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A numerical solution of mixed Volterra Fredholm integral equations of Urysohn type on non-rectangular regions using meshless methods

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Abstract. In this paper, we propose a new numerical method for solution of Urysohn two dimensional mixed Volterra-Fredholm integral equations of the second kind on a non-rectangular domain. The method approximates the solution by the discrete collocation method based on inverse multiquadric radial basis functions (RBFs) constructed on a set of disordered data. The method is a meshless method, because it is independent of the geometry of the domain and it does not require any background interpolation or approximation cells. The error analysis of the method is provided. Numerical results are presented, which confirm the theoretical prediction of the convergence behavior of the proposed method.

Keywords: Mixed Volterra-Fredholm integral equations; collocation method; radial basis functions; Meshless method; numerical treatment.

2010 AMS Subject Classification: 45A99, 45A05.

1. Introduction

The nonlinear mixed Volterra-Fredholm integral equation is given by

$$u(x,t) - \int_0^t \int_\Omega F(x,t,\xi,s,u(\xi,s))d\xi ds = f(x,t), \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

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where $u(x,t)$ is an unknown function and $f(x,t)$ and $F(x,t,\xi,s,u)$ are given analytical real valued functions defined on $D = \Omega \times [0,T]$ and $C(D^2 \times \mathbb{R})$, respectively and Ω is a compact subset of $\mathbb{R}^n (n = 1,2,3)$, with convenient norm $\|.\|$.

Existence and uniqueness results for (1) may be found in [4, 5]. This type of equations arise in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of epidemic models and various physical and biological problems. Detailed descriptions and analysis of these models may be found in [12]. Equation (1) can be written in the abstract form

$$u = \mathcal{T}u,$$

where the integral operator $\mathcal{T} : C(\Omega \times [0,T]) \to C(\Omega \times [0,T])$ is defined as

$$\mathcal{T}u(x,t) = \int_0^t \int_\Omega F(x,t,\xi,s,u(\xi,s))d\xi ds + f(x,t). \tag{2}$$

In this paper we assume that Ω is a bounded two-dimensional non-rectangular domain. Few numerical methods have been given for numerical solution of (1). Kauthen [7] presented continuous time collocation method and time discretization collocation methods, and analyzed the discrete convergence properties. The results of Kauthen have been extended to nonlinear Volterra-Fredholm integral equations by Brunner [6]. Also Banifatemi et al. [10] applied two-dimensional Legendre wavelets method to mixed Volterra-Fredholm integral equations. Maleknejad and Hadizadeh [8] and Wazwaz [9] used a technique based on the Adomian decomposition method for solution of (1). Cardone, Messina and Russo applied an iterative method for this equation[20]. Han and Zhang [13] presented a particular Nystrom method for the linear case of (1). The block pulse functions and Chebyshev polynomials have been used for numerical solution of (1)[14, 15]. Moreover, Laeli et al [16] obtained a numerical solution for linear Mixed Volterra-Fredholm integral equations by using the RBFs on the non-rectangular domain. For solving (1) on a non-rectangular region, the domain must be segmented to small triangles and numerical integration over such segments is needed. Triangulations and mesh refinement are major difficulties in these methods. Therefore, to release from triangulations and mesh refinement, we can use methods based upon the scattered data approximation that approximate a function without any mesh generation on the domain. Radial basis functions (RBFs) are probably best known for their applications to problems with scattered data. The Multiquadric (MQ) RBF interpolation method was developed by Roland Hardy who described and named the method in a paper appeared in 1971 [17]. In 1990 Kansa first used the MQ-RBFs method to solve differential equations [18, 19]. Kansa's method was recently extended to solve various ordinary and partial differential equations[21–23].

In this paper, we will use the inverse multiquadric radial basis function approximation for numerical solution of nonlinear mixed Volterra-Fredholm integral equations on a non-rectangular region.

The remainder of the paper is organized as follows: In section 2 we give a brief survey of radial basis functions. In section 3 the proposed method is introduced and applied on equation (1). In Section 4, we provide error analysis for the proposed method. Three examples are solved to demonstrate the validity of our method in Section 5.

2. Radial basis function approximation

In recent years meshless methods have gained considerable attention in engineering and applied mathematics. A meshless method does not require a mesh to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. Radial basis functions (RBFs) is one of the most developed meshless methods that has attracted attention in recent years and form a primary tool for multivariate interpolation[11]. They are also receiving increased attention for solving PDE's on irregular domains. The main advantage of radial basis functions is that they involve a single independent variable regardless of the dimension of the problem.

In the following, we want to consider a general algorithm of the radial basis function approximation, but prior to that we quote the following preliminaries which follow from [1, 26].

Definition 2.1 A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a univariate function $\phi : [0, \infty) \to \mathbb{R}$ such that $\Phi(\mathbf{x}) = \phi(r)$, where $r = \|\mathbf{x}\|$ and $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^d$.

Some of the most popular RBFs are given in Table 1.

Table 1. Some well-known functions that generate RBFs

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2), \quad c > 0$
Multiquadric (MQ)	$\phi(r) = (-1)^{\lceil\beta\rceil}(r^2 + c^2)^{\beta},\ c, \beta > 0, \beta \notin \mathbb{N}$
Inverse multiquadric (IMQ)	$\phi(r) = (r^2 + c^2)^{-\beta},\ c, \beta > 0$
Thin plate splines	$\phi(r) = (-1)^{k+1} r^{2k} \log(r),\ k \in \mathbb{N}$

In the Table 1, $\lceil\beta\rceil$ indicates the smallest integer greater than β.

Definition 2.2 ([1]): The fill distance of a given set $\mathcal{X} = \{x_1, ..., x_n\}$ consisting of pairwise distinct points in Ω can be defined as

$$h_{\mathcal{X},\Omega} = \sup_{x\in\Omega} \min_{x_j\in\mathcal{X}} \|x - x_j\|,$$

which indicates how well the data in the set $\mathcal{X}$ fill out the domain Ω.

