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Bipolar general Fuzzy automata

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Abstract. In this paper, we define the notion of a bipolar general fuzzy automaton, then we construct some closure operators on the set of states of a bipolar general fuzzy automaton. Also, we construct some topologies on the set of states of a bipolar general fuzzy automaton. Then we obtain some relationships between them.

Keywords: Fuzzy automata, closure operator, bipolar valued Fuzzy set.

1. Introduction

Zadeh [18] introduced the theory of fuzzy sets and, soon after, Wee [16] introduced the concept of fuzzy automata. Automata have a long history both in theory and application [1, 2] and are the prime examples of general computational systems over discrete spaces [5]. In the conventional spectrum of automata (i.e. deterministic finite-state automata, non-deterministic finite-state automata, probabilistic automata and fuzzy finite-state automata), deterministic finite-state automata have found the most application in different areas [3, 10, 11, 14]. Fuzzy automata not only provide a systematic approach for handling uncertainty in such systems, but are can also be used in continuous spaces [15]. Moreover, they are able to create capabilities which are not easily achievable by other mathematical tools [17].
In the traditional fuzzy sets, the membership degrees of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not

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belong to the fuzzy set. The membership degrees on interval (0,1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements.

2. Preliminaries

Bipolar valued fuzzy sets, which are introduced by Lee [7, 8], are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. Lee [7] introduced an extension of fuzzy sets named bipolar valued fuzzy sets. Bipolar valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter property. In a bipolar valued fuzzy sets, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on (0,1] indicate that elements somewhat satisfy the implicit counter property [7]. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter property. This kind of bipolar valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). Let X be the universe of discourse. A bipolar valued fuzzy set φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. The positive membership degree $\varphi^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$ but somewhat satisfies the counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$. It is possible for an element x to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of the domain [8]. For the sake of simplicity, we shall use the symbol $\varphi =< \varphi^-, \varphi^+ >$ for the bipolar valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar valued fuzzy sets. In 2004, M. Doostfatemeh and S. C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [4].

Definition 2.1 [9] Let Σ be a set. A word of Σ is the product of a finite sequence of elements in Σ, Λ is empty word and Σ^* is the set of all words on Σ. In fact, Σ^* is the free monoid on Σ. The length $\ell(x)$ of word $x \in X^*$ is the number of its letters, so $\ell(\Lambda) = 0$.

Definition 2.2 [9] Let X be an arbitrary set. The function $\psi : P(X) \longrightarrow P(X)$ is called a closure operator on X, if for any two elements A and B of $P(X)$, we have

(i) $\psi(\emptyset) = \emptyset$,
(ii) $A \subseteq \psi(A)$,
(iii) $\psi(A \bigcup B) = \psi(A) \bigcup \psi(B)$,
(iv) $\psi(\psi(A)) = \psi(A)$.

3. Bipolar general Fuzzy automata

In this section, we introduce several new concepts and derive related results.

Definition 3.1 A bipolar fuzzy finite-state automaton is a six-tuple denoted as $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta =< \delta^-, \delta^+ >$ is the bipolar fuzzy transition function in $Q \times \Sigma \times Q$ and $\omega : Q \to Z$ is the output function. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j) =< \delta^-(q_i, a_k, q_j), \delta^+(q_i, a_k, q_j) >$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a tow-tuple value, it refers to the weight of the transition. Otherwise, it specifies the transition itself. Also, the set of all transitions of $\tilde{F}$ is denoted as $\Delta =< \Delta^-, \Delta^+ >$.

For a nonempty set X, $\tilde{P}(X)$ denotes the set of all bipolar fuzzy sets on X.

