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Construction of strict Lyapunov function for nonlinear parameterised perturbed systems

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Abstract. In this paper, global uniform exponential stability of perturbed dynamical systems is studied by using Lyapunov techniques. The system presents a perturbation term which is bounded by an integrable function with the assumption that the nominal system is globally uniformly exponentially stable. Some examples in dimensional two are given to illustrate the applicability of the main results.

Keywords: Perturbed systems, Lyapunov function, uniform exponential stability.

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1. Introduction

Any mathematical model adequately describing the reality in terms of differential equations involves (explicitly or implicitly) some parameters whose values are in the typical situation known only approximately, with a given accuracy. That is why the question of the characteristics of the solutions of differential equation under a small change of parameters involved in the equation is of principle interest. Since the classical works of H. Poincare and A.M. Lyapunov, the so-called regular case, has been investigated in details. The concept of stability and boundedness of solutions of parameterised systems cannot be overemphasized in the theory and applications of differential equations.

The second Lyapunov method have long played an important role in the history of stability theory, and it will no doubt continue to serve as an indispensable tool in future research papers. The strength of this methods is that knowledge of the exact solution is

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not necessary and the qualitative behavior of the solution to the system can be investigated without computing the actual solution. This method establishes the stability or instability of the origin by requiring the existence of a Lyapunov function that satisfies certain conditions and so many theorems establishing different kind of stability have been proven.(see[1],[2],...,[17]).

Although for a nonlinear time varying parameterised system, the construction of a Lyapunov function is an intractable problem, the usefulness of the "second method" of Lyapunov is reflected in the study of stability of parameterised perturbed systems. For such systems, the use of the second Lyapunov method is based on the converse theorems that provide, under certain conditions, that if the nominal system is stable, then there exists a Lyapunov function, which can be considred as a Lyapunov function candidate for the perturbed system.

By adopting the idea introduced by [17] to the study of the stability of autonomous cascades systems, another way is open by the new approach given in [11]. In this paper, a new construction of a Lyapunov function for the study of asymptotic stability of parameterised perturbed systems is investigated. The purpose of this paper is to etablish sufficient conditions for the exponential stability of a class of nonlinear time-varying parameterised systems. In the spirit of the idea of [1], [17], we study the exponential stability and we give some examples to illustrate our results.

2. Preliminaries

Consider the perturbed parameterised nonlinear time-varying systems of the form:

$$\dot{x} = f(t, x, \theta) + g(t, x, \theta) \tag{1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $\theta \in \mathbb{R}^m$ is a constant free parameter and $f, g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ are locally Lipschitz in state and piecewise continuous in time such that

$$f(t, 0, \theta) = g(t, 0, \theta) = 0, \,\forall\, t \geqslant 0, \,\forall\, \theta \in \mathbb{R}^m.$$

Lyapunov analysis can be used to shows boundedness of the solution of the state equation, even when there is no equilibrium point at the origin. In our case when the perturbation term is uniformly bounded and $g(t, x, \theta) \neq 0$, for some $t \geqslant 0$ and $\theta \in \mathbb{R}^m$ for perturbed system of the form (1). The asymptotic stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is accepted, thus the notion of practical stability can be more suitable in several situations than Lyapunov stability (see [8], [10]). Suppose that the nominal system

$$\dot{x} = f(t, x, \theta) \tag{2}$$

has a globally uniformly exponentially stable (G.U.E.S) equilibrium point at the origin with $W_\theta(t, x)$ as an associate Lyapunov function, then calculating the derivative of $W_\theta(t, x)$ along the trajectories of the perturbed system (1) one can reach the conclusion about the definiteness of $\dot{W}_\theta(t, x)$ by imposing some restrictions on the perturbation term

$g(t,x,\theta)$. Our approach is to analysis the stability of some classes of perturbed systems which can be shown to be G.U.E.S by using a Lyapunov function of the form

$$V_\theta(t,x) = W_\theta(t,x) + \Psi_\theta(t,x),$$

where the function $\Psi_\theta(t,x)$ is given by:

$$\Psi_\theta(t,x) = \int_t^{+\infty} \frac{\partial W_\theta}{\partial x}(s, \phi_\theta(s,t,x)).g(s, \phi_\theta(s,t,x), \theta)ds, \tag{3}$$

with $\phi_\theta(s,t,x)$ is the solution of the parameterised perturbed system (1) such that $\phi_\theta(t,t,x) = x$. Note that the function $\Psi_\theta(t,x)$ is chosen in such a way that the function $V_\theta(t,x)$ is positive definite, and its derivative along the trajectories of (1) is negative definite. Naturally, the choice of $\Psi_\theta(t,x)$ depends on the perturbation term $g(t,x,\theta)$ and its smoothness is given under some restrictions on the dynamics of the system.
The strict Lyapunov function decay condition $\dot{V}_\theta(t,x) < 0$, for all $x \neq 0$, all $t \geqslant 0$ and all $\theta \in \mathbb{R}^m$, means that $\frac{dV_\theta}{dt}(t, \phi_\theta(t,t_0,x_0)) < 0$, for all $t \geqslant t_0 \geqslant 0$ as long as the trajectory $\phi_\theta(t,t_0,x_0)$ is not at zero. The decay condition is equivalent to the existence of a positive definite function $\alpha(.)$ such that

$$\dot{V}_\theta(t,x) \leqslant -\alpha(\|x\|),\ \forall\, x \in \mathbb{R}^n,\ \forall\, t \geqslant 0,\ \forall\, \theta \in \mathbb{R}^m.$$

3. Definitions and notations

3.1 *Notations*

In this paper the solution of the system (2) (resp (1)), with initial condition $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ is denoted by $x_\theta(., t_0, x_0)$ (resp $\phi_\theta(., t_0, x_0)$). If $r > 0$, we denote the closed ball of $\mathbb{R}^n$ of radius r by

$$\mathcal{B}_r = \{x \in \mathbb{R}^n; \|x\| \leqslant r\}.$$

If $\rho : \mathbb{R} \longrightarrow \mathbb{C}$ is a measurable function, we denote by $\|\rho\|_p = \Big[\int_{\mathbb{R}} |\rho(s)|^p ds\Big]^{\frac{1}{p}}$ and the Lebesgue space

$$L^p = \Big\{\rho :\ \|\rho\|_p < +\infty\Big\}.$$

3.2 *Definitions*

Definition 3.1 A solution of (2) is said to be

(1) Uniformly bounded (U.B) if there exists a positive constant c independent of $t_0 \geqslant 0$ and for every $a \in]0, c[$, there is $\beta = \beta(a) > 0$, independent of t_0, such that:

$$\|x_0\| \leqslant a \Longrightarrow \|x_\theta(t,t_0,x_0)\| \leqslant \beta, \quad \forall\, t \geqslant t_0 \geqslant 0, \quad \theta \in \mathbb{R}^m. \tag{4}$$

(2) Globally uniformly bounded (G.U.B) if (4) holds for all $a > 0$.