Definition 2.3 ([1]) A set Ω is said to satisfy an interior cone condition if there exists an angle $\theta \in (0, \frac{\pi}{2})$ and a radius $r > 0$ such that for every given $x \in \Omega$ a unit vector $\eta(x)$ exists such that the cone

$$C(x, \eta(x), \theta, r) = \{x + \lambda y : y \in \mathbb{R}^d, ||y||_2 = 1, y^T\eta(x) \geqslant \cos(\theta), \lambda \in [0, r]\}$$

is contained in Ω.

Definition 2.4 ([1]) Let H be a real Hilbert space of $u : \Omega \to \mathbb{R}$. A function $K : \Omega\times\Omega \to \mathbb{R}$ is called reproducing kernel for H if

(1) $K(x,.) \in H$ for all $x \in \Omega$
(2) $u(x) =< u, K(x,.) >_H$ for all $u \in H$ and all $x \in \Omega$. If $K(x,y) = \phi(x-y)$, where ϕ is a radial basis function, then it is shown that

$$H_\phi(\Omega) = span\{\phi(.-y) : y \in \Omega\}$$

is the space with the bilinear form

$$< \sum_{j=1}^{N} c_j \phi(.-x_j), \sum_{k=1}^{N} d_k \phi(.-y_k) >_\phi = \sum_{j=1}^{N}\sum_{k=1}^{N} c_j d_k \phi(x_j, y_k) \tag{3}$$

Definition 2.5 ([1]) A real valued continuous even function ϕ is called positive definite on $\mathbb{R}^s$ if

$$\sum_{j=1}^{N}\sum_{k=1}^{N} c_j c_k \phi(x_j - x_k) \geqslant 0, \tag{4}$$

for any N pairwise different points $x_1, x_2, ..., x_N \in \mathbb{R}^s$ and $C = [c_1, c_2, \ldots, c_N]^T \in \mathbb{R}^N$.

The function ϕ is called strictly positive definite on $\mathbb{R}^s$ if the only vector $C = 0$ that turns (4) into an equality is the zero vector.

As a conclusion, if a radial basis function is positive definite function, then the associated interpolation matrix is a positive semi-definite and similarly, a strictly positive definite function is associated with a positive definite interpolation matrix. Therefore, by choosing a strictly positive definite function, the associated interpolation matrix is non-singular. Examples of such functions are inverse multiquadric and Gaussian radial basis functions.

Theorem 2.6 ([26]) Let ϕ be a radial basis and strictly positive definite function. Then the bilinear form $<,>_\phi$ defines an inner product on $H_\phi(\Omega)$. Moreover $H_\phi(\Omega)$ is a pre-Hilbert space with reproducing ϕ.

Definition 2.7 ([1]) The native space $\mathcal{N}_\phi(\Omega)$ of ϕ is now defined to be the completion of $H_\phi(\Omega)$ with respect to the ϕ -norm $\|.\|_\phi$, so that $\|u\|_\phi = \|u\|_{\mathcal{N}_\phi}(\Omega)$ for all $u \in \mathcal{N}_\phi(\Omega)$.

2.1 *Interpolation by radial basis functions*

Radial basis function interpolation to a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ on a set $\{x_1, x_2, \ldots, x_N\}$ starts with a function $\Phi : \mathbb{R}^d \to \mathbb{R}$ that is radial in the sense that $\Phi(x) = \phi(\|x\|)$, where $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^d$. The approximation of a function $u(x)$, using radial basis function $\Phi(x) = \phi(\|x\|)$, may be written as a linear combination, usually it takes the following form

$$\mathcal{P}_N u(x) = \sum_{k=1}^{N} \lambda_k \phi(\|x - x_k\|), \tag{5}$$

where $\mathcal{P}_N : C(\Omega) \to C_N(\Omega)$ is collocation projection operator on the collocation points $\{x_1, x_2, \ldots, x_N\}$, and $C_N(\Omega) = span\{\phi(\|x-x_1\|), \phi(\|x-x_2\|), \ldots, \phi(\|x-x_N\|)\}$ has finite dimension. The interpolation problem is to find $\lambda_i, i = 1, 2, ..., N$ such that the interpolant $\mathcal{P}_N u(x)$ through all data satisfies

$$\mathcal{P}_N u(x_i) = u(x_i), i = 1, 2, ..., N. \tag{6}$$

This interpolation condition leads a linear system

$$A\Lambda = U, \tag{7}$$

where A is $N \times N$ matrix with the elements $A_{i,j} = \phi(\|x_i - x_j\|),\ i, j = 1, 2, ..., N$, and Λ is the vector of coefficients of $\mathcal{P}_N u(x)$ and the components of U are the data $u(x_i), i = 1, 2, ..., N$.

It has been shown that the associated interpolation matrix for inverse Multiquadric RBFs is non-singular[1]. For the theoretical developments of the RBFs in scattered data interpolation, Madych and Nelson [2, 3] showed that the inverse MQ-RBF interpolant employs exponential convergence with minimal semi-norm errors.

3. Implementation of the numerical method

To apply the method, suppose $0 = t_0 < t_1 < ... < t_M = T$ be a scattered set of data in $[0, T]$ and $\{x_0, x_1, ..., x_N\}$ be a scattered set of nodes in Ω. The distribution of nodes could be selected regularly or randomly. We assume that Ω usually has a non-rectangular shape. According to the RBF collocation method, the approximation of the function $u(x, t)$ may be written as a linear combination of inverse multiquadric radial basis functions as follows

$$\mathcal{P}_{M,N} u(x,t) = \sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} \phi_k(x) \eta_j(t), \qquad (x,t) \in \Omega \times [0,T], \tag{8}$$

where

$$\phi_k(x) = \left(\|x - x_k\|^2 + c^2\right)^{-\beta},\ c, \beta > 0$$

$$\eta_j(t) = ((t - t_j)^2 + c^2)^{-\beta},\quad j = 0, ..., M.$$

For simplicity, we can write (8) as

$$\mathcal{P}_{M,N} u(x,t) = \sum_{\mu=1}^{Q} d_\mu \psi_\mu(x,t),\quad (x,t) \in \Omega \times [0,T], \tag{9}$$

where $d_\mu = c_{k,j},\ \psi_\mu(x,t) = \phi_k(x)\eta_j(t)$ and $Q = (M + 1)(N + 1)$.
Note that $\mathcal{P}_{M,N}$ is a collocation projection operator that maps $C(\Omega)$ onto $C_{M,N}$ at the collocation nodes, where $C_{M,N} = span\{\psi_1, \psi_2, \ldots, \psi_Q\}$ has finite dimension. The index μ is determined by $\mu = (N + 1)j + k + 1$.

