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Asymptotic aspect of quadratic functional equations and superstability of higher derivations in multi-Fuzzy normed spaces

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Abstract. In this paper, we introduce the notion of multi-fuzzy normed spaces and establish an asymptotic behavior of the quadratic functional equations in the setup of such spaces. We then investigate the superstability of strongly higher derivations in the framework of multi-fuzzy Banach algebras.

Keywords: Fuzzy normed space, higher derivation, Hyers-Ulam-Rassias stability, multi-normed space, quadratic functional equation.

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1. Introduction

In 1940, a question that was given by Ulam [27] concerning the stability of group homomorphisms gave rise to the stability problem of functional equations. In the following year, Hyers [14] gave a partial affirmative answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. Rassias's paper has significantly influenced in the development of what we now call the Hyers-Ulam-Rassias stability of functional equations.

On the other hand, Dales and Polyakov [8] introduced the notion of multi-normed spaces and presented many nice properties of these spaces. A number of analysts have

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established some results concerning the stability problems of various functional equations in multi-normed spaces (cf. e.g. [9, 20, 21]). Moslehian [22] studied superstability of higher derivations in multi-Banach algebras and proved that every (α, φ)-approximate strongly higher derivation in a multi-Banach algebra is a higher derivation. We recall that an equation is called superstable if each its approximate solution is an exact solution.

The notion of fuzzy norm first was introduced by Katsaras [16]. Thereafter, some mathematicians discussed several notions of fuzzy norms from various points of view (cf. e.g. [11, 17]). In particular in 2003, Bag *et al.*, [1], following Cheng and Mordeson [3] gave an idea of a fuzzy norm and established a decomposition theorem of a fuzzy norm into a family of crisp norms. They also described some nice properties of the fuzzy norm in [2]. In [10], Eshaghi Gordji *et al.*, introduced the notion of fuzzy Banach algebra and studied the generalized Hyers-Ulam stability of Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras.

In this work, inspiring the concepts of multi-normed space, multi-Banach algebra and fuzzy normed space we establish the structure of multi-fuzzy normed space and multi-fuzzy Banach algebra. Then taking some ideas from [20, 22], we extend the Hyers-Ulam stability of the quadratic functional equation for mappings from groups into multi-fuzzy normed spaces and also the superstability of strongly higher derivations in multi-fuzzy Banach algebras.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation and every solution Q of it is said to be a quadratic mapping. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [26] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] showed that the theorem of Skof is true if the relevant domain X is replaced by an abelian group. In [6], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. Furthermore, Park [23] studied the stability of quadratic mappings on Banach modules.

Let $\mathcal{A}$ be an algebra and $n_0 \in \{0,1,\ldots\} \cup \{\infty\}$. A sequence $\{d_j\}_{j=0}^{n_0}$ of linear mappings on $\mathcal{A}$ is called a higher derivation of rank n_0 if for each $0 \leqslant j \leqslant n_0$, $d_j(ab) = \sum_{l=0}^{j} d_l(a)d_{j-l}(b)$ $(a, b \in \mathcal{A})$. Evidently, d_0 is a homomorphism and d_1 is a d_0-derivation in the sense of [19]. If d_0 is the identity operator $id_{\mathcal{A}}$ on $\mathcal{A}$, then $\{d_j\}_{j=0}^{n_0}$ is said to be a strongly higher derivation. The notion of higher derivation was introduced by Hasse and Schmidt [12]. This notion closely related to the concept of homomorphisms [7]. A standard example of a strongly higher derivation is $\{\frac{D^j}{j!}\}_{j=0}^{\infty}$, where D is a derivation on an algebra $\mathcal{A}$. For more details on this issue see [13, 15, 18, 25] and the bibliography quoted there.

In the following we recall some basic definitions and facts which will be used further on. We denote by $\mathbb{T}$ the set $\{-1, 1\}$, which will be used in next sections.

Definition 1.1 [4] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if
$(N1)$ $N(x,t) = 0$ for all $x \in X$ and $t \in \mathbb{R}$ with $t \leqslant 0$;
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $x \in X$ and $t > 0$;
$(N3)$ $N(cx,t) = N(x, \frac{t}{|c|})$ for all $x \in X$ and $c \neq 0$;
$(N4)$ $N(x+y, s+t) \geqslant \min\{N(x,s), N(y,t)\}$ for all $x, y \in X$ and all $s, t \in \mathbb{R}$;
$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$ for all $x \in X$ and $t \in \mathbb{R}$;
$(N6)$ For all $x \in X$ with $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Let (X, N) be a fuzzy normed space. Then a sequence $\{x_n\}$ in X is called:
(i) convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we write $N\text{-}\lim_{n\to\infty} x_n = x$.
(ii) Cauchy if for each $\varepsilon > 0$ and each $\delta > 0$, there exists n_0 such that $N(x_m - x_n, \delta) > 1 - \varepsilon$ for all $n, m \geqslant n_0$.
The fuzzy normed space (X, N) is complete if every Cauchy sequence in X converges in X. In this case fuzzy normed space is called a fuzzy Banach space.

Definition 1.2 [10] Let X be an algebra and (X, N) be complete fuzzy normed space, the pair (X, N) is said to be a fuzzy Banach algebra if for every $x, y \in X$, $t, s \in \mathbb{R}$,

$$N(xy, ts) \geqslant \max\{N(x, t), N(y, s)\}.$$

It is straightforward to check that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences in (X, N) with $N\text{-}\lim_{n\to\infty} x_n = x$ and $N\text{-}\lim_{n\to\infty} y_n = y$, then $N\text{-}\lim_{n\to\infty} x_n y_n = xy$.

