Theorem 2.5(e) and Proposition 2.6, we have \( A = 0 \). The converse is trivial.

For (e), let \( 1 < k \leq n \). Moreover, let \( X = [x_1, \ldots, x_n], Y = [y_1, \ldots, y_n] \in \mathcal{X}_{n \times k} \) be given.
Therefore, we have

\[
\frac{1}{k} \left| \sum_{j=1}^{k} x_j^* A y_j \right| = \frac{1}{k} \left| \sum_{j=1}^{k} \frac{1}{k-1} \sum_{i=1}^{k} x_i^* A y_i \right|
\]

\[
\leq \frac{1}{k} \sum_{j=1}^{k} \frac{1}{k-1} \sum_{i=1 \atop i \neq j}^{k} x_i^* A y_i
\]

\[
\leq \frac{1}{k} \sum_{j=1}^{k} \| A \|_{(k-1)}
\]

\[
= \| A \|_{(k-1)}.
\]

So, \( \| A \|_{(k)} \leq \| A \|_{(k-1)} \). This completes the proof. 

\[\blacksquare\]

References