Now by replacing (9) in (1) we have

$$\sum_{\mu=1}^{Q} d_\mu \psi_\mu(x,t) - \int_0^t \int_\Omega K(x,t,\xi,s,\sum_{\mu=1}^{Q} d_\mu \psi_\mu(\xi,s))d\xi ds = f(x,t), \quad (x,t) \in \Omega \times [0,T]. \tag{10}$$

To approximate the integrals in (10) we can transfer the integration interval $[0,t]$ to the fixed interval $[-1,1]$ by the linear transformation

$$s(t,\theta) = \frac{t}{2}\theta + \frac{t}{2}.$$

Then (10) takes the following form :

$$\sum_{\mu=1}^{Q} d_\mu \psi_\mu(x,t) - \int_{-1}^{1} \int_\Omega \frac{t}{2} K(x,t,\xi,s(t,\theta),\sum_{\mu=1}^{Q} d_\mu \psi_\mu(\xi,s(t,\theta)))d\xi d\theta = f(x,t), \quad (x,t) \in \Omega \times [0,T]. \tag{11}$$

Now, assuming that Ω has a normal non-rectangular shape as

$$\Omega = \{\xi = (\sigma,\tau) \in \mathbb{R}^2 : -1 \leqslant \tau \leqslant 1, v_1(\tau) \leqslant \sigma \leqslant v_2(\tau)\}, \tag{12}$$

equation (11) becomes

$$\sum_{\mu=1}^{Q} d_\mu \psi_\mu(x,t) - \int_{-1}^{1} \int_{-1}^{1} \int_{v_1(\tau)}^{v_2(\tau)} \frac{t}{2} K\big(x,t,\sigma,\tau,s(t,\theta),\sum_{\mu=1}^{Q} d_\mu \psi_\mu(\sigma,\tau,s(t,\theta))\big) d\sigma d\tau d\theta = f(x,t). \tag{13}$$

Then converting the interval $[v_1(\tau), v_2(\tau)]$ to the fixed interval [-1,1] by the following linear transformation:

$$\sigma(\tau,z) = \frac{v_2(\tau) - v_1(\tau)}{2} z + \frac{v_2(\tau) + v_1(\tau)}{2},$$

equation (13) may be restated as:

$$\sum_{\mu=1}^{Q} d_\mu \psi_\mu(x,t) - \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} K_1\big(x,t,\sigma(\tau,z),\tau,s(t,\theta),\sum_{\mu=1}^{Q} d_\mu \psi_\mu(\sigma(\tau,z),\tau,s(t,\theta))\big) dz d\tau d\theta = f(x,t), \tag{14}$$

where

$$K_1(\bullet) = t\frac{(v_2(\tau) - v_1(\tau))}{4} K(\bullet). \tag{15}$$

Then collocating (14) at points $(x_i, t_r), i = 0, ..., N, r = 0, ..., M$, we have

$$\begin{aligned}&\sum_{\mu=1}^{Q} d_\mu \psi_\mu(x_i, t_r)\\ &\quad - \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K_1\big(x_i, t_r, \sigma(\tau, z), \tau, s(t_r, \theta), \sum_{\mu=1}^{Q} d_\mu \psi_\mu(\sigma(\tau, z), \tau, s(t_r, \theta))\big) dz d\tau d\theta \\ &\quad = f(x_i, t_r).\end{aligned} \tag{16}$$

Now using an m point Gauss-Legendre quadrature formula with the points $\{\theta_l\}, \{\tau_p\}, \{z_q\}$ in the interval $[-1, 1]$ and weights $\{w_l\}, \{w_p\}, \{w_q\}$ we may approximate (16) as

$$\begin{gathered}\sum_{\mu=1}^{Q} \hat{d}_\mu \psi_\mu(x_i, t_r) - \\ \sum_{l=1}^{m}\sum_{p=1}^{m}\sum_{q=1}^{m} w_p w_q w_l K_1(x_i, t_r, \sigma(\tau_p, z_q), \tau_p, s(t_r, \theta_l), \sum_{\mu=1}^{Q} \hat{d}_\mu \psi_\mu(\sigma(\tau_p, z_q), \tau_p, s(t_r, \theta_l))) \\ = f(x_i, t_r).\end{gathered} \tag{17}$$

Eq (17) is a nonlinear algebraic system of equations that can be solved by iteration methods, such as Newton's method, to determine the unknown coefficients $\hat{d}_\mu$. So the values of $u(x, t)$ at any point of $\Omega \times [0, T]$ can be approximated by

$$\hat{u}_{M,N}(x, t) = \sum_{\mu=1}^{Q} \hat{d}_\mu \psi_\mu(x, t), \quad (x, t) \in \Omega \times [0, T].$$

Then we consider $\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_d$, where $\Omega_i,\ 1 \leqslant i \leqslant d$ are domains of the form (12). So we may write the final system as

$$\begin{aligned}&\sum_{\mu=1}^{Q} \hat{d}_\mu \psi_\mu(x_i, t_r)\\ &- \sum_{e=1}^{d}\sum_{l=1}^{m}\sum_{p=1}^{m}\sum_{q=1}^{m} w_p w_q w_l K_1\big(x_i, t_r, \sigma_e(\tau_p, z_q), \tau_p, s(t_r, \theta_l), \sum_{\mu=1}^{Q} \hat{d}_\mu \psi_\mu(\sigma_e(\tau_p, z_q), \tau_p, s(t_r, \theta_l))\big)\\ &= f(x_i, t_r).\end{aligned} \tag{18}$$

where

$$\sigma_e(\tau, z) = \frac{v_{2,e}(\tau) - v_{1,e}(\tau)}{2} z + \frac{v_{2,e}(\tau) + v_{1,e}(\tau)}{2}. \tag{19}$$

and $v_{1,e}$ and $v_{2,e}$ are continuous functions on the sub-domain Ω_e.