Definition 3.2 A continuous function $\alpha : [0, a) \longrightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$.

Definition 3.3 A continuous function $\beta : [0, a[\times[0, \infty[\longrightarrow [0, \infty[$ is said to belong to class $\mathcal{KL}$ if; for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \longrightarrow 0$ as $s \longrightarrow +\infty$.

Definition 3.4 The equilibrium point $x = 0$ of (2) is globally uniformly asymptotically stable (G.U.A.S) if there exist a $\mathcal{KL}$ function $\beta(.,.)$ such that

$$\|x_\theta(t, t_0, x_0)\| \leqslant \beta(\|x_0\|, t - t_o)\,; \forall\, t \geqslant t_0, \geqslant 0\, \forall\, x_0 \in \mathbb{R}^n. \tag{5}$$

Definition 3.5 The equilibrium point $x = 0$ of (2) is globally uniformly exponentially stable (G.U.E.S) if (5) is satisfies with $\beta(r, s) = kre^{-\lambda s}$; $k,\ r,\ \lambda > 0$.

4. Mains results

Let $x = 0$ be an equilibrium point for the nonlinear nominal system (2) where the jacobian matrix $[\partial f/\partial x]$ is bounded on $\mathbb{R}^n$, uniformly in t. Let $K, \lambda > 0$, and assume that the trajectories of the nominal system satisfy:

$$\|x_\theta(t, t_0, x_0)\| \leqslant K\|x_0\| \exp(-\lambda(t - t_0)),\ \forall\, x_0 \in \mathbb{R}^n,\ \forall t \geqslant t_0 \geqslant 0,\ \forall\, \theta \in \mathbb{R}^m.$$

Then, by a classical Theorem in [7], there is a continuously differentiable Lyapunov function

$$W_\theta : [0, +\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}_+,$$

that satisfies the assumption **(H1)**, which will be indicated in the rest. Let us consider the followings assumptions.

(**H1**) The Lyapunov function W_θ satisfies:

i) $\exists\, c_1, c_2 > 0 :\ c_1\|x\|^2 \leqslant W_\theta(t, x) \leqslant c_2\|x\|^2,\ \forall\,(t, x) \in [0, +\infty[\times\mathbb{R}^n,\ \theta \in \mathbb{R}^m,$

ii) $\exists\, c_3 > 0 :\ \dot{W}_\theta(t, x) = \dfrac{\partial W_\theta}{\partial t}(t, x) + \dfrac{\partial W_\theta}{\partial x}(t, x).f(t, x, \theta) \leqslant -c_3\|x\|^2,$ for all $(t, x) \in [0, +\infty[\times\mathbb{R}^n$ and $\theta \in \mathbb{R}^m$,

iii) $\exists\, c_4 > 0;\ \|\dfrac{\partial W_\theta}{\partial x}(t, x)\| \leqslant c_4\|x\|,\ \forall(t, x) \in [0, +\infty[\times\mathbb{R}^n,\ \theta \in \mathbb{R}^m.$

(**H2**) There exists a continuous function $\rho : [0, +\infty[\longrightarrow [0, +\infty[$, such that

$$\|g(t, x, \theta)\| \leqslant \rho(t)\|x\|,\ \forall\, t \geqslant 0,\ \forall\, x \in \mathbb{R}^n, \forall\, \theta \in \mathbb{R}^m.$$

The following result gives a sufficient condition, in terms of a Lyapunov function, for the dynamical parameterised system to be globally uniformly exponentially stable.

Proposition 4.1 If $\rho \in L^p$, $p \geqslant 1$, then under assumptions **(H1)** and **(H2)**, the equilibrium point $x = 0$ of the perturbed system (1) is globally uniformly exponentially stable.

Proof. Let $(t_0, x_0) \in [0, +\infty[\times \mathbb{R}^n \setminus \{0\}$ an initial condition. In this proof we will discuss two cases

Case 1: $p = 1$ We have

$$\begin{aligned}\frac{d}{dt}(W_\theta(t, \phi(t, t_0, x_0))) &= \dot{W}_\theta(t, \phi_\theta(t, t_0, x_0)) + \frac{dW_\theta}{dx}(t, \phi_\theta(t, t_0, x_0)).g(t, \phi_\theta(t, t_0, x_0), \theta)\\ &\leqslant -c_3\|\phi_\theta(t, t_0, x_0)\|^2 + \frac{dW_\theta}{dx}(t, \phi_\theta(t, t_0, x_0)).g(t, \phi_\theta(t, t_0, x_0), \theta)\\ &\leqslant -c_3\|\phi(t, t_0, x_0)\|^2 + c_4\rho(t)\|\phi(t, t_0, x_0)\|^2,\\ &\leqslant -\frac{c_3}{c_2}W_\theta(t, \phi_\theta(t, t_0, x_0)) + \frac{c_4}{c_1}\rho(t)W_\theta(t, \phi_\theta(t, t_0, x_0)).\end{aligned}$$

It follows that,

$$\frac{\frac{d}{dt}(W_\theta(t, \phi_\theta(t, t_0, x_0)))}{W_\theta(t, \phi_\theta(t, t_0, x_0))} \leqslant -\frac{c_3}{c_2} + \frac{c_4}{c_1}\rho(t),$$

by integration between t and t_0, we obtain the inequality, for all $t \geqslant t_0 \geqslant 0$

$$\|\phi_\theta(t, t_0, x_0)\| \leqslant \sqrt{\frac{c_2}{c_1}}\, e^{\frac{c_4}{2c_1}\|\rho\|_1}\|x_0\|e^{-\frac{c_3}{2c_2}(t-t_0)}, \ \forall\, \theta \in \mathbb{R}^m.$$