Definition 3.2 A bipolar general fuzzy automaton $\tilde{F}$ is an eight-tuple machine denoted as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R} =< \tilde{R}^-, \tilde{R}^+ >$ is the set of bipolar fuzzy start states,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} =< \tilde{\delta}^-, \tilde{\delta}^+ >$ is a bipolar fuzzy set, where $\tilde{\delta}^- : (Q \times [-1, 0]) \times \Sigma \times Q \to [-1, 0]$ and $\tilde{\delta}^+ : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$,
(vii) $F_1 =< F_1^-, F_1^+ >$, where $F_1^- : [-1, 0] \times [-1, 0] \to [-1, 0]$ and $F_1^+ : [0, 1] \times [0, 1] \to [0, 1]$ are called membership assignment functions.
As we can notice, the function $F_1^+(\mu^+, \delta^+)$, is motivated by two parameters μ^+ and δ^+, where μ^+ is the membership value of a predecessor and δ^+ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) =< \mu^{t+1}(q_j)^-, \mu^{t+1}(q_j)^+ >$$

where

$$\mu^{t+1}(q_j)^- = \tilde{\delta}^-((q_i, \mu^t(q_i)^-), a_k, q_j) = F_1^-(\mu^t(q_i)^-, \delta^-(q_i, a_k, q_j))$$

and

$$\mu^{t+1}(q_j)^+ = \tilde{\delta}^+((q_i, \mu^t(q_i)^+), a_k, q_j) = F_1^+(\mu^t(q_i)^+, \delta^+(q_i, a_k, q_j)).$$

Which means that membership value (mv) of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

(viii) $F_2 =< F_2^-, F_2^+ >$, where $F_2^- : [-1, 0]^* \to [-1, 0]$ and $F_2^+ : [0, 1]^* \to [0, 1]$ are called multi-membership resolution functions.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i) =< Q_{act}(t_i)^-, Q_{act}(t_i)^+ >$ be the bipolar fuzzy set of all active states at time t_i, $\forall i \geqslant 0$. We have $Q_{act}(t_0) = \tilde{R}$,

$$Q_{act}(t_i)^- = \{(q, \mu^{t_i}(q)^-) : \exists q' \in Q_{act}(t_{i-1})^-, \exists a \in \Sigma, \delta^-(q', a, q) \in \Delta^-\}, \forall i \geqslant 1,$$

$$Q_{act}(t_i)^+ = \{(q, \mu^{t_i}(q)^+) : \exists q' \in Q_{act}(t_{i-1})^+, \exists a \in \Sigma, \delta^+(q', a, q) \in \Delta^+\}, \forall i \geqslant 1.$$

Since $Q_{act}(t_i)^+$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)^+$ and T is a subset of $Q_{act}(t_i)^+$, we should write:

$$q \in Domain(Q_{act}(t_i)^+) \text{ and } T \subset Domain(Q_{act}(t_i)^+).$$

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)^+$ and $T \subset Q_{act}(t_i)^+$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 3.3. (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t + 1$, the following algorithm will assign a unified membership value to that:

(1) Each transition weight $\delta(q_i, a_k, q_j) =< \delta^-(q_i, a_k, q_j), \delta^+(q_i, a_k, q_j) >$ together with $\mu^t(q_i) =< \mu^t(q_i)^-, \mu^t(q_i)^+ >$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this $v_i =< v_i^-, v_i^+ >$,

$$v_i^- = \tilde{\delta}^-((q_i, \mu^t(q_i)^-), a_k, q_j) = F_1^-(\mu^t(q_i)^-, \delta^-(q_i, a_k, q_j)),$$

$$v_i^+ = \tilde{\delta}^+((q_i, \mu^t(q_i)^+), a_k, q_j) = F_1^+(\mu^t(q_i)^+, \delta^+(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j)^- = \underset{i=1}{\overset{n}{F_2^-}}[v_i^-] = \underset{i=1}{\overset{n}{F_2^-}}[F_1^-(\mu^t(q_i)^-, \delta^-(q_i, a_k, q_j))],$$

$$\mu^{t+1}(q_j)^+ = \underset{i=1}{\overset{n}{F_2^+}}[v_i^+] = \underset{i=1}{\overset{n}{F_2^+}}[F_1^+(\mu^t(q_i)^+, \delta^+(q_i, a_k, q_j))].$$

Where

- n: is the number of simultaneous transitions to the active state q_j at time $t + 1$.
- $\delta(q_i, a_k, q_j)$: is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t + 1$.