Finally, for the case

$$\Omega = \{(\sigma, \tau) \in \mathbb{R}^2 : -1 \leqslant \sigma \leqslant 1, v_1(\sigma) \leqslant \tau \leqslant v_2(\sigma)\}, \tag{20}$$

where v_1 and v_2 are continuous functions of σ, the method could be used straightforward similarly by commuting the variables.

4. Convergence analysis

In this section, an error estimate of the applied method for the smooth solutions of (1) will be provided.

Define the approximating operator $\mathcal{T}_n$ on $C(\Omega)$ by

$$\mathcal{T}_n u(x,t) = \sum_{l=1}^{m_n}\sum_{p=1}^{m_n} w_l w_p \frac{t}{2} K(x,t,\xi_p, s(t,\theta_l), u(\xi_p, s(t,\theta_l))) + f(x,t), n \geqslant 1.$$

Following are hypotheses for the approximating operators $\{\mathcal{T}_n \mid n \geqslant 1\}$[24].

- H_1 : $\mathcal{T}$ and $\mathcal{T}_n$ are completely continuous nonlinear operators on $C(\Omega)$ into $C(\Omega)$.
- H_2 : $\mathcal{T}_n$ is a collectively compact family on $C(\Omega)$ i.e., for every bounded set $B \subset C(\Omega)$ the closure of the set $\cup_{n=1}^{\infty} \mathcal{T}_n(B)$ is compact in $C(\Omega)$.
- H_3 :$\mathcal{T}_n$ is pointwise convergent to $\mathcal{T}$ on $C(D)$.
- H_4 : At each point of $C(D)$, $\mathcal{T}_n$ is an equicontinuous family.
- H_5 : $\mathcal{T}$ and $\mathcal{T}_n$ are twice Frechet differentiable on $B(u_0, r) = \{u : \| u - u_0 \| \leqslant r, r > 0\}$ and $\| \mathcal{T}''_n \| \leqslant \alpha < \infty$.

Now, we can represent the following theorem about the error bound for approximating a function $u \in \mathcal{N}_\phi(\Omega)$ by the inverse multiquadric radial basis function[1].

Theorem 4.1 Let $\phi(x)$ be a inverse multiquadric radial basis function. Suppose that $\Omega \subset \mathbf{R}^2$ be open and bounded satisfying an interior cone condition. Denote the interpolant of a function $u \in \mathcal{N}_\phi(\Omega)$ by $\mathcal{P}_{M,N}u$. Then for some constant c,

$$\|u - \mathcal{P}_{M,N}u\|_{L^\infty(\Omega)} \leqslant e^{\frac{-c}{h_{\mathcal{X},D}}} \|u\|_{\mathcal{N}_\phi(\Omega)}. \tag{21}$$

We can rewrite Eq. (17) in the operator form

$$\hat{u}_n = \mathcal{P}_n \mathcal{T}_n \hat{u}_n, \tag{22}$$

In order to obtain the more accurate approximated solution we define

$$\tilde{u}_n = \mathcal{T}_n \hat{u}_n, \tag{23}$$

then we can write

$$\mathcal{P}_n \tilde{u}_n = \hat{u}_n. \tag{24}$$

So, we have

$$\tilde{u}_n = \mathcal{T}_n \mathcal{P}_n \tilde{u}_n. \tag{25}$$

It is worth noting that if $\mathcal{T}_n$ satisfies H1-H5, then $\mathcal{T}_n\mathcal{P}_n$ also satisfies H1-H5[24]. We give the following theorem from [25] that is used to obtain the error analysis of the proposed method.

Theorem 4.2 Assume H1-H4, and u_0 be a solution of (1). Moreover assume that 1 is not an eigenvalue of $\mathcal{T}'(u_0)$, where $\mathcal{T}'(u_0)$ denotes the Frechet derivative of $\mathcal{T}$ at u_0. If H5 is satisfied on $B(u_0, r) \subseteq C(\Omega)$, then u_0 is an isolated solution of (1), of nonzero index. Furthermore, there are ε, Λ such that for every $n \geqslant \Lambda, \mathcal{T}_n\mathcal{P}_n$ has a unique fixed point $\tilde{u}_n$ in $B(u_0, \varepsilon)$. Also, there is a constant λ_1 such that

$$\|\tilde{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant \lambda_1 \|\mathcal{T}u_0 - \mathcal{T}_n\mathcal{P}_n u_0\|, \quad n \geqslant \Lambda. \tag{26}$$

The following theorem gives the convergence of the approximate solution $\hat{u}_n$ to u.

Theorem 4.3 Assume that the conditions of Theorem 4.1 and 4.2 hold, and let $u_0 \in \mathcal{N}_\phi(\Omega)$ be a unique solution of (2). Then there exists $M, \epsilon > 0$ such that for every $n > M$ the method has a unique solution $\hat{u}_n \in B(u_0, \epsilon)$ and

$$\|\hat{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant e^{\frac{-c}{h_{\mathcal{X},\Omega}}} \|u_0\|_{\mathcal{N}_\phi(\Omega)}(1 + \lambda_1\lambda_2\lambda_3) + \lambda_1\lambda_3\|\mathcal{T}u_0 - \mathcal{T}_n u_0\|_{L^\infty(\Omega)}. \tag{27}$$

Proof. Using Theorem (4.2) we have

$$\begin{aligned}\|\tilde{u}_n - u_0\|_{L^\infty(\Omega)} &\leqslant \lambda_1\|\mathcal{T}u_0 - \mathcal{T}_n\mathcal{P}_n u_0\|_{L^\infty(\Omega)} \\ &\leqslant \lambda_1\{\|\mathcal{T}u_0 - \mathcal{T}_n u_0\|_{L^\infty(\Omega)} + \|\mathcal{T}_n(u_0 - \mathcal{P}_n u_0)\|_{L^\infty(\Omega)}\}.\end{aligned} \tag{28}$$