Case 2: $p > 1$ Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, by assumption **(H1)** we have,

$$\begin{aligned}\frac{d}{dt}(W_\theta(t, \phi_\theta(t, t_0, x_0))) &\leqslant -c_3\|\phi_\theta(t, t_0, x_0)\|^2 + \frac{dW_\theta}{dx}(t, \phi_\theta(t, t_0, x_0)).g(t, \phi(t, t_0, x_0), \theta)\\ &\leqslant -c_3\|\phi_\theta(t, t_0, x_0)\|^2 + c_4\rho(t)\|\phi_\theta(t, t_0, x_0)\|^2,\\ &\leqslant -c_3\|\phi_\theta(t, t_0, x_0)\|^2 + c_4\Big[\frac{\rho(t)}{\varepsilon}\|\phi_\theta(t, t_0, x_0)\|^{\frac{2}{p}}\varepsilon\|\phi_\theta(t, t_0, x_0)\|^{\frac{2}{q}}\Big],\ \varepsilon > 0.\end{aligned}$$

By Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}(W_\theta(t, \phi_\theta(t, t_0, x_0))) &\leqslant -c_3\|\phi_\theta(t, t_0, x_0)\|^2 + c_4\Big[\frac{\rho^p(t)}{\varepsilon^p}\|\phi_\theta(t, t_0, x_0)\|^2 + \varepsilon^q\|\phi_\theta(t, t_0, x_0)\|^2\Big]\\ &\leqslant -(c_3 - \varepsilon^q)\|\phi_\theta(t, t_0, x_0)\|^2 + \frac{c_4}{\varepsilon^p}\rho^p(t)\|\phi_\theta(t, t_0, x_0)\|^2.\end{aligned}$$

If one chooses $0 < \varepsilon < \sqrt[q]{c_3}$, then we obtain the following inequality

$$\frac{d}{dt}(W_\theta(t, \phi_\theta(t, t_0, x_0))) \leqslant -\frac{c_3 - \varepsilon^q}{c_2}W_\theta(t, \phi_\theta(t, t_0, x_0) + \frac{c_4}{c_1\varepsilon^p}\rho^p(t)W_\theta(t, \phi_\theta(t, t_0, x_0).$$

By integration the above inequality between t and t_0, we obtain

$$\int_{t_0}^{t} \frac{\frac{d}{ds}\Big(W_\theta(s,\phi_\theta(s,t_0,x_0))\Big)}{W_\theta(s,\phi_\theta(s,t_0,x_0)}ds \leqslant -\frac{c_3-\varepsilon^q}{c_2}(t-t_0)+\int_{t_0}^{t}\frac{c_4}{c_1\varepsilon^p}\rho^p(s)ds$$
$$\leqslant -\frac{c_3-\varepsilon^q}{c_2}(t-t_0)+\frac{c_4}{c_1\varepsilon^p}\|\rho\|_p^p$$

Thus, we obtain the estimation

$$\ln\Big[\frac{W_\theta(t,\phi(t,t_0,x_0))}{W_\theta(t_0,x_0)}\Big] \leqslant -\frac{c_3-\varepsilon^q}{c_2}(t-t_0)+\frac{c_4}{c_1\varepsilon^p}\|\rho\|_p^p$$

It follows that, for all $t \geqslant t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$

$$\|\phi_\theta(t,t_0,x_0)\| \leqslant \sqrt{\frac{c_2}{c_1}}e^{\frac{c_4}{2c_1\varepsilon^p}\|\rho\|_p^p}\|x_0\|e^{-\frac{c_3-\varepsilon^q}{2c_2}(t-t_0)},\ \forall\,\theta\in\mathbb{R}^m.$$

■

Now, we study the case when the perturbation term satisfies the following assumption

(**H3**) There exists a continuous and integrable function $\rho : [0,+\infty[\longrightarrow [0,+\infty[$, such that

$$\|g(t,x,\theta)\| \leqslant \rho(t)\|x\|^\alpha,\ 0<\alpha<1,\ \forall\,t\geqslant 0,\ \forall\,x\in\mathbb{R}^n,\ \forall\,\theta\in\mathbb{R}^m.$$

Remark 1 *In this case the Lyapunov function $W_\theta(t,x)$ does not directly that the perturbed system is globally asymptotically stable. That is why we will seek a Lyapunov function having the following form*

$$V_\theta(t,x) = W_\theta(t,x)+\Psi_\theta(t,x)$$

witch satisfies,

$$d_1(\|x\|) \leqslant V_\theta(t,x) \leqslant d_2(\|x\|),$$

and

$$\dot{V}_\theta(t,x) = \dot{W}_\theta(t,x) \leqslant -c_3\|x\|^2,$$

with d_1, d_2 are two $\mathcal{K}_\infty$ functions.

The next proposition shows the global uniform boundedness of solution of perturbed system (1) under conditions **(H1)** and **(H3)**.

Proposition 4.2 Under the assumptions **(H1)** and **(H3)**, the solution $\phi_\theta(.,t_0,x_0)$ of the perturbed system (1) is globally uniformly bounded.

Proof. Let $(t_0, x_0) \in [0, +\infty[\times\mathbb{R}^n\backslash\{0\}$ an initial condition. The derivative of $W_\theta(t,x)$ along the trajectories of (1) is given by, for all $t \geqslant t_0 \geqslant 0$

$$\frac{d}{dt}(W_\theta(t,\phi_\theta(t,t_0,x_0))) = \frac{dW_\theta}{dt}(t,\phi_\theta(t,t_0,x_0)) + \frac{dW_\theta}{dx}(t,\phi_\theta(t,t_0,x_0)).f(t,\phi_\theta(t,t_0,x_0),\theta) + \frac{dW_\theta}{dx}(t,\phi_\theta(t,t_0,x_0)).g(t,\phi_\theta(t,t_0,x_0),\theta).$$