Definition 3.4. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a bipolar general fuzzy automaton, which is defined in Definition 3.2. We define max-min bipolar general fuzzy automata of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

where $Q^-_{act} = \{Q_{act}(t_0)^-, Q_{act}(t_1)^-, Q_{act}(t_2)^-, \dots\}$,
$Q^+_{act} = \{Q_{act}(t_0)^+, Q_{act}(t_1)^+, Q_{act}(t_2)^+, \dots\}$ and $\tilde{\delta}^* =< \tilde{\delta}^{*-}, \tilde{\delta}^{*+} >$ is a bipolar fuzzy set in $Q_{act} \times \Sigma^* \times Q$ and let for every i, $i \geqslant 0$

$$\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), \Lambda, p) = \begin{cases} -1, \, q = p, \\ 0, \quad otherwise \end{cases}$$

$$\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), \Lambda, p) = \begin{cases} 1, \, q = p, \\ 0, \, otherwise \end{cases}$$

and for every i, $i \geqslant 1$
$\tilde{\delta}^{*-}((q, \mu^{t_{i-1}}(q)^-), u_i, p) = \tilde{\delta}^{-}((q, \mu^{t_{i-1}}(q)^-), u_i, p)$,
$\tilde{\delta}^{*+}((q, \mu^{t_{i-1}}(q)^+), u_i, p) = \tilde{\delta}^{+}((q, \mu^{t_{i-1}}(q)^+), u_i, p)$,
$\tilde{\delta}^{*-}((q, \mu^{t_{i-1}}(q)^-), u_i u_{i+1}, p)$
$= \bigwedge_{q' \in Q_{act}(t_i)^-} (\tilde{\delta}^{-}((q, \mu^{t_{i-1}}(q)^-), u_i, q') \vee \tilde{\delta}^{-}((q', \mu^{t_i}(q')^-), u_{i+1}, p))$,

$\tilde{\delta}^{*+}((q, \mu^{t_{i-1}}(q)^+), u_i u_{i+1}, p)$
$= \bigvee_{q' \in Q_{act}(t_i)^+} (\tilde{\delta}^{+}((q, \mu^{t_{i-1}}(q)^+), u_i, q') \wedge \tilde{\delta}^{+}((q', \mu^{t_i}(q')^+), u_{i+1}, p))$,
and recursively
$\tilde{\delta}^{*-}((q, \mu^{t_0}(q)^-), u_1 u_2 \dots u_n, p)$
$= \wedge\{\tilde{\delta}^{-}((q, \mu^{t_0}(q)^-), u_1, p_1) \vee \tilde{\delta}^{-}((p_1, \mu^{t_1}(p_1)^-), u_2, p_2) \vee \dots$
$\vee \tilde{\delta}^{-}((p_{n-1}), \mu^{t_{n-1}}(p_{n-1})^-), u_n, p) | p_1 \in Q_{act}(t_1)^-, p_2 \in Q_{act}(t_2)^-, \dots,$
$p_{n-1} \in Q_{act}(t_{n-1})^-\}$,

$\tilde{\delta}^{*+}((q, \mu^{t_0}(q)^+), u_1 u_2 \dots u_n, p)$
$= \vee\{\tilde{\delta}^{+}((q, \mu^{t_0}(q)^+), u_1, p_1) \wedge \tilde{\delta}^{+}((p_1, \mu^{t_1}(p_1)^+), u_2, p_2) \wedge \dots$
$\wedge \tilde{\delta}^{+}((p_{n-1}), \mu^{t_{n-1}}(p_{n-1})^+), u_n, p) | p_1 \in Q_{act}(t_1)^+, p_2 \in Q_{act}(t_2)^+, \dots,$
$p_{n-1} \in Q_{act}(t_{n-1})^+\}$,
in which $u_i \in \Sigma$, $\forall 1 \leqslant i \leqslant n$ and assuming that the entered input at time t_i be u_i, $\forall 1 \leqslant i \leqslant n-1$.