Let $(E, \|.\|)$ be a complex normed space and let $k \in \mathbb{N}$. We denote by E^k, the linear space $E \oplus \ldots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, \ldots, k\}$ and denote by $\mathfrak{S}_k$ the group of permutations on k symbols. For $\sigma \in \mathfrak{S}_k$, $x = (x_1, \ldots, x_k) \in E^k$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$ define $A_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and $M_\alpha(x) = (\alpha_1 x_1, \ldots, \alpha_k x_k)$.

Definition 1.3 [8] Let $(E, \|.\|)$ be a complex (respectively, real) normed space. A multi-norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{\|.\|_k\} = \{\|.\|_k, k \in \mathbb{N}\}$ such that $\|.\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in E$ (so that $\|.\|_1$ is the initial norm), and the following axioms $(A1)$-$(A4)$ are satisfied for each $k \in \mathbb{N}$ with $k \geqslant 2$:
$(A1)$ for each $\sigma \in \mathfrak{S}_k$ and $x \in E^k$, $\|A_\sigma(x)\|_k = \|x\|_k$;
$(A2)$ for each $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ $(\mathbb{R})$ and $x \in E^k$, $\|M_\alpha(x)\|_k \leqslant (\max_{1\leqslant i\leqslant k} |\alpha_i|)\|x\|_k$;
$(A3)$ for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$;
$(A4)$ for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$.
In this case, we say that $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is a multi-normed space. Moreover, if $(E, \|.\|_1)$ is a Banach space, then $(E^k, \|.\|_k)$ is a Banach space for each $k = 2, 3, \ldots$, in this case, $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is called a multi-Banach space.

Theorem 1.4 [20] Let $\{(E^n, \|.\|_n) : n \in \mathbb{N}\}$ be a multi-normed space and let $\{(F^n, \|.\|_n) : n \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $f : E \to F$ is a mapping. Then f is quadratic if and only if for every $k \in \mathbb{N}$,
$\|f(x_1+y_1)+f(x_1-y_1)-2f(x_1)-2f(y_1), \ldots, f(x_k+y_k)+f(x_k-y_k)-2f(x_k)-2f(y_k)\|_k \to 0$ as $\|(x_1, \ldots, x_k)\|_k + \|(y_1, \ldots, y_k)\|_k \to \infty$.

2. Asymptotic aspect of quadratic the functional equations in multi-fuzzy normed spaces

In this section, we deal with the asymptotic behavior of the quadratic equations in the framework of multi-fuzzy normed spaces.

Definition 2.1 Let (E, N) be a fuzzy normed space. A multi-fuzzy norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{N_k\}$ such that N_k is a fuzzy norm on E^k for each $k \in \mathbb{N}$, $N_1(x, t) = N(x, t)$

for each $x \in E$ and $t \in \mathbb{R}$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geqslant 2$:
$(MF1)$ for each $\sigma \in \mathfrak{S}_k$, $x \in E^k$ and $t \in \mathbb{R}$, $N_k(A_\sigma(x), t) = N_k(x, t)$;
$(MF2)$ for each $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $x \in E^k$, $t \in \mathbb{R}$,

$$N_k(M_\alpha(x), t) \geqslant N_k(\max_{i \in \mathbb{N}_k} |\alpha_i| x, t);$$

$(MF3)$ for each $x_1, \ldots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}\Big((x_1, \ldots, x_k, 0), t\Big) = N_k\Big((x_1, \ldots, x_k), t\Big);$$

$(MF4)$ for each $x_1, \ldots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}\Big((x_1, \ldots, x_k, x_k), t\Big) = N_k\Big((x_1, \ldots, x_k), t\Big).$$

In such a case $\{(E^k, N_k), k \in \mathbb{N}\}$ is called a multi-fuzzy normed space.

***Example* 2.2** Let (E, N) be a fuzzy normed space. For each $k \in \mathbb{N}$, set

$$N_k\Big((x_1, \ldots, x_k), t\Big) = \min\Big\{N(x_i, t), i = 1, \ldots, k\Big\}, \quad x_1, \ldots, x_k \in E,\ t \in \mathbb{R}.$$

Then $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy normed space.

Proposition 2.3 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $k, n \in \mathbb{N}, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n} \in E$ and $\eta_1, \ldots, \eta_k \in \mathbb{T}$, then we have
(i) $N_k\Big((\eta_1 x_1, \ldots, \eta_k x_k), t\Big) = N_k\Big((x_1, \ldots, x_k), t\Big)$.
(ii) $N_k\Big((x_1, \ldots, x_k), t\Big) \geqslant N_{k+1}\Big((x_1, \ldots, x_k, x_{k+1}), t\Big)$.
(iii) $N_{k+n}\Big((x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n}), t\Big) \geqslant$

$$\min\Big\{N_k\Big((x_1, \ldots, x_k), \alpha t\Big), N_n\Big((x_{k+1}, \ldots, x_{k+n}), \beta t\Big)\Big\},$$

where $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$.
(iv) $\min_{i \in \mathbb{N}_k} N(x_i, t) \geqslant N_k\Big((x_1, \ldots, x_k), t\Big) \geqslant \min_{i \in \mathbb{N}_k} N(x_i, \alpha_i t)$, where $\alpha_1, ..., \alpha_k \geqslant 0$ and $\sum_{i=1}^{k} \alpha_i = 1$. In particular,

$$N_k\Big((x_1, \ldots, x_k), t\Big) \geqslant \min_{i \in \mathbb{N}_k} N(k x_i, t).$$

Here is a direct consequence of Proposition 2.3.