It follows by assumption **(H1)**, that

$$\begin{aligned}\frac{d}{dt}(W_\theta(t,\phi_\theta(t,t_0,x_0))) &\leqslant \frac{dW_\theta}{dx}(t,\phi_\theta(t,t_0,x_0)).g(t,\phi_\theta(t,t_0,x_0),\theta),\\ &\leqslant c_4\rho(t)\|\phi_\theta(t,t_0,x_0)\|^{1+\alpha}\\ &\leqslant c_4\rho(t)\Big[(\|\phi_\theta(t,t_0,x_0)\|)^2\Big]^{\frac{1+\alpha}{2}}\\ &\leqslant c_4\rho(t)\beta\Big(\frac{W_\theta(t,\phi(t,t_0,x_0))}{c_1}\Big);\ \beta(r)=r^{\frac{1+\alpha}{2}}.\end{aligned}$$

By integration between t and t_0, we obtain the inequality

$$\int_{t_0}^{t}\frac{\frac{d}{ds}\Big(W_\theta(s,\phi_\theta(s,t_0,x_0))\Big)}{c_1\beta\Big(\frac{W_\theta(s,\phi_\theta(s,t_0,x_0)}{c_1}\Big)}ds \leqslant \int_{t_0}^{t}\frac{c_4}{c_1}\rho(s)ds \leqslant \frac{c_4}{c_1}\|\rho\|_1. \tag{6}$$

Let us consider the following function H defined by,

$$H(r) = \int_a^r \frac{ds}{\beta(s)} = \frac{2}{1-\alpha}\Big(r^{\frac{1-\alpha}{2}} - a^{\frac{1-\alpha}{2}}\Big); r \geqslant 0,$$

with

$$a = \Big(\frac{c_4(1-\alpha)}{2c_1}\|\rho\|_1\Big)^{\frac{2}{1-\alpha}}.$$

$H(.)$ is a continuous and strictly nondecreasing function from $[0, +\infty[$ to $[\frac{-2}{1-\alpha}a^{\frac{1-\alpha}{2}}, +\infty[$, which implies that it is invertible. The explicit form of H^{-1} is given by

$$H^{-1}(r) = \Big[\frac{1-\alpha}{2}r + a^{\frac{1-\alpha}{2}}\Big]^{\frac{2}{1-\alpha}}.$$

By using the inequality (6), we obtain

$$\Big[H\Big(\frac{W_\theta(s,\phi(s,t_0,x_0))}{c_1}\Big)\Big]_{t_0}^{t} \leqslant \frac{c_4}{c_1}\|\rho\|_1.$$

Thus

$$\|\phi_\theta(t,t_0,x_0)\|^2 \leqslant H^{-1}\Big[H\Big(\frac{c_2}{c_1}\|x_0\|^2\Big) + \frac{c_4}{c_1}\|\rho\|_1\Big].$$

Let us consider the set

$$I = \Big\{ r \geqslant 0;\ H\Big(\frac{c_2}{c_1} r^2\Big) + \frac{c_4}{c_1}\|\rho\|_1 \leqslant 0 \Big\}.$$

Since $a = \Big(\frac{c_4(1-\alpha)}{2c_1}\|\rho\|_1\Big)^{\frac{2}{1-\alpha}}$, then we have the following equivalence

$$r \in I \Longleftrightarrow r = 0.$$

It follows that for all $\|x_0\| > 0$, we have

$$H\Big(\frac{c_2}{c_1}\|x_0\|^2\Big) + \frac{c_4}{c_1}\|\rho\|_1 > 0.$$

By using the fact that the function $H^{-1}(.)$ is strictly increasing function, the solution $\phi_\theta(., t_0, x_0)$ satisfies

$$\begin{aligned}
\|\phi_\theta(s, t_0, x_0)\|^2 &\leqslant H^{-1}\Big[2H\Big(\frac{c_2}{c_1}\|x_0\|^2\Big) + \frac{2c_4}{c_1}\|\rho\|_1\Big] \\
&= \Big[2\Big(\frac{c_2}{c_1}\Big)^{\frac{1-\alpha}{2}}\|x_0\|^{1-\alpha} - a^{\frac{1-\alpha}{2}} + \frac{c_4(1-\alpha)}{2c_1}\|\rho\|_1\Big]^{\frac{2}{1-\alpha}} \\
&= 4^{\frac{1}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)\|x_0\|^2.
\end{aligned}$$

Therefore, the solution $\phi_\theta(., t_0, x_0)$ of the perturbed system (1) satisfies the estimation

$$\|\phi_\theta(s, t_0, x_0)\| \leqslant 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\|x_0\|,\ \forall\, s \geqslant t_0 \geqslant 0,\ \forall\, \theta \in \mathbb{R}^m.$$

■

So, we have the following corollary, whose proof is obvious.

Corollary 4.3 The equilibrium point $x = 0$ of the perturbed system (1) is uniformly stable.

Now, some proprieties on the cross term Ψ can be given.

Proposition 4.4 [1] Under the assumptions **(H1)** and **(H3)**, the function $\Psi_\theta(t, x)$ exists and continuous on $[0, +\infty[\times\mathbb{R}^n$. Moreover, the derivative of $\Psi_\theta(t, x)$ along the trajectories of the perturbed system (1) exists and it is given by:

$$\forall t \geqslant 0,\ \forall x \in \mathbb{R}^n,\ \forall\, \theta \in \mathbb{R}^m,\ \dot{\Psi}_\theta(t, x) = -\frac{\partial W_\theta}{\partial x}(t, x).g(t, x, \theta). \tag{7}$$

The derivative of $V_\theta(t, x)$ along the trajectories of the system (1) is given by

$$\dot{V}_\theta(t, x) = \dot{W}_\theta(t, x).$$

Proof. For the proof you can see [1]. ■

5. Asymptotic stability of perturbed system

In this section, assume that the function f satisfies the Lipschitz condition where the Lipschitz constant is a function that varies with time. Let us consider the following assumption

(**H4**) There exists a function $L : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that:
i) $\|f(t,x,\theta) - f(t,y,\theta)\| \leqslant L(t)\|x-y\|, \forall\, t \geqslant 0,\ x,y \in \mathbb{R}^n,\ \theta \in \mathbb{R}^m$
ii) $\int_t^{t+h} L(u)du \leqslant \varphi(h), \quad \forall\, t, \quad h \geqslant 0$, where $\varphi(.)$ is a strictly increasing function such that $\varphi(0) = 0$ and $\lim_{h\longrightarrow+\infty} \varphi(h) = +\infty$

Remark 2

*(1) The assumption **(H4)** generalizes the case when the function is globally uniformly Lipschitzian ($L(t) = L$) and $\varphi(h) = Lh$.*
(2) If there exists $p > 1$, such that $L \in L^p([0,+\infty[)$, then $\varphi(h) = h^{\frac{p-1}{p}}\|L\|_p$.

