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A note on quasi irresolute topological groups

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Abstract. In this study, we investigate the further properties of quasi irresolute topological groups defined in [20]. We show that if a group homomorphism f between quasi irresolute topological groups is irresolute at e_G, then f is irresolute on G. Later we prove that in a semi-connected quasi irresolute topological group $(G, *, \tau)$, if V is any symmetric semi-open neighborhood of e_G, then G is generated by V. Moreover it is proven that a subgroup H of a quasi irresolute topological group $(G, *, \tau)$ is semi-discrete if and only if it has a semi-isolated point.

Keywords: Semi-open set, semi-closed set, irresolute mapping, semi-homeomorphism, quasi irresolute topological group.

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1. Introduction and Preliminaries

In 1963, semi-open sets and semi-continuous functions were introduced by Levine [16], after then many mathematicians investigate the basic concepts by replacing open set and continuity by semi-open set and semi-continuity. A subset A of a topological space X is said to be semi-open if there exists an open set U in such that $U \subset A \subset Cl(U)$, or equivalently if $U \subset Cl(Int(U))$. Every open set is semi-open and the union of any collection of semi-open sets is a semi-open set, while the intersection of two semi-open sets need not be semi-open.

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Semi-closed sets, semi-closure and semi-interior were defined in a manner analogous to corresponding concepts of closed sets, closure and interior in [8, 9]. The complement of a semi-open set is said to be semi-closed. Every closed set is semi-closed. Semi-closure of $A \subset X$, denoted by $sCl(A)$, is the intersection of all semi-closed subsets of X containing A and $x \in sCl(A)$ if and only if for any semi-open set U containing x, $U \cap A \neq \emptyset$. $x \in X$ is called a semi-interior point of A if there exist a semi-open set U such that $x \in U \subset A$. The set of all semi-interior points of A is called the semi-interior of A, denoted $sInt(A)$. Moreover A is semi-open if and only if $A = sInt(A)$ and A is semi-closed if and only if $A = sCl(A)$. If a semi-open set U contains a point $x \in X$ we say that U is a semi-open neighbourhood of x.

A mapping $f : X \longrightarrow Y$ between topological spaces X and Y is called:

(i) semi-continuous (resp. irresolute ([10])) if for each open (resp, semi-open) set $V \subset Y$, the set $f^{-1}(V)$ is semi-open in X [16]. Equivalently, the mapping f is ([2]) semi-continuous (resp.irresolute ([15])) if for each $x \in X$ and for each open (semi-open) neighborhood V of $f(x)$ there exists a semi-open neighborhood U of x such that $f(U) \subset V$;

(ii) pre-semi-open if for every semi-open set A of X, $f(A)$ is semi-open in Y [10];

(iii) semi-homeomorphism if f is bijective, irresolute and pre -semi-open [10];

(iv) S-homeomorphism if it is semi-continuous and pre-semi-open [6].

In [10], it was shown that a homeomophism is a also a semi-homeomorphism (Thoerem 1.9) but the inverse is not true (Example 1.2).

A topological group $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ satisfies the following two axioms :

(1) The multiplication map $m : G \times G \longrightarrow G, m(x, y) = x * y$ is continuous.

(2) The symmetry map $i : G \longrightarrow G, i(x) = x^{-1}$ is continuous [7].

(1) and (2) are equivalent to saying that the mapping $m_1 : G \times G \longrightarrow G, m_1(x, y) = x * y^{-1}$ is continuous..

(1) also implies that the left translations ${}_gl : G \longrightarrow G, {}_gl(x) = g * x$ and the right translations $r_g : G \longrightarrow G, r_g(x) = x * g$ are continuous. But the inverse is not true.

A semi topological group (or quasi topological group in [1]) $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ such that left translations, the right translations and the symmetry map are continuous (hence homeomorphism) [7].

Semi topological groups were also introduced in [3] as group $(G, *)$ with a topology τ satisfies the following axiom:

(3) For each $x, y \in G$ and each neighbourhood W of $x * y^{-1}$ there are semi-open neighbourhoods U of x and V of y such that $U * V^{-1} \subset W$.

Later these spaces were called as s-topological group in [6]. Although (3) is not equivalent to saying that m_1 is semi continuous, (3) implies that m_1 is semi-continuous (Theorem 6 in [3]) and the multiplication map and the symmetry map are semi-continuous (Theorem 7 in [3]). (3) also implies the left and the right translations are S-homeomorphism by Theorem 3,1 in [6]. Moreover they and the symmetry map are semi-homeomorphism (see Remark *1*).

In [6], authors were also introduced $S-$topological groups by replacing the continuity by semi-continuity in topological groups as follows:

An S-topological group $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ satisfies the following two axioms:

(a) The multiplication map $m : G \times G \longrightarrow G, m(x, y) = x * y$ is semi-continuous.