By using H_1, we have that $\mathcal{T}_n$ is uniformly bounded on $C(\Omega)$, for some large n, i.e., there exists some $\lambda_2 > 0$ such that $\|\mathcal{T}_n\| \leqslant \lambda_2 < \infty$. So we may write

$$\|\tilde{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant \lambda_1\{\|\mathcal{T}u_0 - \mathcal{T}_n u_0\|_{L^\infty(\Omega)} + \lambda_2\|u_0 - \mathcal{P}_n u_0\|_{L^\infty(\Omega)}\}. \tag{29}$$

On the other hand

$$\|\hat{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant \|u_0 - \mathcal{P}_n u_0\|_{L^\infty(\Omega)} + \|\mathcal{P}_n\|\|\tilde{u}_n - u_0\|_{L^\infty(\Omega)}. \tag{30}$$

Combining (28) and (30) yields:

$$\|\hat{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant \|u_0 - \mathcal{P}_n u_0\|_{L^\infty(\Omega)} + \|\mathcal{P}_n\|_{L^\infty(\Omega)}\{\lambda_1\{\|\mathcal{T}u_0 - \mathcal{T}_n u_0\|_{L^\infty(\Omega)} + \lambda_2\|u_0 - \mathcal{P}_n u_0\|_{L^\infty(\Omega)}\}, \tag{31}$$

Since $\mathcal{P}_n$ is interpolary operator, so it is bounded and we suppose $\|\mathcal{P}_n\| \leqslant \lambda_3 < \infty$. Therefore, by Theorem (4.1) we have

$$\|\hat{u}_n - u_0\|_{L^\infty(\Omega)} \leqslant e^{\frac{-c}{h_{\mathcal{X},\Omega}}} \|u_0\|_{\mathcal{N}_\phi(\Omega)}(1 + \lambda_1\lambda_2\lambda_3) + \lambda_1\lambda_3\|\mathcal{T}u_0 - \mathcal{T}_n u_0\|_{L^\infty(\Omega)}. \tag{32}$$

This completes the proof. ∎

5. Numerical results

In this section, some examples are provided to show the strength of the proposed method in approximating the solution of nonlinear mixed Volterra Fredholm integral equations.

Clearly, a good choice of the radial basis function is important for the quality of the approximation. In this paper, we utilize inverse multiquadric radial basis function with $\beta = \frac{1}{2}, \beta = \frac{3}{2}$ and shape parameter $c = 4$. Also, for the numerical quadrature rule, we have used the 5-point Gauss-Legendre quadrature formula.To measure the accuracy of the method, we have used the maximum absolute error:

$$\|e\|_\infty = max|u(x,t) - \hat{u}(x,t)|, \quad (x,t) \in \Omega \times [0,T].$$

All the computations were carried out using the Maple 16 software.

***Example* 5.1** Consider the nonlinear Mixed Volterra Fredholm integral equation

$$u(x,y,t) - \int_0^t \int_\Omega k(x,y,t,\sigma,\tau,s,u(\sigma,\tau,s))d\sigma d\tau ds = f(x,y,t),$$

where $\Omega = \left\{(\sigma,\tau) \in \mathbb{R}^2 : 0 \leqslant \sigma \leqslant \frac{\pi}{4}, \sin(\sigma) \leqslant \tau \leqslant \cos(\sigma)\right\}$ and

$$k(x,y,t,\sigma,\tau,s,u) = \sin(x)u^2.$$

The exact solution of this equation is $u(x,y,t) = x$ and $f(x,y,t)$ is defined accordingly. The numerical results for different values of M, N and β are given in Table 2. The distribution of nodes is depicted in Figure 1. Also the maximum errors for $M = 2, N = 27$ and $\beta = 0.5, 1.5$ are graphically shown in Figure 2. As we expected, the numerical results gradually converge to the exact values along with the increase of the nodes.

Table 2. Maximum absolute errors for different values of M, N and β with $t = 0.5$ for Example 1.

M, N	$\beta = 0.5$	$\beta = 1.5$
2, 3	2.5×10^{-3}	5.0×10^{-3}
3, 7	8.0×10^{-4}	1.5×10^{-3}
3, 12	4.0×10^{-6}	1.0×10^{-5}
2, 21	2.0×10^{-6}	4.0×10^{-6}
2, 27	1.5×10^{-8}	6.0×10^{-8}

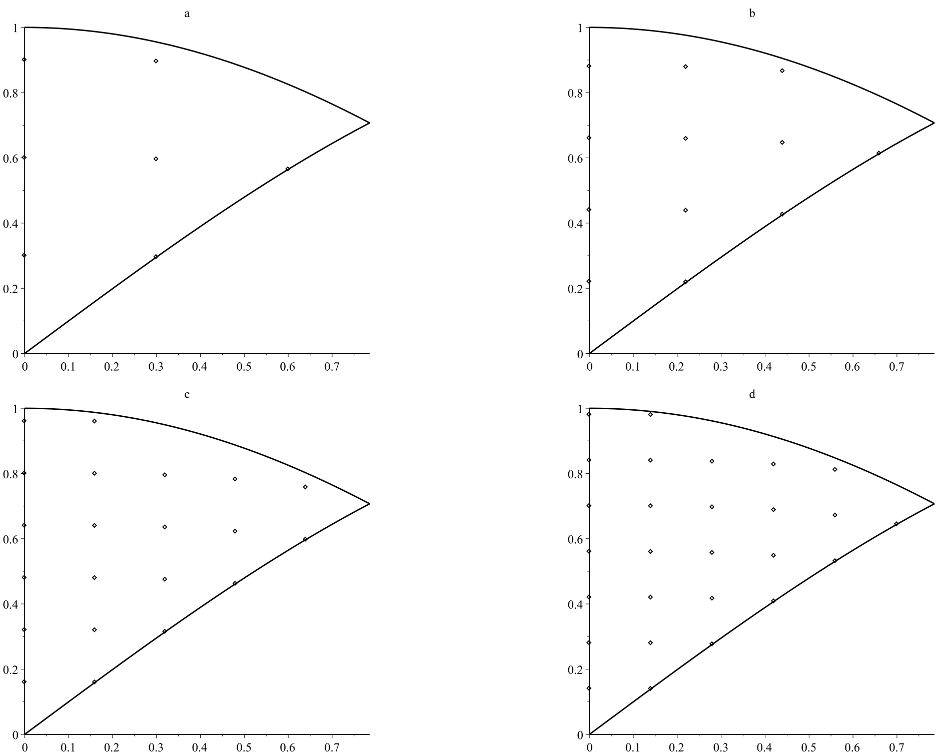

Figure 1. Node distribution for Example 1 (a: $N = 7$, b: $N = 12$, c: $N = 21$, d: $N = 27$).