Corollary 2.4 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space and (E, N_1) be a fuzzy Banach space. Then for each $k \in \mathbb{N}$, (E^k, N_k) is a fuzzy Banach space, too.

In the light of the above corollary, the following definition is reasonable.

Definition 2.5 Suppose that $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy normed space, for which (E, N_1) is a fuzzy Banach space, then we say $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy Banach space.

The following result plays an essential role in the proofs of this section. We include the proof for the benefit of the reader.

Lemma 2.6 Let E be a group, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space, (Z, N) be a fuzzy-normed space and $f : E \to F$ be a mapping. Suppose that α is a nonzero fixed vector in Z such that

$$N_k\Big(\big(f(x_1+y_1)+f(x_1-y_1)-2f(x_1)-2f(y_1),\ldots,f(x_k+y_k)+$$

$$f(x_k-y_k)-2f(x_k)-2f(y_k)\big),t\Big) \geqslant N(\alpha,t), \tag{1}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k \in E$ and all $t > 0$, then there exists a unique quadratic mapping $Q : E \to F$ such that

$$N_k\Big(\big(Q(x_1)-f(x_1)+\frac{f(0)}{3},\ldots,Q(x_k)-f(x_k)+\frac{f(0)}{3}\big),3t\Big) \geqslant N(\alpha,t), \tag{2}$$

where $x_1,\ldots,x_k \in E$ and $t > 0$.

Proof. Replacing x_i and y_i by $2^n x_i$, $i = 1,\ldots,k$, in (1), we deduce that

$$N_k\Big(\big(\frac{f(2^{n+1}x_1)}{4^{n+1}}+\frac{f(0)}{4^{n+1}}-\frac{f(2^nx_1)}{4^n},\ldots,\frac{f(2^{n+1}x_k)}{4^{n+1}}+\frac{f(0)}{4^{n+1}}-\frac{f(2^nx_k)}{4^n}\big),\frac{t}{4^{n+1}}\Big)$$

$$\geqslant N(\alpha,t). \tag{3}$$

An induction argument implies that

$$N_k\Big(\big(\frac{f(2^{n+m}x_1)}{4^{n+m}}-\frac{f(2^nx_1)}{4^n}+f(0)(\sum_{i=n+1}^{n+m}4^{-i}),\ldots,\frac{f(2^{n+m}x_k)}{4^{n+m}}-\frac{f(2^nx_k)}{4^n}+$$

$$f(0)(\sum_{i=n+1}^{n+m}4^{-i})\big),t(\sum_{i=n+1}^{n+m}4^{-i})\Big) \geqslant N(\alpha,t), \tag{4}$$

for nonnegative integer numbers n,m. Substituting $y_i = 0$ for $i = 1,\ldots,k$, in (1), we obtain

$$N(f(0),\frac{t}{2}) \geqslant N(\alpha,t). \tag{5}$$

From (4), (5) and the properties of fuzzy norm we get

$$N_k\Big(\big(\tfrac{f(2^{n+m}x_1)}{4^{n+m}}-\tfrac{f(2^nx_1)}{4^n},\ldots,\tfrac{f(2^{n+m}x_k)}{4^{n+m}}-\tfrac{f(2^nx_k)}{4^n}\big),t(\sum_{i=n+1}^{n+m}4^{-i})\Big)$$

$$\geqslant \min\{N_k\Big(\big(\frac{f(2^{n+m}x_1)}{4^{n+m}}-\frac{f(2^nx_1)}{4^n}+f(0)(\sum_{i=n+1}^{n+m}4^{-i}),\ldots,\frac{f(2^{n+m}x_k)}{4^{n+m}}-\frac{f(2^nx_k)}{4^n}+$$

$$f(0)(\sum_{i=n+1}^{n+m}4^{-i})\big),\frac{t}{2}(\sum_{i=n+1}^{n+m}4^{-i})\Big),N(f(0),\frac{t}{2})\}\geqslant N(2\alpha,t). \tag{6}$$

Replacing $x_1,\ldots,x_k$ by x in (6), it follows that

$$N(\frac{f(2^{n+m}x)}{4^{n+m}}-\frac{f(2^nx)}{4^n},t(\sum_{i=n+1}^{n+m}4^{-i}))\geqslant N(2\alpha,t).$$

Let $\varepsilon>0$ and $\delta>0$ be given. Since $\lim_{t\to\infty}N(2\alpha,t)=1$, there is some $t_0>0$ such that $N(2\alpha,t_0)>1-\varepsilon$. Since $\sum_{i=1}^{\infty}t_04^{-i}<\infty$, there is some $n_0\in\mathbb{N}$ such that $\sum_{i=n+1}^{n+m}t_04^{-i}<\delta$ for all $n\geqslant n_0$. It implies that

$$N(\frac{f(2^{n+m}x)}{4^{n+m}}-\frac{f(2^nx)}{4^n},\delta)\geqslant N(\frac{f(2^{n+m}x)}{4^{n+m}}-\frac{f(2^nx)}{4^n},t_0(\sum_{i=n+1}^{n+m}4^{-i}))$$

$$\geqslant N(2\alpha,t_0)>1-\varepsilon.$$

Consequently, $\{\frac{f(2^nx)}{4^n}\}$ is a Cauchy sequence in (F,N). Since (F,N) is complete $\{\frac{f(2^nx)}{4^n}\}$ converges to some $Q(x)\in F$. Thus the mapping $Q:E\to F$ given by

$$Q(x):=N-\lim_{n\to\infty}\frac{f(2^nx)}{4^n},$$

is well-defined and according to Proposition 2.3, we deduce

$$N\text{-}\lim_{n\to\infty}\Big(\frac{f(2^nx_1)}{4^n},\ldots,\frac{f(2^nx_k)}{4^n}\Big)=\Big(Q(x_1),\ldots,Q(x_k)\Big). \tag{7}$$