(b) The symmetry map $i : G \longrightarrow G, i(x) = x^{-1}$ is semi-continuous.

But (a) and (b) do not imply (3) and also do not imply the semi-continuity of the left and the right translations (Remark 3.5 in [6]).

In similar manner, in [13], authors were introduced irresolute-topological and Irr-topological groups as follows:

An irresolute-topological group $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ satisfies the following axiom:

(c) For each $x, y \in G$ and each semi-open neighbourhood W of $x * y^{-1}$ there are semi-open neighbourhoods U of x and V of y such that $U * V^{-1} \subset W$.

An Irr-topological group $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ satisfies the following two axioms:

(d) The multiplication map $m : G \times G \longrightarrow G, m(x, y) = x * y$ is irresolute.

(e) The symmetry map $i : G \longrightarrow G, i(x) = x^{-1}$ is irresolute.

Here clearly (c) implies that m_1 is irresolute but the converse is not true. Also (c) implies (d) and (e) (Theorem 3.1 in [13]) and the left and the right translations and the symmetry map are semi-homeomorphism (Lemma 3.1, and Lemma 3.13 in [12]).

For further properties of s-topological, S-topological, irresolute topological , Irr-topological groups see [3, 5, 6, 12, 13, 17, 18].

In [20], authors introduced the notions of semi topological groups with respect to semi-continuity (or quasi s-topological groups in [14]) and irresoluteness by replacing the continuity with semi-continuity and irresoluteness as follows:

A semi-topological group with respect to semi-continuity $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ such that the left translations, the right translations and the symmetry map are semi-continuous.

A semi-topological group with respect to irresoluteness $(G, *, \tau)$ is a group $(G, *)$ endowed with a topology τ such that the left translations, the right translations and the symmetry map are irresolute.

Semi topological group with respect to irresoluteness was defined independently by Bosan et al. and known as quasi irresolute topological group (see Definition 1 in [4]).

In this study we called semi-topological group with respect to semi-continuity as quasi s-topological group and semi-topological group with respect to irresoluteness as quasi irresolute topological group for the consistency.

Note that the irresoluteness of the left translations, the right translations and the symmetry map in quasi irresolute topological group is equivalent to semi-homeomorphism (see [20]).

For further properties of quasi s-topological and quasi irresolute topological group, see [4, 14, 20, 21].

Since a irresolute function is also semi-continuous, every quasi irresolute topological group is also a quasi s-topological group. Thus by extending the Remark 3.17 in [12], we can summarize the relations of these notions as follows:

$$
\begin{array}{lcl}
\text{topological group} & \Longrightarrow & \text{s-topological group} \\
\Downarrow & & \Downarrow \\
\text{Irr-topological group} & \Longrightarrow & \text{S-topological group} \\
\Uparrow & & \Uparrow \\
\text{Irresolute topological} & \Longrightarrow & \text{s-topological group} \\
\Uparrow & & \Downarrow \\
\text{topological group} & \Longrightarrow & \text{quasi irresolute topological group} \\
 & & \Downarrow \\
 & & \text{quasi } s\text{-topological group}
\end{array}
$$

In this study, we investigate the further properties of quasi irresolute topological groups.

We show that if a group homomorphism f between quasi irresolute topological groups is irresolute at e_G, then f is irresolute on G. Later we prove that in a semi-connected quasi irresolute topological group $(G,*,\tau)$, if V is any symmetric semi-open neighborhood of e_G, then G is generated by V. Moreover it is proven that a subgroup H of a quasi irresolute topological group $(G,*,\tau)$ is semi-discrete if and only if it has a semi-isolated point.

We need following definitions and theorems in sequel.

Definition 1.1 [19]Let A be a subset of a topological space X. Then
(a) A point $x \in A$ is said to be a semi-isolated point of A if there is a semi-open set U such that $U \cap A = \{x\}$.
(b) A set A is said to be semi-discrete if each point of A is semi-isolated.

Definition 1.2 [11]A topological space X is said to be semi-connected if X cannot be expressed as the union of two disjoint nonempty semi-open sets in X. Maximal semi-connected subsets of a topological space X are called semi-component of X and for an element x of X, the semi-component including x is denoted by $SC(x)$.

Theorem 1.3 [20]A subgroup of a quasi irresolute topological group is semi-open if and only if it has a semi-interior point.

Theorem 1.4 [21]Let $(G,*,\tau)$ be a quasi irresolute topological group, then every semi-open subgroup H of G is also semi-closed.

Theorem 1.5 [4]Let U be any symmetric semi-open neighbourhood of e_G in a quasi irresolute topological group $(G,*,\tau)$. Then the set $L = \bigcup_{n=1}^{\infty} U^n$ is a semi-open and a semi-closed subgroup of G.