Definition 3.5. Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton. The response function $r^{\tilde{F}^*} =< r^{\tilde{F}^* -}, r^{\tilde{F}^* +} >$ of $\tilde{F}^*$ is defined by

$$r^{\tilde{F}^* -}(x, q) = \bigwedge_{q' \in Q_{act}(t_0)^-} \tilde{\delta}^{*-}((q', \mu^{t_0}(q')^-), x, q),$$

$$r^{\tilde{F}^* +}(x, q) = \bigvee_{q' \in Q_{act}(t_0)^+} \tilde{\delta}^{*+}((q', \mu^{t_0}(q')^+), x, q),$$

for any $x \in \Sigma^*$, $q \in Q$, where $r^{\tilde{F}^*-} : \Sigma^* \times Q \to [-1, 0]$, $r^{\tilde{F}^*+} : \Sigma^* \times Q \to [0, 1]$.

Definition 3.6. Let $q \in Q$ and $0 \leqslant c < 1$. Then q is called an accessible state of $\tilde{F}^*$ with threshold c if there exist $x, y \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, q)^- < -c$ and $r^{\tilde{F}^*}(y, q)^+ > c$.

Definition 3.7. Let $A \subseteq Q$. Then $\tilde{F}^*$ is said to be connected with threshold c on A, if $A = \bar{Q}_c$, where $\bar{Q}_c$ is the set of all accessible states with threshold c.

Definition 3.8. Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton, $p \in Q, q \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+, i \geqslant 0$ and $0 \leqslant c < 1$. Then p is called a bipolar successor of q with threshold c if there exist $x, y \in \Sigma^*$ such that $\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), x, p) < -c$ and $\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), y, p) > c$.

Definition 3.3 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton, $q \in Q_{act}(t_i), i \geqslant 0$ and $0 \leqslant c < 1$. Also let $S_c(q)$ denote the set of all bipolar successors of q with threshold c. If $T \subseteq Q$, then $S_c(T)$ the set of all bipolar successors of T with threshold c is defined by

$$S_c(T) = \bigcup \{S_c(q) : q \in T\}.$$

Theorem 3.4 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and $0 \leqslant c < 1$. Then
(i) $q \in S_c(q), \forall q \in Q$.
(ii) If $r \in S_c(p)$, $p \in S_c(q)$, then $r \in S_c(q)$.

Proof. (i) Since for all $q \in Q$, there exists $i \geqslant 0$ such that $q \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+$, $\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), \Lambda, q) = -1 < -c$ and $\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), \Lambda, q) = 1 > c$, then $q \in S_c(q)$.
(ii) Since $p \in S_c(q)$, then $q \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+$ and there exist $x, y \in \Sigma^*$ such that $\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), x, p) < -c$ and $\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), y, p) > c$. Also, since $r \in S_c(p)$, then $p \in Q_{act}(t_j)^- \cap Q_{act}(t_j)^+$ and there exist $z, t \in \Sigma^*$ such that $\tilde{\delta}^{*-}((p, \mu^{t_j}(p)^-), z, r) < -c$ and $\tilde{\delta}^{*+}((p, \mu^{t_j}(p)^+), t, r) > c$. Thus, we have

$$\begin{aligned} &\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), xz, r) \\ = &\bigwedge_{q' \in Q_{act}(t_j)^-} [\tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), x, q') \bigvee \tilde{\delta}^{*+}((q', \mu^{t_j}(q')^-), z, r)] \\ &\leqslant \tilde{\delta}^{*-}((q, \mu^{t_i}(q)^-), x, p) \bigvee \tilde{\delta}^{*-}((p, \mu^{t_j}(p)^-), z, r) < -c, \\ &\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), yt, r) \\ = &\bigvee_{q' \in Q_{act}(t_j)^+} [\tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), y, q') \bigwedge \tilde{\delta}^{*+}((q', \mu^{t_j}(q')^+), t, r)] \\ &\geqslant \tilde{\delta}^{*+}((q, \mu^{t_i}(q)^+), y, p) \bigwedge \tilde{\delta}^{*+}((p, \mu^{t_j}(p)^+), t, r) > c. \end{aligned}$$