Moreover, put $n=0$ in (4), using $(N5)$ it follows that

$$N_k\Big(\big(\frac{f(2^mx_1)}{4^m}-f(x_1)+f(0)(\sum_{i=1}^{m}4^{-i}),\ldots,\frac{f(2^mx_k)}{4^m}-f(x_k)+f(0)(\sum_{i=1}^{m}4^{-i})\big),t\Big)$$
$$\geqslant N(\alpha,t). \tag{8}$$

Therefore,

$$\begin{aligned}N_k\Big(\big(Q(x_1)-f(x_1)+\frac{f(0)}{3},\dots,Q(x_k)-f(x_k)+\frac{f(0)}{3}\big),3t\Big) \geqslant \min\Big\{&N_k\big((\frac{f(2^m x_1)}{4^m}-f(x_1)\\ &+f(0)(\sum_{i=1}^{m}4^{-i}),\dots,\frac{f(2^m x_k)}{4^m}-f(x_k)+f(0)(\sum_{i=1}^{m}4^{-i})),t\big),\\ &N_k\big((\frac{f(2^m x_1)}{4^m}-Q(x_1),\dots,\frac{f(2^m x_k)}{4^m}-Q(x_k)),t\big),\\ &N\big(f(0)(\sum_{i=1}^{m}4^{-i}-\frac{1}{3}),t\big)\Big\}.\end{aligned}$$

Applying (5), (7) and (8), for m large enough, we derive

$$N_k\Big(\big(Q(x_1)-f(x_1)+\frac{f(0)}{3},\dots,Q(x_k)-f(x_k)+\frac{f(0)}{3}\big),3t\Big) \geqslant N(\alpha,t).$$

Next we will show that Q is quadratic. Let $x,y\in E$, put

$$\begin{aligned}x_1=\ldots=x_k=2^n x,\\ y_1=\ldots=y_k=2^n y,\end{aligned}$$

and replace t by $4^n t$ in (1), to obtain

$$N_k\Big(\frac{f(2^n x+2^n y)}{4^n}+\frac{f(2^n x-2^n y)}{4^n}-\frac{2f(2^n x)}{4^n}-\frac{2f(2^n y)}{4^n},t\Big) \geqslant N(\alpha,4^n t). \tag{9}$$

On the other hand,

$$\begin{aligned}N\big(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),5t\big) \geqslant \min\Big\{&N\big(Q(x+y)-\frac{f(2^n x+2^n y)}{4^n},t\big),\\ &N\big(Q(x-y)-\frac{f(2^n x-2^n y)}{4^n},t\big),N\big(2Q(x)-\frac{2f(2^n x)}{4^n},t\big),N\big(2Q(y)-\frac{2f(2^n y)}{4^n},t\big),\\ &N\big(\frac{f(2^n x+2^n y)}{4^n}+\frac{f(2^n x-2^n y)}{4^n}-\frac{2f(2^n x)}{4^n}-\frac{2f(2^n y)}{4^n},t\big)\Big\},\end{aligned}$$

for each $x,y\in E$ and $t>0$. The first four terms on the right hand side of the above inequality tend to 1 as $n\to\infty$ and fifth term, by (9) is greater than or equal $N(\alpha,4^n t)$, which tends to 1 as $n\to\infty$. Therefore $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$. This means that Q is quadratic. We are going to show that Q is unique. Suppose that Q' is another quadratic mapping from E into F, which satisfies the required inequality. Then for each

$x \in E$ and $t > 0$ we have

$$\begin{aligned} N\big(Q(x) - Q'(x), 6t\big) &= N\big(\frac{Q(2^n x)}{4^n} - \frac{Q'(2^n x)}{4^n}, 6t\big) \\ &\geqslant \min\Big\{N\big(Q(2^n x) - f(2^n x) + \frac{f(0)}{3}, 3t \times 4^n\big), \\ &\qquad N(Q'(2^n x) - f(2^n x) + \frac{f(0)}{3}, 3t \times 4^n)\Big\} \\ &\geqslant N(\alpha, 4^n t). \end{aligned}$$

Taking the limit as $n \to \infty$, we observe that $Q = Q'$. ∎

In the following we demonstrate the stability of quadratic equation on a restricted domain. We denote by $\mathfrak{B}^k(E^k)$ the open ball in E^k with center $0 \in E^k$, radius $0 < r < 1$ and with $t' > 0$, more precisely $\mathfrak{B}^k(E^k) = B^k(0, r, t') = \{z \in E^k, N_k(z, t') > 1 - r\}$.