Proposition 5.1 Suppose assumption **(H1)**, **(H3)** and **(H4)** hold. If $\rho \in L^1 \cap L^p$, $p \geqslant 1$, then the origin $x = 0$ of the perturbed system (1) is globally uniformly asymptotically stable.

Proof. To prove that the origin of perturbed system (1) is globally uniformly asymptotically stable , it suffices to prove that the Lyapunov function $V_\theta(t,x)$ is positive definite and its derivative along the trajectories of the perturbed system (1) is negative definite. Firstly by proposition 4.4, we have

$$\dot{V}_\theta(t,x) = \dot{W}_\theta(t,x) + \frac{\partial W_\theta}{\partial x}(t,x)g(t,x,\theta) + \dot{\Psi}_\theta(t,x) = \dot{W}_\theta(t,x) \leqslant -c_3\|x\|^2,$$

and

$$\begin{aligned} V_\theta(t,x) &\leqslant c_2\|x\|^2 + \int_0^{+\infty} \Big|\frac{\partial W_\theta}{\partial x}(s,\phi_\theta(s,t,x))g(s,\phi_\theta(s,t,x),\theta)\Big|ds \\ &\leqslant c_2\|x\|^2 + c_4\int_0^{+\infty} \rho(s)\|\phi_\theta(s,t,x)\|^{1+\alpha}ds \\ &\leqslant c_2\|x\|^2 + \|\rho\|_1 2^{\frac{1+\alpha}{1-\alpha}}\left(\frac{c_2}{c_1}\right)^{\frac{1+\alpha}{2}}\|x\|^{1+\alpha} := d_2(\|x\|) \end{aligned}$$

with

$$d_2(r) = c_2r^2 + \|\rho\|_1 2^{\frac{1+\alpha}{1-\alpha}}\left(\frac{c_2}{c_1}\right)^{\frac{1+\alpha}{2}} r^{1+\alpha},$$

which is a class $\mathcal{K}_\infty$ function. Secondly, by Theorem 1 in [1], the Lyapunov function $V_\theta(t,x)$ satisfies the inequality

$$V_\theta(t,x) \geqslant \int_t^{+\infty} c_3\|\phi_\theta(s,t,x)\|^2 ds.$$

Since

$$\phi_\theta(s,t,x) = x + \int_t^s \dot{\phi}_\theta(u,t,x)du = x + \int_t^s f(u,\phi_\theta(u,t,x),\theta) + g(u,\phi_\theta(u,t,x),\theta)du,$$

thus

$$\begin{aligned}\|x\| &\leqslant \|\phi_\theta(s,t,x)\| + \int_t^s L(u)\|\phi_\theta(u,t,x)\|du + \int_t^s \rho(u)\|\phi_\theta(u,t,x)\|^\alpha du \\ &\leqslant \|\phi_\theta(s,t,x)\| + 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\|x\| \int_t^s L(u)du + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}}\|x\|^\alpha \int_t^s \rho(u)du.\end{aligned}$$

We will discuss two cases

Case 1: For $\|x\| \geqslant 1$, the above inequality implies that,

$$\|x\| \leqslant \|\phi_\theta(s,t,x)\| + 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\|x\| \int_t^s L(u)du + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}}\|x\| \int_t^s \rho(u)du.$$

Therefore

$$\|\phi_\theta(s,t,x)\| \geqslant \|x\| - \|x\|\Big[2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}} \int_t^s L(u)du + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}} \int_t^s \rho(u)du\Big].$$

We have: $s \geqslant t \geqslant 0$, thus we can right $s = t+h$, $h \in \mathbb{R}_+$.
We obtain the following inequality:

$$\begin{aligned}\|\phi_\theta(t+h,t,x)\| &\geqslant \|x\| - \|x\|\Big[2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}} \int_t^{t+h} L(u)du + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}} \int_t^{t+h} \rho(u)du\Big] \\ &\geqslant \|x\| - \|x\|\Big[2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\varphi(h) + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}} \int_t^{t+h} \rho(u)du\Big], \forall\, h \geqslant 0 \\ &\geqslant \|x\| - \|x\|\Big[2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\varphi(h) + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}} \|\rho\|_p h^{\frac{p-1}{p}}\Big], \forall\, h \geqslant 0\end{aligned}$$

Consider now the function,

$$\overline{\varphi}(h) = 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\varphi(h) + 2^{\frac{\alpha}{1-\alpha}}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha}{2}} \|\rho\|_p h^{\frac{p-1}{p}},\ h \in \mathbb{R}_+.$$

The map

$$h \longmapsto \overline{\varphi}(h),$$

is a continuous function from $[0, +\infty[$ to $[0, +\infty[$. Since $\overline{\varphi}(0) = 0$ and $\lim_{h \longrightarrow +\infty} \overline{\varphi}(h) = +\infty$, then we can conclude that there exists a positive real $h_0 > 0$, such that

$$\overline{\varphi}(h_0) = \frac{1}{2}.$$

and for all $h \in [0, h_0]$, we have

$$\overline{\varphi}(h) \leqslant \frac{1}{2}.$$

Therefore

$$\|\phi_\theta(t+h, t, x)\| \geqslant \frac{\|x\|}{2}, \forall\, h \in [0, h_0],$$

witch implies that

$$\|\phi_\theta(s, t, x)\| \geqslant \frac{\|x\|}{2}, \forall s \in [t, t+h_0].$$

It follows that the Lyapunov function $V_\theta(t, x)$ verified,

$$\begin{aligned} V_\theta(t, x) &\geqslant \int_t^{+\infty} c_3 \|\phi_\theta(s, t, x)\|^2 ds \\ &\geqslant \int_t^{t+h_0} c_3 \|\phi_\theta(s, t, x)\|^2 ds \\ &\geqslant \frac{h_0 c_3}{4} \|x\|^2. \end{aligned}$$