So we get that $r \in S_c(q)$. ■

***Example* 3.5** Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min bipolar general fuzzy automaton, where
$Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b\}$, $Q_{act}(t_0)^- = \{(q_0, \mu^{t_0}(q_0)^-)\} = \{(q_0, -1)\}$,
$Q_{act}(t_0)^+ = \{(q_0, \mu^{t_0}(q_0)^+)\} = \{(q_0, 1)\}$, $F_1^-(\mu^-, \delta^-) = Min(\mu^-, \delta^-)$,
$F_1^+(\mu^+, \delta^+) = Max(\mu^+, \delta^+)$, $Z = \emptyset$, ω and F_2 are not applicable,
$\delta(q_0, a, q_1) =< -0.4, 0.4 >$, $\delta(q_0, b, q_2) =< -0.5, 0.5 >$, $\delta(q_1, a, q_2) =< -0.3, 0.3 >$, $\delta(q_2, a, q_2) =< -0.2, 0.2 >$. If we choose the input string $x = aa \ldots a$, then we have

$$Q_{act}(t_1)^- = \{(q_1, \mu^{t_1}(q_1)^-)\}, \qquad Q_{act}(t_i)^- = \{(q_2, \mu^{t_i}(q_2)^-)\}, \forall i \geqslant 2,$$
$$Q_{act}(t_1)^+ = \{(q_1, \mu^{t_1}(q_1)^+)\}, \qquad Q_{act}(t_i)^+ = \{(q_2, \mu^{t_i}(q_2)^+)\}, \forall i \geqslant 2,$$

$\mu^{t_1}(q_1)^- = \tilde{\delta}^-((q_0, \mu^{t_0}(q_0)^-), a, q_1) = F_1^-(\mu^{t_0}(q_0)^-, \delta^-(q_0, a, q_1))$
$= F_1^-(-1, -0.4) = -1,$
$\mu^{t_1}(q_1)^+ = \tilde{\delta}^+((q_0, \mu^{t_0}(q_0)^+), a, q_1) = F_1^+(\mu^{t_0}(q_0)^+, \delta^+(q_0, a, q_1))$
$= F_1^+(1, 0.4) = 1,$
$\mu^{t_2}(q_2)^- = \tilde{\delta}^-((q_1, \mu^{t_1}(q_1)^-), a, q_2) = F_1^-(\mu^{t_1}(q_1)^-, \delta^-(q_1, a, q_2))$
$= F_1^-(-1, -0.3) = -1,$
$\mu^{t_2}(q_2)^+ = \tilde{\delta}^+((q_1, \mu^{t_1}(q_1)^+), a, q_2) = F_1^+(\mu^{t_1}(q_1)^+, \delta^+(q_1, a, q_2))$
$= F_1^+(1, 0.3) = 1,$
$\mu^{t_3}(q_2)^- = \tilde{\delta}^-((q_2, \mu^{t_2}(q_2)^-), a, q_2) = F_1^-(\mu^{t_2}(q_2)^-, \delta^-(q_2, a, q_2))$
$= F_1^-(-1, -0.2) = -1,$
$\mu^{t_3}(q_2)^+ = \tilde{\delta}^+((q_2, \mu^{t_2}(q_2)^+), a, q_2) = F_1^+(\mu^{t_2}(q_2)^+, \delta^+(q_2, a, q_2))$
$= F_1^+(1, 0.2) = 1,$
$\mu^{t_i}(q_2)^- = -1, \forall i \geqslant 4,\ \mu^{t_i}(q_2)^+ = 1, \forall i \geqslant 4,$
$\tilde{\delta}^{*-}((q_0, \mu^{t_0}(q_0)^-), a, q_1) = -1,$
$\tilde{\delta}^{*+}((q_0, \mu^{t_0}(q_0)^+), a, q_1) = 1,$
$\tilde{\delta}^{*-}((q_0, \mu^{t_0}(q_0)^-), a^2, q_2) = -1$
$\tilde{\delta}^{*+}((q_0, \mu^{t_0}(q_0)^+), a^2, q_2) = 1.$
Also, we have $\tilde{\delta}^{*-}((q_0, \mu^{t_0}(q_0)^-), \Lambda, q_0) = -1$ and $\tilde{\delta}^{*+}((q_0, \mu^{t_0}(q_0)^+), \Lambda, q_0) = 1$. Thus $S_c(q_0) = Q$, $\forall c$, $0 \leqslant c < 1$.