Proposition 2.7 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space, $0 < r < 1$, $p > 1$, $t' > 0$ and $\psi_k : E^{2k} \times \mathbb{R} \to [0, 1]$ $(k \in \mathbb{N})$ be a family of functions such that $\psi_k(x, x, .)$ ba a decreasing function of $\mathbb{R}$, $\lim\limits_{t\to\infty} \psi_k(x, y, t) = 0$, $\lim\limits_{t\to 0}\psi_k(x, y, t) = 1$ and $\psi_k(\frac{x}{2}, \frac{y}{2}, t) \leqslant \psi_k(x, y, 4^{2p}t)$ for all $x, y \in \mathfrak{B}^k(E^k)$, all $t \in \mathbb{R}$ and all $k \in \mathbb{N}$. Suppose that $f : E \to F$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} N_k\Big(\big(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1) - 2f(y_1), \ldots, f(x_k + y_k)+ \\ f(x_k - y_k) - 2f(x_k) - 2f(y_k)\big), t\Big) \geqslant 1 - \psi_k(x, y, t), \end{aligned} \tag{10}$$

for all $k \in \mathbb{N}$ and all $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathfrak{B}^k(E^k)$ with $x \pm y \in \mathfrak{B}^k(E^k)$ and $t > 0$. Then there exists a unique quadratic mapping $Q : E \to F$ such that

$$N_k\Big(\big(f(x_1) - Q(x_1), \ldots, f(x_k) - Q(x_k)\big), 2t\Big) \geqslant 1 - \psi_k(x, x, 4^{2p}t), \tag{11}$$

where $x = (x_1, \ldots, x_k) \in \mathfrak{B}^k(E^k)$ and $t > 0$.

Proof. Let $x = (x_1, \ldots, x_k) \in \mathfrak{B}^k(E^k)$. The argument used in the proof of Lemma 2.6, shows that $\{4^n f(\frac{x}{2^n})\}$ is Cauchy and so is convergent in the complete fuzzy normed space F. In addition, the mapping

$$\tilde{Q}(x) := N\text{-}\lim_{n\to\infty} 4^n f(2^{-n}x),\ (x \in \mathfrak{B}^1(E)),$$

satisfies

$$N_k\Big(\big(\tilde{Q}(x_1) - f(x_1), \ldots, \tilde{Q}(x_k) - f(x_k)\big), 2t\Big) \geqslant 1 - \psi_k(x, x, 4^{2p}t),$$

where $x = (x_1, \ldots, x_k) \in \mathfrak{B}^k(E^k)$. Define a mapping $Q : E \to F$ by $Q(x) := 4^{n_0}\tilde{Q}(2^{-n_0}x)$, where n_0 is the least nonnegative integer such that $2^{-n_0}x \in \mathfrak{B}^1(E)$. It is readily verified that

$$Q(x) = N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \quad (x \in E)$$

and $\tilde{Q}|_{\mathfrak{B}^1(E)} = Q(x)$. Now let $x, y \in E$. There is a large enough n such that $2^{-n}x, 2^{-n}y, 2^{-n}(x+y), 2^{-n}(x-y) \in \mathfrak{B}^1(E)$. Put $x_1 = \cdots = x_k = 2^{-n}x$, $y_1 = \cdots = y_k = 2^{-n}y$ in (10), to obtain
$N\Big(f(2^{-n}(x+y)) + f(2^{-n}(x-y)) - 2f(2^{-n}x) - 2f(2^{-n}y), t\Big) \geqslant$

$$1 - \psi_k\Big((x, \ldots, x), (y, \ldots, y), 4^{2np}t\Big).$$

We observe that
$N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \geqslant$
$\min\Big\{N(Q(x+y) - 4^n f(2^{-n}(x+y)), \frac{t}{5}), N(Q(x-y) - 4^n f(2^{-n}(x-y)), \frac{t}{5}),$
$N(2 \times 4^n f(2^{-n}x) - 2Q(x), \frac{t}{5}), N(2 \times 4^n f(2^{-n}y) - 2Q(y), \frac{t}{5}),$
$N(4^n f(2^{-n}(x+y)) + 4^n f(2^{-n}(x-y)) - 2 \times 4^n f(2^{-n}x) - 2 \times 4^n f(2^{-n}y), \frac{t}{5})\Big\}.$
Whence by taking the limit as $n \to \infty$, we get $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$. Hence Q is quadratic.
It remains to prove the uniqueness assertion. Suppose that Q' is another quadratic mapping satisfying (11). Then for each $x \in \mathfrak{B}^1(E)$ and $t > 0$ we have

$$N(Q(x) - Q'(x), 4t) = N(4^n Q(\frac{x}{2^n}) - 4^n Q'(\frac{x}{2^n}), 4t)$$

$$\geqslant \min\Big\{N(Q(\frac{x}{2^n}) - f(\frac{x}{2^n}), \frac{2t}{4^n}), N(Q'(\frac{x}{2^n}) - f(\frac{x}{2^n}), \frac{2t}{4^n})\Big\}$$
$$\geqslant 1 - \psi_k(x, x, 4^{n(2p-1)}t).$$

Passing the limit as $n \to \infty$, we obtain $Q = Q'$. ■

The following lemma is a version of [20, Lemma 4.1] in the framework of multi-fuzzy normed spaces.

Lemma 2.8 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space and (Z, N), $(\mathbb{R}, N)$ be fuzzy-normed spaces. Suppose that $\{\beta_k\}$ is a sequence of positive real numbers, α is a nonzero fixed vector in Z and $f : E \to F$ is a mapping satisfying

$$N_k\Big(\big(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1) - 2f(y_1), \ldots,$$

$$f(x_k + y_k) + f(x_k - y_k) - 2f(x_k) - 2f(y_k)\big), t\Big) \geqslant N(\alpha, 5t), \qquad (12)$$

for all $k \in \mathbb{N}$, $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$, and all $t > 0$ with

$$\min\Big\{N_k\Big((x_1, \ldots, x_k), t\Big), N_k\Big((y_1, \ldots, y_k), t\Big)\Big\} \leqslant N(\beta_k, t).$$

Then there exists a unique quadratic mapping $Q : E \to F$ such that

$$N_k\Big(\big(Q(x_1) - f(x_1) + \frac{f(0)}{3}, \ldots, Q(x_k) - f(x_k) + \frac{f(0)}{3}\big), 15t\Big) \geqslant N(\alpha, 5t),$$

for all $x_1, \ldots, x_k \in E$ and all $t > 0$.