2. Results

Remark 1 *Above we mentioned that in an s-topological group $(G,*,\tau)$, the left, the right translations are S-homeomorphism. Here we want to emphasize that they and the symmetry map are actually semi-homeomorphism since for any semi-open set B in G and any $a \in G$, ${}_a l^{-1}(B) = a^{-1}*B = {}_{a^{-1}} l(B), r_a^{-1}(B) = B*a^{-1} = r_{a^{-1}}(B)$ and $s^{-1}(B) = s(B)$ are semi-open by Theorem 5 and Theorem 8 in [3]. Hence some properties of s-topological groups and irresolute-topological groups on which depends the semi-homeomorphism of the left, the right translations and the symmetry map are coincide with the properties of quasi irresolute topological groups. Nevertheless we give their proofs.*

Proposition 2.1 Let $(G,*,\tau)$ be a quasi irresolute topological group and $g \in G$. Then for all semi-open neighbourhood W of g, there exits semi-open neighbourhoods V and U of e_G such that $g*V \subset W$ and $U*g \subset W$.

Proof. Since ${}_g l$ and r_g are irresolute at e_G and ${}_g l(e_G) = g, r_g(e_G) = g$, for all semi-open neighbourhood W of g, there exits semi-open neighbourhoods V and U of e_G such that $g*V \subset W$ and $U*g \subset W$. ■

Theorem 2.2 Let $(G,*,\tau_G)$ and $(H,\circ,\tau_H)$ be quasi irresolute topological groups and $f:(G,*,\tau_G) \longrightarrow (H,\circ,\tau_H)$ be a group homomorphism. If f is irresolute at e_G, then f is irresolute on G.

Proof. Let $g \in G$ and W be any semi-open neighbourhood of $f(g)$. By Proposition 2.1, there exits a semi-open neighbourhood V of e_H such that $f(g) \circ V \subset W$. Since f irresolute at e_G and $f(e_G) = e_H$, for semi-open neighbourhood V of e_H, there exits semi-open neighbourhoods U of e_G such that $f(U) \subset V$. Hence we have

$$f(g * U) = f(g) \circ f(U) \subset f(g) \circ V \subset W,$$

where $g * U$ is also semi-open neighbourhood of g. Therefore f is irresolute on G. ■

Theorem 2.3 Let $(G, *, \tau)$ be a quasi irresolute topological group, $SO(e_G)$ be semi-connected component of e_G and U be any symmetric semi-open neighbourhood of e_G. Then $SO(e_G) \subset \bigcup_{n=1}^{\infty} U^n$, in particular, if G is semi-connected, then $G = \bigcup_{n=1}^{\infty} U^n$.

Proof. By Theorem 1.5, $L = \bigcup_{n=1}^{\infty} U^n$ is semi-open as well as semi-closed subgroup of G. Since $SO(e_G)$ is semi-connected component of e_G, we have $SO(e_G) \subset \bigcup_{n=1}^{\infty} U^n$. If G is semi-connected $G = \bigcup_{n=1}^{\infty} U^n$. ■

Theorem 2.4 Let $(G, *, \tau)$ be a semi-connected quasi irresolute topological group and H be a subgroup which contains a semi-interior point, then $H = G$. In particular, a semi-open subgroup of G equals G.

Proof. Since H contains a semi-interior point, H is semi-open by Theorem 1.3. H is also semi-closed by Theorem 1.4. Since G is semi-connected $G = H$. ■

Theorem 2.5 Let $(G, *.\tau)$ be a semi-connected quasi irresolute topological group. If V is any symmetric semi-open neighborhood of e_G, then G is generated by V.

Proof. The subgroup generated by V, which is denoted by V^∞ consists of all products $u_1 * u_2 * \cdots * u_n$ of finite sequences of elements of V. V^∞ is semi-open, since it has e_g as a semi-interior point. By Theorem 2.4, $V^\infty = G$ which means that G is generated by U. ■

Theorem 2.6 A subgroup H of a quasi irresolute topological group $(G, *, \tau)$ is semi-discrete if and only if it has a semi-isolated point.

Proof. If H is semi-discrete, then every point of H is semi-isolated. Let a be a semi-isolated point in H. There exist a semi-open set U such that $H \cap U = \{a\}$. Then for any $h \in H$, we have

$$\begin{aligned} h &=_{ha^{-1}} l(a) =_{ha^{-1}} l(H \cap U) =_{ha^{-1}} l(H) \cap_{ha^{-1}} l(U) \\ &= ha^{-1}H \cap ha^{-1}U \\ &= H \cap ha^{-1}U \end{aligned}$$

where $ha^{-1}U$ is semi-open. Therefore H is semi-discrete. ■

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