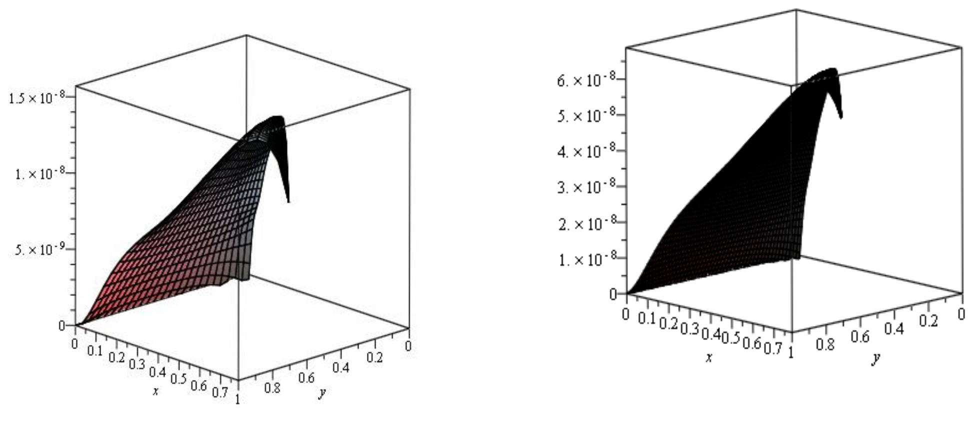

Figure 2. The absolute errors for Example 1 (Left:$M = 2, N = 27, \beta = 0.5$, Right:$M = 2, N = 27, \beta = 1.5$).

***Example* 5.2** We consider the following nonlinear Mixed Volterra Fredholm integral equation

$$u(x,y,t)-\int_0^t\int_\Omega x\cos(x)u^2(\sigma,\tau,s)d\sigma d\tau ds=f(x,y,t),$$

which is defined on a non-rectangular domain

$$\Omega=\left\{(\sigma,\tau)\in\mathbb{R}^2:-1\leqslant\tau\leqslant 1, v_1(\tau)\leqslant\sigma\leqslant v_2(\tau)\right\},$$

where

$$v_1(\tau)=\begin{cases}0, & -1\leqslant\tau\leqslant 0,\\ -\sqrt{\tau-\tau^2}\,, & 0\leqslant\tau\leqslant 1,\end{cases}$$

and

$$v_2(\tau)=\sqrt{1-\tau^2}.$$

$$f(x,y,t)=xt-\frac{17}{384}\pi x\cos(x)t^3.$$

The exact solution of this equation is $u(x,y,t)=xt$. The traditional methods have some difficulties in numerical solution of this problem due to the irregularity of its domain. But this problem could be solved using our meshless method by using some nodes scattered over the Ω. Table 3 shows numerical results at different numbers of M,N and β in term of $\|e\|_\infty$. The maximum errors for $M=2,N=25$ and $\beta=0.5,1.5$ are graphically shown in Figure 3. Numerical results confirm theoretical error estimates as the number of nodes increases. The distribution of nodes is depicted in Figure 4.

Table 3. Maximum absolute errors for different values of M,N and β with $t=0.5$ for Example 2.

M,N	$\beta=0.5$	$\beta=1.5$
2, 5	5.0×10^{-2}	9.0×10^{-2}
3, 9	3.0×10^{-3}	7.0×10^{-3}
3, 19	2.5×10^{-4}	7.0×10^{-3}
2, 25	3.0×10^{-4}	6.0×10^{-4}

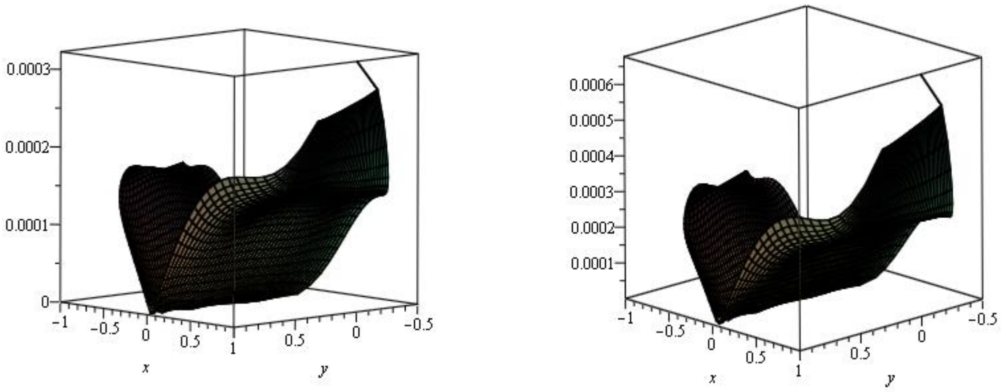

Figure 3. The absolute errors for Example 2 (Left:$M = 2, N = 25, \beta = 0.5$, Right:$M = 2, N = 25, \beta = 1.5$).