Proof. Let us fix $k \in \mathbb{N}$ and $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Assume that

$$\min\Big\{N_k(x,t), N_k(y,t)\Big\} > N(\beta_k, t).$$

For $x = y = 0$, take $z = (z_1, \ldots, z_k)$ to be a vector of E^k with $N_k(z,t) \leqslant N(\beta_k, t)$. If x and y are nonzero, then there exist sufficiently large natural numbers k_1 and k_2, in which $N_k(x, \frac{2t}{k_1-3}) \leqslant N(\beta_k, t)$ and $N_k(y, \frac{2t}{k_2-1}) \leqslant N(\beta_k, t)$. Put $z_1 = k_1 x$, $z_2 = k_2 y$, and also set

$$z := \begin{cases} \frac{z_1 - x}{2}, \text{ if } & N_k(x,t) \leqslant N_k(y,t), \\ \frac{z_2 - y}{2}, \text{ if } & N_k(y,t) < N_k(x,t), \end{cases}$$

for all $t > 0$. Clearly, $N_k(z,t) \leqslant N(\beta_k, t)$.
One observes that the following relations hold.

$$\min\Big\{N_k(x,t), N_k(y+2z,t)\Big\} \leqslant N(\beta_k, t).$$

$$\min\Big\{N_k(x-z,t), N_k(y+z,t)\Big\} \leqslant N(\beta_k, t).$$

$$\min\Big\{N_k(x+z,t), N_k(y+z,t)\Big\} \leqslant N(\beta_k, t).$$

$$\min\Big\{N_k(y+z,t), N_k(z,t)\Big\} \leqslant N(\beta_k, t).$$

$$\min\Big\{N_k(x,t), N_k(z,t)\Big\} \leqslant N(\beta_k, t). \tag{13}$$

If $x = 0$ and $y \neq 0$ (or $x \neq 0$ and $y = 0$), then easy computations shows that (13) holds, too. From (13) we get

$N_k\Big(\big(f(x_1+y_1) + f(x_1-y_1) - 2f(x_1) - 2f(y_1), \ldots, f(x_k+y_k) + f(x_k-y_k) - 2f(x_k) - 2f(y_k)\big), 5t\Big) \geqslant$
$\min\Big\{N_k\Big(\big(f(x_1+y_1) + f(x_1-y_1-2z_1) - 2f(x_1-z_1) - 2f(y_1+z_1), \ldots, f(x_k+y_k) + f(x_k-y_k-2z_k) - 2f(x_k-z_k) - 2f(y_k+z_k)\big), t\Big),$
$N_k\Big(\big(f(x_1+y_1+2z_1) + f(x_1-y_1) - 2f(x_1+z_1) - 2f(y_1+z_1), \ldots, f(x_k+y_k+2z_k) + f(x_k-y_k) - 2f(x_k+z_k) - 2f(y_k+z_k)\big), t\Big),$
$N_k\Big(\big(2f(y_1+2z_1) + 2f(y_1) - 4f(y_1+z_1) - 4f(z_1) \ldots, 2f(y_k+2z_k) + 2f(y_k) - 4f(y_k+z_k) - 4f(z_k)\big), t\Big),$
$N_k\Big(\big(f(x_1+y_1+2z_1) + f(x_1-y_1-2z_1) - 2f(x_1) - 2f(y_1+2z_1), \ldots, f(x_k+y_k+2z_k) +$

$f(x_k - y_k - 2z_k) - 2f(x_k) - 2f(y_k + 2z_k)\big), t\Big)$,
$N_k\Big(\big(2f(x_1 + z_1) + 2f(x_1 - z_1) - 4f(x_1) - 4f(z_1), \ldots, 2f(x_k + z_k) + 2f(x_k - z_k) - 4f(x_k) - 4f(z_k)\big), t\Big)\Big\} \geqslant N(\alpha, 5t)$.

This inequality holds for all $k \in \mathbb{N}$, all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and all $t > 0$ (if $\min\{N_k(x,t), N_k(y,t)\} \leqslant N(\beta_k, t)$ we get it immediately by (12)). Now the result is deduced from Lemma 2.6. ■

Now we are ready to prove the main result of this section.

Theorem 2.9 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space. Suppose that $f : E \to F$ is a mapping such that $f(0) = 0$. Then f is quadratic if and only if for every $k \in \mathbb{N}$

$$N_k\Big(\big(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1) - 2f(y_1), \ldots,$$

$$f(x_k + y_k) + f(x_k - y_k) - 2f(x_k) - 2f(y_k)\big), t\Big) \to 1$$

as

$$\min\Big\{N_k\Big(\big(x_1, \ldots, x_k\big), t\Big), N_k\Big(\big(y_1, \ldots, y_k\big), t\Big)\Big\} \to 0. \tag{14}$$

Proof. If f is quadratic, then evidently (14) holds. Conversely, let (Z, N) and $(\mathbb{R}, N)$ be fuzzy-normed spaces. Fix nonzero vector α in Z. Using the limits (14), we can find for every $n \in \mathbb{N}$ a sequence $\{\beta_{n_k}\}$ in $\mathbb{R}$ such that

$$N_k\Big(\big(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1) - 2f(y_1), \ldots, f(x_k + y_k)$$

$$+ f(x_k - y_k) - 2f(x_k) - 2f(y_k)\big), t\Big) \geqslant N(\frac{\alpha}{n}, 5t),$$

for all $k \in \mathbb{N}$ and all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$, with

$$\min\Big\{N_k\Big(\big(x_1, \ldots, x_k\big), t\Big), N_k\Big(\big(y_1, \ldots, y_k\big), t\Big)\Big\} \leqslant N(\beta_{n_k}, t).$$