Case 2 : For $\|x\| < 1$, which can be have

$$\|x\| \leqslant \|\phi_\theta(s, t, x)\| + 2^{\frac{1}{1-\alpha}} \sqrt{\frac{c_2}{c_1}} \|x\| \int_t^s L(u) du + 2^{\frac{\alpha}{1-\alpha}} \left(\frac{c_2}{c_1}\right)^{\frac{\alpha}{2}} \|x\|^\alpha \int_t^s \rho(u) du. \tag{8}$$

Since,

$$2^{\frac{-1}{1-\alpha}} \sqrt{\frac{c_1}{c_2}} \|\phi_\theta(s, t, x)\| \leqslant \|x\| \leqslant 1,$$

then

$$2^{\frac{-1}{1-\alpha}} \sqrt{\frac{c_1}{c_2}} \|\phi_\theta(s, t, x)\| \leqslant \left[2^{\frac{-1}{1-\alpha}} \sqrt{\frac{c_1}{c_2}} \|\phi_\theta(s, t, x)\|\right]^\alpha.$$

Therefore the estimation (8) implies that

$$2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|x\| \leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(s,t,x)\|\Big]^\alpha + \|x\|^\alpha \int_t^s L(u)du + \frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha-1}{2}}\|x\|^\alpha \int_t^s \rho(u)du.$$

This yields

$$\begin{aligned}
2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|x\| &\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^\alpha\Big[\int_t^{t+h} L(u)du + \frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha-1}{2}}\int_t^{t+h}\rho(u)du\Big]\\
&\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^\alpha\Big[\varphi(h) + \frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha-1}{2}}\int_t^{t+h}\rho(u)du\Big]\\
&\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^\alpha\Omega(h);\ h \geqslant 0
\end{aligned}$$

with

$$\Omega(h) = \varphi(h) + \frac{1}{2}\Big(\frac{c_2}{c_1}\Big)^{\frac{\alpha-1}{2}}\|\rho\|_p h^{\frac{p-1}{p}},\ h \geqslant 0.$$

Let us consider $\beta, \gamma > 0$ such that $1+\beta = \alpha+\gamma$. We have

$$2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|x\| \leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^\alpha\Omega(h);\ h \geqslant 0,$$

then

$$\begin{aligned}
2^{\frac{-1}{1-\alpha}}\Big(\frac{c_1}{c_2}\Big)^{\frac{\alpha}{2}}\|x\|^{1+\beta} &\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^{\alpha+\beta}\Omega(h)\\
&\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^\alpha + \|x\|^{\alpha}\Omega(h)\\
&\leqslant \Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^{\alpha+\gamma} + \|x\|^{\alpha+\gamma}\Omega(h),\ \forall\, h \geqslant 0
\end{aligned}$$

Now, let $h_2 > 0$ such that

$$\Omega(h) \leqslant \frac{2^{\frac{-1}{1-\alpha}}\Big(\frac{c_1}{c_2}\Big)^{\frac{\alpha}{2}}}{2},\ \forall\, h \in [0, h_2],$$

and

$$\Omega(h_2) = \frac{2^{\frac{-1}{1-\alpha}}\Big(\frac{c_1}{c_2}\Big)^{\frac{\alpha}{2}}}{2}.$$

It follows that

$$\Big[2^{\frac{-1}{1-\alpha}}\sqrt{\frac{c_1}{c_2}}\|\phi_\theta(t+h,t,x)\|\Big]^{\alpha+\gamma} \geqslant \frac{2^{\frac{-1}{1-\alpha}}\left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}}{2}\|x\|^{\alpha+\gamma},\ \forall\, h \in [0,h_2].$$

Thus

$$\|\phi_\theta(t+h,t,x)\| \geqslant 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\Big(\frac{2^{\frac{-1}{1-\alpha}}\left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}}{2}\Big)^{\frac{1}{\alpha+\gamma}}\|x\|,\ \forall\, h \in [0,h_2],$$

therefore

$$\|\phi_\theta(s,t,x)\| \geqslant 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\Big(\frac{2^{\frac{-1}{1-\alpha}}\left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}}{2}\Big)^{\frac{1}{\alpha+\gamma}}\|x\|,\ \forall\, s \in [t,t+h_2].$$

Hence, we can conclude that,

$$V_\theta(t,x) \geqslant \int_t^{t+h_2} c_3\|\phi_\theta(s,t,x)\|^2 ds \geqslant c_3 h_2 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\Big(\frac{2^{\frac{-1}{1-\alpha}}\left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}}{2}\Big)^{\frac{1}{\alpha+\gamma}}\|x\|^2.$$

Let

$$d_1 = \inf\Big(c_3 h_2 2^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\Big(\frac{2^{\frac{-1}{1-\alpha}}\left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}}{2}\Big)^{\frac{1}{\alpha+\gamma}}, \frac{h_0 c_3}{4}\Big)$$

It follows that the function $V_\theta(t,x)$ satisfies

$$d_1\|x\|^2 \leqslant V_\theta(t,x) \leqslant d_2(\|x\|),\ \forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \forall\,\theta \in \mathbb{R}^m,$$

and

$$\dot{V}_\theta(t,x) = \dot{W}_\theta(t,x) \leqslant -c_3\|x\|^2,\ \forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\},\ \forall\,\theta \in \mathbb{R}^m.$$

Hence, that the equilibrium point $x = 0$ of (1) is globally uniformly asymptotically stable. ■

6. Strictification

The strict Lyapunov functions are of great importance in the study of the stability of systems, and are a key tool for robustness analysis. In general, it is more difficult to construct strict Lyapunov functions for time-varying systems than it is for time-invariant systems. In this section we give a method for constructing strict Lyapunov functions for time-varying systems. The challenge is then to transform non-strict Lyapunov functions satisfying this more complicated decay condition into explicit strict Lyapunov functions. Assume the following assumptions.