Theorem 3.6 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and $0 \leqslant c < 1$. We define

$$\begin{aligned} S_c : P(Q) &\longrightarrow P(Q) \\ A &\longrightarrow S_c(A). \end{aligned}$$

Then S_c is a closure operator on Q.

Proof. (i) $S_c(\emptyset) = \emptyset$.
(ii) Let $q \in A \subseteq Q$. By Theorem 3.10, $q \in S_c(q)$. Then $q \in S_c(q) \subseteq S_c(A)$, Thus, $A \subseteq S_c(A)$.
(iii)$S_c(A \bigcup B) = \bigcup_{q \in A \bigcup B} S_c(q) = (\bigcup_{q \in A} S_c(q)) \bigcup (\bigcup_{q \in B} S_c(q)) = S_c(A) \bigcup S_c(B)$.
(iv) By (ii), we have $S_c(A) \subseteq S_c(S_c(A))$. Conversely, let $p \in S_c(S_c(A))$. Then there exists $q' \in S_c(A)$ such that $p \in S_c(q')$. Thus $q' \in S_c(q'')$, for some $q'' \in A$. Consequently, by Theorem 3.10, $p \in S_c(q'')$. Hence, $p \in S_c(A)$. Therefore $S_c(S_c(A)) \subseteq S_c(A)$. ■

Corollary 3.7 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and λ be a fuzzy subset on Q. Then $\tau = \{A^C : A \subseteq Q, S_c(A) = A\}$ is a topology on Q.

Proof. (i) Since $S_c(\emptyset) = \emptyset$, then $Q = (\emptyset)^C \in \tau$.
(ii) Since S_c is a closure operator on Q, then $Q \subseteq S_c(Q)$. On the other hand, since $S_c(Q) \in P(Q)$, then $S_c(Q) \subseteq Q$. Thus, $S_c(Q) = Q$. Therefore we conclude that $\emptyset = (Q)^C \in \tau$.
(iii) Let A_1^C and A_2^C belong to τ. Then $S_c(A_1) = A_1$ and $S_c(A_2) = A_2$. Thus, we have

$$S_c(A_1 \bigcup A_2) = S_c(A_1) \bigcup S_c(A_2) = A_1 \bigcup A_1.$$

That is $A_1^C \bigcap A_2^C = (A_1 \bigcup A_2)^C \in \tau$.
(iv) Let $A_i^C \in \tau$, $\forall i \in I$. Then $S_c(A_i) = A_i$, $\forall i \in I$. Since S_c is a closure operator on Q, then $\bigcap_{i \in I} A_i \subseteq S_c(\bigcap_{i \in I} A_i)$. On the other hand, since $A_i \bigcup (\bigcap_{i \in I} A_i) = A_i$, so we get that $S_c(A_i) \bigcup (S_c(\bigcap_{i \in I} A_i)) = S_c(A_i)$. Then $S_c(\bigcap_{i \in I} A_i) \subseteq S_c(A_i) = A_i$. Thus $S_c(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} A_i$. Hence, $S_c(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} A_i$. That is $\bigcup_{i \in I} A_i^C = (\bigcap_{i \in I} A_i)^C \in \tau$. ■

Definition 3.8 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and $0 \leqslant c < 1$. Then we say that $\tilde{F}^*$ is good with threshold c, if $\forall q \in Q$, $\exists q' \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+$: $q \in S_c(q')$.

***Example* 3.9** Let $\tilde{F}^*$ be the max-min bipolar general fuzzy automaton in Example 3. 11 and $0 \leqslant c < 1$. Since $S_c(q_0) = Q$, then $\tilde{F}^*$ is good with threshold c.

Theorem 3.10 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and $0 \leqslant c < 1$. Then $\tilde{F}^*$ is good with threshold c if and only if $\tilde{F}^*$ is connected with threshold c on Q.