In view of Lemma 2.8, for every $n \in \mathbb{N}$ there exists a unique quadratic mapping Q_n such that

$$N(Q_n(x) - f(x), 15t) \geqslant N(\frac{\alpha}{n}, 5t) \tag{15}$$

for all $x \in E$ and all $t > 0$. Since $N(Q_1(x) - f(x), 15t) \geqslant N(\alpha, 5t)$ and $N(Q_n(x) - f(x), 15t) \geqslant N(\frac{\alpha}{n}, 5t) \geqslant N(\alpha, 5t)$ by the uniqueness of Q_1, we conclude that $Q_n = Q_1$ for all $n \in \mathbb{N}$. Now, tending with n to infinity in (15), we deduce that $f = Q_1$ and hence f is quadratic. ■

3. Superstability of higher derivations in multi-fuzzy Banach algebras

In this section, we define the notion of a (ϕ, ϕ')-approximate strongly higher derivation in multi-fuzzy Banach algebras. Our scope is obtaining the superstability of strongly higher derivations under the frame of multi-fuzzy Banach algebras. To achieve our goal, we give the following result which can be proved by a similar argument to that in Lemma 2.6.

Lemma 3.1 Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space and $\phi : E \to F$ be a map such that $\phi(2x) = \frac{1}{2}\phi(x)$ for all $x \in E$. Let $f : E \to F$ be a mapping such that $f(0) = 0$ and

$$N_k\Big(\big(f(\mu x_1 + y_1) - \mu f(x_1) - f(y_1), \ldots, f(\mu x_k + y_k) - \mu f(x_k) - f(y_k)\big), t\Big)$$

$$\geqslant N_k\Big(\big(\phi(x_1 + y_1), \ldots, \phi(x_k + y_k)\big), t\Big),$$

for all $\mu \in \mathbb{T}$, all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and all $t > 0$. Then there exists a unique linear mapping $T : E \to F$ such that

$$N_k\Big(\big(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)\big), 2t\Big) \geqslant N_k\Big(\big(\phi(x_1), \ldots, \phi(x_k)\big), t\Big)$$

for all $x_1, \ldots, x_k \in E$ and all $t > 0$.

Note that in the above lemma $T(x) := N - \lim\limits_{n\to\infty} \dfrac{f(2^n x)}{2^n}$ for each $x \in E$.

Definition 3.2 Let $\mathcal{A}$ be an algebra and $\{(\mathcal{A}^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space. Then $\{(\mathcal{A}^k, N_k), k \in \mathbb{N}\}$ is said to be a multi-fuzzy normed algebra if for every $k \in \mathbb{N}$, $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathcal{A}$ and $t, s \in \mathbb{R}$

$$\begin{aligned} N_k\Big(\big((a_1, \ldots, a_k)(b_1, \ldots, b_k)\big), ts\Big) &= N_k\Big(\big(a_1 b_1, \ldots, a_k b_k\big), ts\Big) \\ &\geqslant \max\Big\{N_k\Big(\big(a_1, \ldots, a_k\big), t\Big)\ ,\ N_k\Big(\big(b_1, \ldots, b_k\big), s\Big)\Big\}. \end{aligned}$$

Further, the multi-fuzzy normed algebra $\{(\mathcal{A}^k, N_k), k \in \mathbb{N}\}$ is said to be a multi-fuzzy Banach algebra if $\{(\mathcal{A}^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy Banach space.

Definition 3.3 Let $\{(\mathcal{A}^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach algebra, $\phi : \mathcal{A} \to \mathcal{A}$ be a map such that $\phi(2a) = \frac{1}{2}\phi(a)$ for all $a \in \mathcal{A}$ and let $\phi' : \mathcal{A} \times \mathcal{A} \times \mathbb{R} \to [0, 1]$ be a control function such that

$$\phi'(2^m a, 2^n b, t) \geqslant \phi'(a, b, \frac{t}{\beta^{m+n}}),$$

for some $0 < \beta < 1$, all nonnegative numbers m, n, all $a, b \in \mathcal{A}$ and all $t > 0$. Moreover, assume that $\phi'(a, b, .)$ is an increasing function of $\mathbb{R}$, $\lim\limits_{t\to 0}\phi'(a, b, t) = 0$ and $\lim\limits_{t\to\infty}\phi'(a, b, t) = 1$ for all $a, b \in \mathcal{A}$. A (ϕ, ϕ')-approximate strongly higher derivation of rank n_0 is a sequence $\{f_j\}_{j=0}^{n_0}$ of mappings $f_j : \mathcal{A} \to \mathcal{A}$ with $f_j(0) = 0$, $f_0 = id_{\mathcal{A}}$ and such that

$$N_k\Big(\big(f_j(\mu x_1+y_1)-\mu f_j(x_1)-f_j(y_1),\dots,f_j(\mu x_k+y_k)-\mu f_j(x_k)-f_j(y_k)\big),t\Big)$$

$$\geqslant N_k\Big(\big(\phi(x_1+y_1),\dots,\phi(x_k+y_k)\big),t\Big), \tag{16}$$

for all $0\leqslant j\leqslant n_0$, all $\mu\in\mathbb{T}$, all $x_1,\dots,x_k,y_1,\dots,y_k\in\mathcal{A}$ and all $t>0$ and also

$$N\Big(f_j(ab)-\sum_{l=0}^{j}f_l(a)f_{j-l}(b),t\Big)\geqslant\phi'(a,b,t), \tag{17}$$

for all $0\leqslant j\leqslant n_0$, all $a,b\in\mathcal{A}$ and all $t>0$.