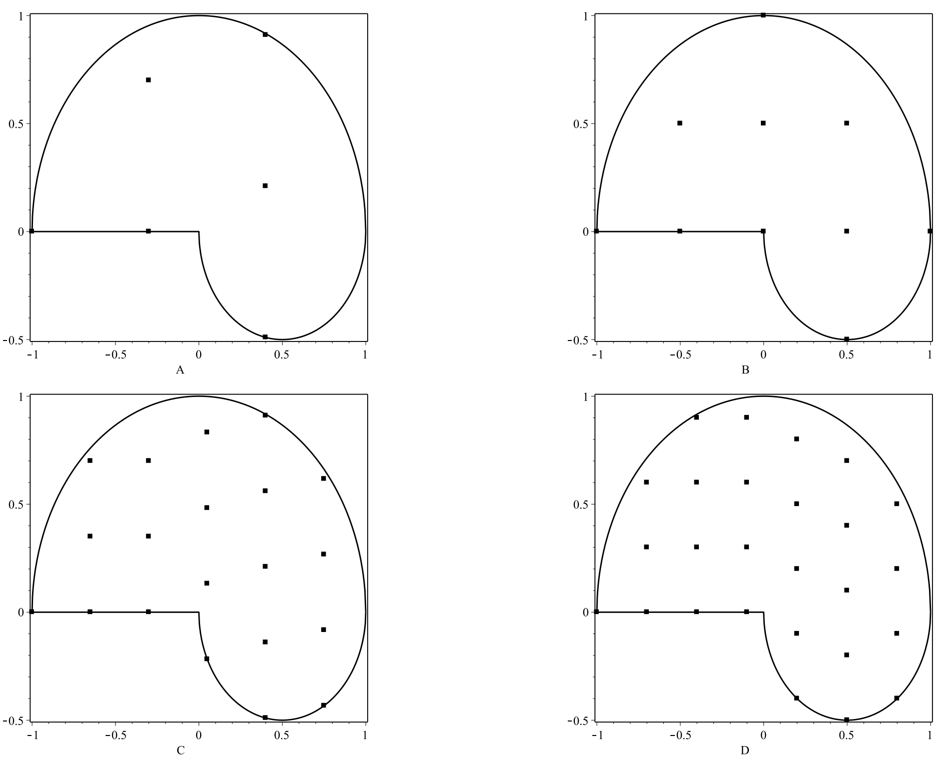

Figure 4. Node distribution for Example 2 (A: $N = 5$, B: $N = 9$, C: $N = 19$, D: $N = 25$).

***Example* 5.3** Consider the nonlinear Mixed Volterra Fredholm integral equation

$$u(x,y,t)-\int_0^t\int_\Omega k(x,y,t,\sigma,\tau,s,u(\sigma,\tau,s))d\sigma d\tau ds=f(x,y,t),$$

where $\Omega=\left\{(\sigma,\tau)\in\mathbb{R}^2:0\leqslant\sigma\leqslant 1,\sigma^2\leqslant\tau\leqslant\sqrt{\sigma}\right\}$ and

$$k(x,y,t,\sigma,\tau,s,u)=x(1-\exp(-u)),$$

$$f(x,y,t)=-\ln(1+xt)+\frac{3}{40}xt^2$$

The exact solution of this equation is $u(x,y,t)=-\ln(1+xt)$. The numerical results for different values of M,N and β are given in Table 4. The distribution of nodes is depicted in Figure 5. Moreover the absolute errors for $M=3,N=8$ and $\beta=0.5,1.5$ are shown in Figure 6.

Table 4. Maximum absolute errors for different values of M,N and β with $t=0.5$ for Example 3.

M,N	$\beta=0.5$	$\beta=1.5$
3, 5	3.0×10^{-3}	3.0×10^{-3}
3, 12	5.0×10^{-4}	6.0×10^{-4}
3, 18	4.0×10^{-4}	5.0×10^{-4}
3, 25	5.0×10^{-4}	6.0×10^{-4}

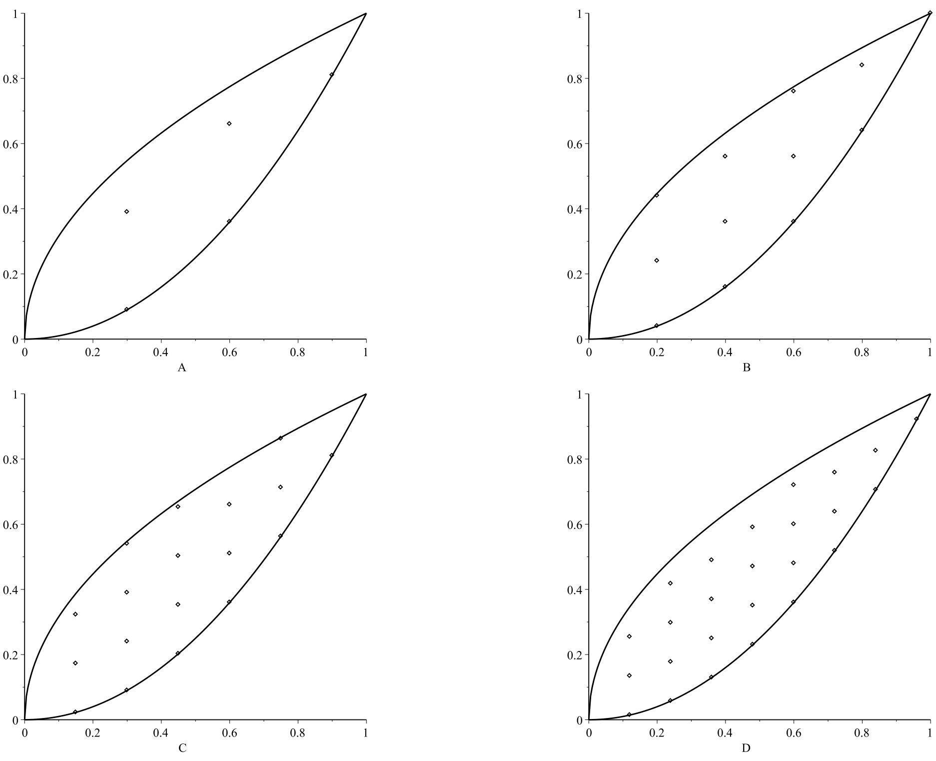

Figure 5. Node distribution for Example 3 (A: $N = 5$, B: $N = 12$, C: $N = 18$, D: $N = 25$).