(H'1) There is a Lyapunov function $W_\theta(t,x)$, periodic in time and of period T and a non-negative bounded continuous function $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

i) $c_1\|x\|^2 \leqslant W_\theta(t,x) \leqslant c_2\|x\|^2$, $\forall\,(t,x) \in [0,+\infty[\times\mathbb{R}^n$, $\forall\,\theta \in \mathbb{R}^m$,

ii) $\dot{W}_\theta(t,x) = \dfrac{\partial W_\theta}{\partial t}(t,x) + \dfrac{\partial W_\theta}{\partial x}(t,x).f(t,x,\theta) \leqslant -p(t)\|x\|^2 \leqslant 0$, $\forall\,\theta \in \mathbb{R}^m$,

iii) $\|\dfrac{\partial W_\theta}{\partial x}(t,x)\| \leqslant c_4\|x\|$, $\forall(t,x) \in [0,+\infty[\times\mathbb{R}^n$, $\forall\,\theta \in \mathbb{R}^m$.

(H'2) The constant $\displaystyle\int_0^T p(s)ds > 0$.

Theorem 6.1 If $\rho \in L^p$, then under assumption **(H2)**,**(H'1)** and **(H'2)** the system (1) is globally uniformly exponentially stable.

Proof. Consider the following function

$$P(t) = -t\int_0^T p(s)ds + T\int_0^t p(s)ds$$

$P(t)$ is a T-periodic and continuous function, it follows that is bounded, thus there is $\overline{P} > 0$ such that for all $t \in \mathbb{R}_+$

$$|P(t)| \leqslant \overline{P}.$$

Let us consider the function

$$U_\theta(t,x) = (a_1 + a_2P(t))W_\theta(t,x)^2, \tag{9}$$

with $a_1, a_2 > 0$, such that

$$a_1 \geqslant \max(2Ta_2c_2, 2\overline{P}a_2). \tag{10}$$

and consider the following two functions defined on $\mathbb{R}_+$ by $\Gamma(v) = a_1v^2$, $\lambda(v) = a_2v^2$. It's clear that $\Gamma(.), \lambda(.) \in C^1 \cap \mathcal{K}_\infty$ and $\lambda(.)$ is positive define function, in addition $\lambda'(v) = 2a_2v \geqslant 0$. By (10), we can conclude that

$$\begin{cases} \Gamma(v) \geqslant 2\overline{P}\lambda(v) \\ \frac{1}{2}\Gamma'(v) \geqslant \overline{P}\lambda'(v) \end{cases} \tag{11}$$

witch implies that

$$\frac{1}{4}\Gamma'(W_\theta(t,x))\|x\|^2 \geqslant T\lambda(W_\theta(t,x)). \tag{12}$$

Since

$$U_\theta(t,x) = \Gamma(W_\theta(t,x)) + P(t)\lambda(W_\theta(t,x)),$$

then the function $U_\theta(t,x)$ satisfies the inequality

$$a_2\overline{P}c_1^2\|x\|^4 \leqslant U_\theta(t,x) \leqslant \frac{3}{2}a_1c_2^2\|x\|^4.$$

The derivative of $U_\theta(t,x)$ along the trajectories of (2) satisfies

$$\begin{aligned}\dot{U}_\theta(t,x) &= \Gamma'(W_\theta(t,x))\dot{W}_\theta(t,x) + \Big[-\int_0^T p(s)ds + Tp(t)\Big]\lambda(W_\theta(t,x)) + P(t)\lambda'(W_\theta(t,x))\dot{W}_\theta(t,x)\\ &\leqslant -\frac{1}{2}\Gamma'(W_\theta(t,x))p(t)\|x\|^2 + \Big[-\int_0^T p(s)ds + Tp(t)\Big]\lambda(W_\theta(t,x))\\ &\leqslant -\frac{1}{4}\Gamma'(W_\theta(t,x))p(t)\|x\|^2 - \int_0^T p(s)ds\lambda(W_\theta(t,x))\\ &\leqslant -\int_0^T p(s)ds\lambda(c_1\|x\|^2) = -a_2c_1^2\int_0^T p(s)ds\|x\|^4.\end{aligned}$$

It follows that the nominal system (2) is globally uniformly exponentially stable. The derivative of $U_\theta(t,x)$ along the trajectories of the perturbed system (1) is given by

$$\begin{aligned}\dot{U}_\theta(t,x) &= \dot{U}_\theta(t,x) + \frac{\partial U_\theta}{\partial x}(t,x).g(t,x,\theta)\\ &= \dot{U}_{\theta(2)}(t,x) + 2\Big(a_1 + a_2P(t)\Big)W_\theta(t,x)\frac{\partial W_\theta}{\partial x}(t,x).g(t,x,\theta)\\ &\leqslant -a_2c_1^2\int_0^T p(s)ds\|x\|^4 + 2c_4(a_1+a_2\overline{P})\rho(t)\|x\|^4\\ &\leqslant -\frac{2a_2c_1^2}{3a_1c_2^2}\int_0^T p(s)dsU_\theta(t,x) + \frac{2c_4(a_1+a_2\overline{P})}{a_2\overline{P}c_1^2}\rho(t)U_\theta(t,x).\end{aligned}$$

Let

$$k_1 = \frac{2a_2c_1^2}{3a_1c_2^2}\int_0^T p(s)ds,\ k_2 = \frac{2c_4(a_1+a_2\overline{P})}{a_2\overline{P}c_1^2}.$$

If $p=1$, by integration between t and t_0 we obtain the following estimation

$$\|\phi_\theta(s,t_0,x_0)\| \leqslant \sqrt[4]{\frac{3a_1c_2^2}{2a_2\overline{P}c_1^2}}e^{\frac{k_2}{4}\|\rho\|_1}\|x_0\|e^{-\frac{k_1}{4}(s-t_0)}.$$

If $p>1$, let q such that $\dfrac{1}{p}+\dfrac{1}{q}=1$.