Theorem 3.4 Every (ϕ,ϕ')-approximate strongly higher derivation in a multi-fuzzy Banach algebra is a higher derivation.

Proof. Let $\{(\mathcal{A}^k,N_k),k\in\mathbb{N}\}$ be a multi-fuzzy Banach algebra and let $\{f_j\}_{j=0}^{n_0}$ be a (ϕ,ϕ')-approximate higher derivation. According to Lemma 3.1, for each $j=0,\dots,n_0$, there is a linear mapping D_j defined by $D_j(a):=N\text{-}\lim\limits_{n\to\infty}\dfrac{f_j(2^na)}{2^n}$ $(a\in\mathcal{A})$ such that

$$N(D_j(a)-f_j(a),2t)\geqslant N(\phi(a),t),$$

for all $a\in\mathcal{A}$ and all $t>0$. Trivially $D_0=f_0=id_{\mathcal{A}}$. Making use of (17), it follows that

$$N\Big(\frac{f_j(4^nab)}{4^n}-\sum_{l=0}^{j}\frac{f_l(2^na)f_{j-l}(2^nb)}{4^n},t\Big)\geqslant\phi'(2^na,2^nb,4^nt)\geqslant\phi'(a,b,\frac{t}{\beta^{2n}}),$$

for all $0\leqslant j\leqslant n_0$, all $a,b\in\mathcal{A}$ and all $t>0$. Now on taking limit as $n\to\infty$, we deduce

$$N\Big(\frac{f_j(4^nab)}{4^n}-\sum_{l=0}^{j}\frac{f_l(2^na)f_{j-l}(2^nb)}{4^n},t\Big)\to 1.$$

On the other hand, we have

$$\begin{aligned}N\big(D_j(ab)-\sum_{l=0}^{j}D_l(a)D_{j-l}(b),3t\big)\geqslant\min\Big\{&N\big(D_j(ab)-\frac{f_j(4^nab)}{4^n},t\big),\\&N\big(\frac{f_j(4^nab)}{4^n}-\sum_{l=0}^{j}\frac{f_l(2^na)f_{j-l}(2^nb)}{4^n},t\big),\\&\min\big\{N\big(\frac{f_l(2^na)f_{j-l}(2^nb)}{4^n}-D_l(a)D_{j-l}(b),\frac{t}{j+1}\big),\\&\quad l=0,\dots,n_0\big\}\Big\}.\end{aligned}$$

Letting $n \to \infty$, we derive $D_j(ab) = \sum_{l=0}^{j} D_l(a)D_{j-l}(b)$. Thus $\{D_j\}_{j=0}^{n_0}$ is a strongly higher derivation. In what follows, we show that $f_j = D_j$ for all $j = 1, \ldots, n_0$. On using (16), (17) and Lemma 3.1, we obtain

$$
\begin{aligned}
N\big(f_1(2^m a) - 2^m f_1(a), 4t\big) &\geqslant N\big(2^n f_1(2^m a) - 2^{m+n} f_1(a), 4t\big) \\
&\geqslant \min\Big\{N\big(f_1((2^n 1)(2^m a)) - 2^n 1 f_1(2^m a) - f_1(2^n 1)2^m a, 2t\big), \\
&N\big(f_1((2^n 1)(2^m a)) - f_1(2^n 1)2^m a - 2^{n+m} 1 f_1(a), 2t\big)\Big\} \\
&\geqslant \min\Big\{\phi'\big(a, 1, \frac{2t}{\beta^{m+n}}\big), N\big(f_1((2^n 1)(2^m a)) - D_1((2^n 1)(2^m a)), t\big), \\
&N\big(D_1((2^n 1)(2^m a)) - 2^{n+m} 1 f_1(a) - f_1(2^n 1)2^m a, t\big)\Big\} \\
&\geqslant \min\Big\{\phi'\big(a, 1, \frac{2t}{\beta^{m+n}}\big), N\big(\phi(a), \frac{2^{n+m} t}{2}\big), \\
&N\big(D_1((2^n 1)(2^m a)) - f_1(2^n 1a)2^m, \frac{t}{2}\big), \\
&N\big(f_1(2^n 1a)2^m - f_1(2^n 1)2^m a - 2^{n+m} 1 f_1(a), \frac{t}{2}\big)\Big\} \\
&\geqslant \min\Big\{\phi'\big(a, 1, \frac{2t}{\beta^{m+n}}\big), N\big(\phi(a), 2^{n+m-1} t\big), \\
&N\big(\phi(a), \frac{t}{2^{m-n+2}}\big), \phi'\big(1, a, \frac{t}{2^{m+1}\beta^n}\big)\Big\},
\end{aligned}
$$

for all nonnegative integers m, n and all $t > 0$. Fix m and let n tends to infinity in the above inequality, thus $f_1(2^m a) = 2^m f_1(a)$ for all m and all $a \in \mathcal{A}$. Hence $D_1(a) = N\text{-}\lim_{m\to\infty} \frac{f_1(2^m a)}{2^m} = f_1(a)$ for all $a \in \mathcal{A}$. By utilizing induction on j, one obtains that $D_j = f_j$ for all $j = 1, \ldots, n_0$. This completes the proof. ■

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