Proof. Let $\tilde{F}^*$ be good with threshold c and $q \in Q$. Then there exists $q' \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+$ such that $q \in S_c(q')$. Thus, there exist $x, y \in \Sigma^*$ such that
$\tilde{\delta}^{*-}((q', \mu^{t_i}(q')^-), x, q) < -c$ and $\tilde{\delta}^{*+}((q', \mu^{t_i}(q')^+), y, q) > c$.
So

$$r^{\tilde{F}^*-}(x, q) = \bigwedge_{q' \in Q_{act}(t_0)^-} \tilde{\delta}^{*-}((q', \mu^{t_0}(q')^-), x, q) < -c,$$

$$r^{\tilde{F}^*+}(y, q) = \bigvee_{q' \in Q_{act}(t_0)^+} \tilde{\delta}^{*+}((q', \mu^{t_0}(q')^+), y, q) > c,$$

Consequently, by Definitions 3.6, 3.7, $\tilde{F}^*$ is connected with threshold c on Q. Conversely, let $\tilde{F}^*$ be connected with threshold c on Q and $q \in Q$. Then there exist $x, y \in \Sigma^*$ such that

$$r^{\tilde{F}^*-}(x, q) = \bigwedge_{q' \in Q_{act}(t_0)^-} \tilde{\delta}^{*-}((q', \mu^{t_0}(q')^-), x, q) < -c,$$

$$r^{\tilde{F}^*+}(y, q) = \bigvee_{q' \in Q_{act}(t_0)^+} \tilde{\delta}^{*+}((q', \mu^{t_0}(q')^+), y, q) > c,$$

Hence, there exists $q' \in Q_{act}(t_i)^- \cap Q_{act}(t_i)^+$ such that $\tilde{\delta}^{*-}((q', \mu^{t_i}(q')^-), x, q) < -c$ and $\tilde{\delta}^{*+}((q', \mu^{t_i}(q')^+), y, q) > c$. So $q \in S_c(q')$. Therefore $\tilde{F}^*$ is good with threshold c. ■

Theorem 3.11 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton, $0 \leqslant c < 1$ and suppose that

$$pR_cq \Leftrightarrow p \in S_c(q), q \in S_c(p).$$

Then R_c is an equivalence relation on Q.

Proof. By Theorem 3.10 the proof is obvious. ■

Theorem 3.12 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton, $0 \leqslant c < 1$, $B_c(q) = \{p \in Q : pR_cq\}$, $B_c(A) = \bigcup_{q \in A} B_c(q)$. We define

$$B_c : P(Q) \longrightarrow P(Q)$$
$$A \longrightarrow B_c(A).$$

Then B_c is a closure operator on Q.

Proof. (i) $B_c(\emptyset) = \emptyset$.
(ii) Let $q \in A$. Since qR_cq, then $q \in B_c(q) \subseteq B_c(A)$. Thus, $A \subseteq B_c(A)$.
(iii)$B_c(A \bigcup D) = \bigcup_{q \in A \bigcup D} B_c(q) = (\bigcup_{q \in A} B_c(q)) \bigcup (\bigcup_{q \in D} B_c(q)) = B_c(A) \bigcup B_c(D)$.
(iv) By (ii), we have $B_c(A) \subseteq B_c(B_c(A))$. Conversely, let $q \in B_c(B_c(A))$. Then there exists $q' \in B_c(A)$ such that $q \in B_c(q')$. Thus $q' \in B_c(q'')$, for some $q'' \in A$. Consequently, qR_cq' and $q'R_cq''$. By Theorem 3.17, qR_cq''. Thus $q \in B_c(q'') \subseteq B_c(A)$. Therefore $B_c(B_c(A)) \subseteq B_c(A)$. ■

Corollary 3.13 Let $\tilde{F}^*$ be a max-min bipolar general fuzzy automaton and λ be a fuzzy subset on Q. Then $\xi = \{A^C : A \subseteq Q, B_c(A) = A\}$ is a topology on Q.

Proof. The proof is similar to Theorem 3.13, by using suitable modification. ■

4. Conclusions

In this paper, we have defined the notion of a bipolar general fuzzy automaton, then we have constructed some closure operators on the set of states of a bipolar general fuzzy automaton. Also, we have constructed some topologies on the set of states of a bipolar general fuzzy automaton. Then we have obtained some relationships between them.

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