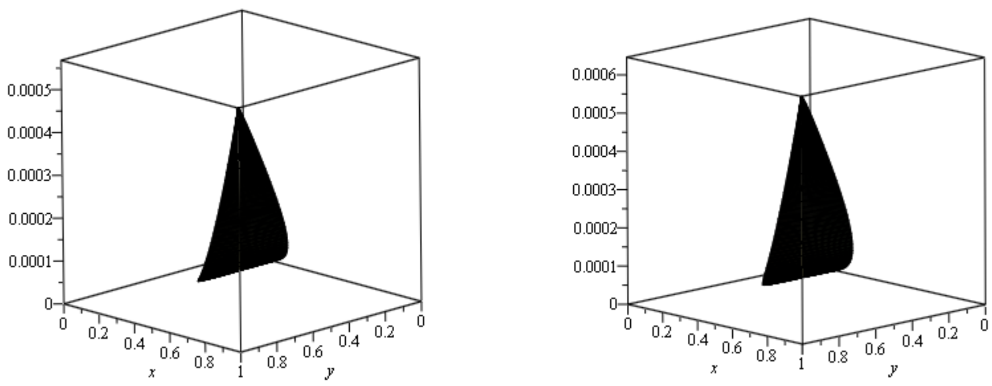

Figure 6. The absolute errors for Example 3 (Left:$M = 3, N = 25, \beta = 0.5$, Right:$M = 3, N = 25, \beta = 1.5$).

6. Conclusion

In this paper, we performed a meshless collocation method for the numerical solution of Urysohn mixed Volterra-Fredholm integral equations of the second kind on a non-rectangular domain. The most important novelty of this work is that the method is meshless and does not need to any background interpolation or approximation cells. It is shown that the proposed scheme is easy to implement, computationally attractive and offers a highly accurate approximate solutions.

References

[1] H. Wendland, *Scattered Data Approximation*, Cambridge University Press, 2005.

[2] W.R. Madych, S.A. Nelson, *Multivariate interpolation and conditionally positive definite functions II*, Math. Comput. 54 (189) (1990) 211-230.

[3] W.R. Madych, S.A. Nelson, *Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation*, J. Approx. Theory 70 (1992) 94-114.

[4] P.J. Kauthen, *Continuous time collocation methods for Volterra-Fredholm integral equations*, Numer. Math. 56 (1989) 409-424.

[5] H. Brunner, P.J. van der Houwen, *The numerical solution of Volterra equations,* CWI Monographs, vol. 3, North-Holland, Amsterdam, 1986.

[6] H. Brunner, *On the numerical solution of nonlinear Volterra-Fredholm integral equations by collocation methods,* SIAM J. Numer. Anal. 27 (4) (1990) 987-1000.

[7] J.P. Kauthen, *Continuous time collocation method for Volterra-Fredholm integral equations*, Numer. Math. 56 (1989) 409-424.

[8] K. Maleknejad, M. Hadizadeh, *A new computational method for Volterra-Fredholm integral equations,* Comput. Math. Appl. 37 (1999) 1-8.

[9] A.M. Wazwaz, *A reliable treatment for mixed Volterra-Fredholm integral equations,* Appl. Math. Comput. 127 (2002) 405-414.

[10] E. Banifatemi, M. Razzaghi, S. Yousefi, *Two-dimensional Legendre Wavelets Method for the mixed Volterra-Fredholm integral equations,* J. Vibr. Control. 13 (2007) 1667-1675.

[11] R.L. Hardy, *Multiquadric equations of topography and other irregular surfaces,* J. Geophys. Res. 76 (1971) 1905-1915.

[12] H.R. Thieme, *A model for the spatio spread of an epidemic,* J. Math. Biol. 4 (1977) 337-351.

[13] G. Han, L. Zhang, *Asymptotic expansion for the trapezoidal Nystrom method of linear VolterraFredholm equations*, J. Comput. Appl. Math. 51 (1994) 339-348.

[14] M. Hadizadeh, M. Asgari, *An effective numerical approximation for the linear class of mixed integral equations,* Appl. Math. Comput. 167 (2005) 1090-1100.

[15] E. Babolian, A.J. Shaerlar, *Two dimensional block pulse functions and application to solve Volterra-Fredholm integral equations with Galerkin method,* Int. J. Contemp. Math. Sci. 6 (2011) 763-770.

[16] H. Laeli Dastjerdi, F. M. Maalek Ghaini, M. Hadizadeh, *A meshless approximate solution of mixed Volterra-Fredholm integral equations,* Int. J. Comput. Math. 90 (2013) 527-538.

[17] R.L. Hardy, *Theory and applications of the multiquadric-biharmonic method. 20 years of discovery 1968-1988,* Comput. Math. Appl. 19 (8-9) (1990) 163-208.

[18] E.J. Kansa, *Multiquadrics-A scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivative estimates,* Comput. Math. Appl. 19 (8-9) (1990) 127-145.

[19] E.J. Kansa, *Multiquadrics-A scattered data approximation scheme with applications to computational fluid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations,* Comput. Math. Appl. 19 (8-9) (1990) 147-161.

[20] A. Cardone, E. Messina, E. Russo, *A fast iterative method for discretized Volterra-Fredholm integral equations,* J. Comput. Appl. Math. 189 (2006) 568-579.

[21] Y.C. Hon, X.Z. Mao, *An efficient numerical scheme for Burgers equation*, Appl. Math. Comput. 95 (1998) 37-50.

[22] Y.C. Hon, K.F. Cheung, X.Z. Mao, E.J. Kansa, *Multiquadric solution for shallow water equations*, ASCE J. Hydraul. Eng. 125 (1999) 524-533.

[23] M. Zerroukat, H. Power, C.S. Chen, *A numerical method for heat transfer problem using collocation and radial basis functions*, Int. J. Numer. Meth. Eng. 42 (1992) 1263-1278.

[24] K. E. Atkinson, F. A. Potra, *Projection and iterated projection methods for nonlinear integral equations*, SIAM J. Numer. Anal. 24 (1987) 1352-1373.

[25] K. Atkinson, J. Flores, *The discrete collocation method for nonlinear integral equations*, IMA J. Numer. Anal. 13 (1993) 195-213.

[26] G.E. Fasshauer, *Meshfree methods.* In: Rieth, M., Schommers, W. (eds.) Handbook of Theoretical and Computational Nanotechnology, American Scientific Publishers, 27 (2006) 33-97.