We have

$$\begin{aligned}\dot{U}_\theta(t,x) &\leqslant -k_1 U_\theta(t,x) + k_2\rho(t)U_\theta(t,x)\\ &\leqslant -k_1 U_\theta(t,x) + k_2\Big[\frac{\rho(t)}{\varepsilon}U_\theta^{\frac{1}{p}}(t,x)\times \varepsilon U_\theta^{\frac{1}{q}}(t,x)\Big];\ \varepsilon>0.\end{aligned}$$

By Young's inequality, we obtain the inequality

$$\dot{U}_\theta(t,x) \leqslant -\Big(k_1 - k_2\varepsilon^q\Big)U_\theta(t,x) + \frac{k_2}{\varepsilon^p}\rho^p(t)U_\theta(t,x),$$

therefore, if

$$0<\varepsilon<\sqrt[q]{\frac{k_1}{k_2}},$$

then the solution of the perturbed system satisfies the estimation

$$\|\phi_\theta(s,t_0,x_0)\| \leqslant \sqrt[4]{\frac{3a_1c_2^2}{a_2\overline{P}c_1^2}e^{\frac{k_2\|\rho\|_p^p}{4\varepsilon^p}}}\|x_0\|e^{-\frac{k_1-k_2\varepsilon^q}{4}(s-t_0)},$$

This completes the proof of the Theorem 6.1. ■

Remark 3 *One can extend Theorem 6.1 to the case of systems which are not periodic but when there exist $T>0$ and $\delta>0$ such that, for all $t\geqslant 0$, $\int_t^{t+T} p(s)ds \geqslant \delta$. In this case the function $P(t)$ is given by*

$$P(t) = \int_{t-T}^{t}\int_s^T p(r)drds$$

and the strict Lyapunov function $U_\theta(t,x)$ satisfies the following conditions

$$a_1c_1^2\|x\|^4 \leqslant \Gamma(W_\theta(t,x)) \leqslant U_\theta(t,x) \leqslant \frac{3}{2}\Gamma(W_\theta(t,x)) \leqslant \frac{3a_1c_2^2}{2}\|x\|^4,$$

and

$$\dot{U}_{\theta(2)}(t,x) \leqslant -\delta a_2 c_1^2\|x\|^4.$$

7. Examples

In this section, we give three examples. The first one to illustrate the exponential stability given by proposition 4.1, and the second to illustrate the asymptotic stability given by proposition 4.4. The third one is given to illustrate the case when the derivative of the Lyapunov function along the trajectories of the nominal system is negative non-define.

***Example* 7.1** Let us consider the following system

$$\begin{cases} \dot{x}_1 = -(1+\theta^2+t^2)x_1 + \frac{\sqrt[3]{\rho(t)}}{1+2\theta^4}x_2 \\ \dot{x}_2 = -(1+2\theta^2+t^2)x_2 + \frac{\sqrt[3]{\rho(t)}}{2+3\theta^4}x_1 \end{cases} \tag{13}$$

where ρ is a continuous, integrable and unbounded function defined by

$$\rho(t) = \begin{cases} 0 & , t \in [0, 2-\frac{1}{8}] \\ n^4 t + (n - n^5) & , t \in [n - \frac{1}{n^3}, n] \\ -n^4 t + (n + n^5) & , t \in [n, n + \frac{1}{n^3}] \\ 0 & , t \in [n + \frac{1}{n^3}, n + 1 - \frac{1}{(n+1)^3}] \end{cases}$$

The system satisfies the conditions of proposition 4.1, thus it's globally uniformly asymptotically stable.
For simulation, if we select $[x_1, x_2]^T = [2, -2]^T$ as initial condition and $\theta = 0$, then we obtain the following result (see figure 1).

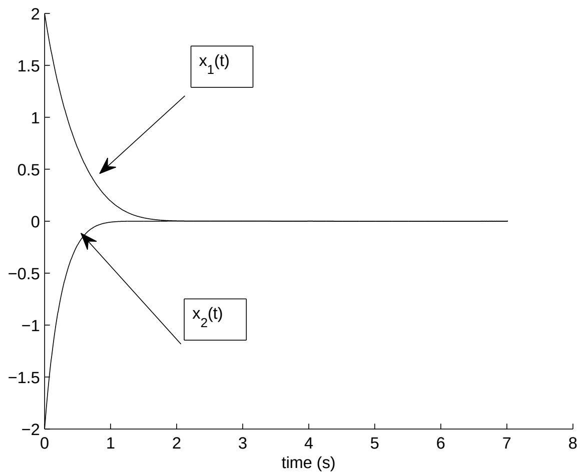

Figure 1. *The trajectories of the system (13)*

***Example* 7.2** Let us consider the following system

$$\dot{x} = f(t,x,\theta) + g(t,x,\theta), \tag{14}$$

with

$$f(t,x,\theta) = \begin{bmatrix} -\frac{\sqrt{\rho(t)}+1.2}{1+\theta^2}x_1 \\ \\ -\frac{\sqrt{\rho(t)}+1}{1+\theta^4}x_2 \end{bmatrix}$$

and

$$g(t,x,\theta) = \rho(t)\begin{bmatrix} -\frac{x_2}{1+\theta^2+(x_1^2+x_2^2)^{\frac{3}{8}}} \\ \\ -\frac{x_1}{1+\theta^4+(x_1^2+x_2^2)^{\frac{3}{8}}} \end{bmatrix}.$$

when $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and integrable function.
Consider the Lyapunov function $W_\theta(t,x) = \|x\|^2 = x_1^2 + x_2^2$. The derivative of $W_\theta(t,x)$ along the trajectories of the nominal system is given by

$$\dot{W}_\theta(t,x) = 2(x_1\dot{x}_1 + x_2\dot{x}_2) \leqslant -2W_\theta(t,x),$$

it follows that the nominal system is globally uniformly exponentially stable.
The functions f and g satisfie the conditions

$$\|g(t,x,\theta)\| \leqslant \rho(t)\|x\|^{\frac{1}{4}},\ \forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

and

$$\|f(t,x,\theta) - f(t,y,\theta)\| \leqslant (1.2 + \sqrt{\rho(t)})\|x-y\| := L(t)\|x-y\|.$$

The function $L(.)$ satisfies

$$\int_t^{t+h} L(u)du = \int_t^{t+h} 1.2 + \sqrt{\rho(u)}du \leqslant 1.2h + \|\rho\|_1\sqrt{h} := \varphi(h)$$

If we choose $\rho(t)$ as in example 7.1, $[x_1, x_2]^T = [3, -2]^T$ as initial condition and $\theta = 0$, then we obtain the following result (see figure 2).

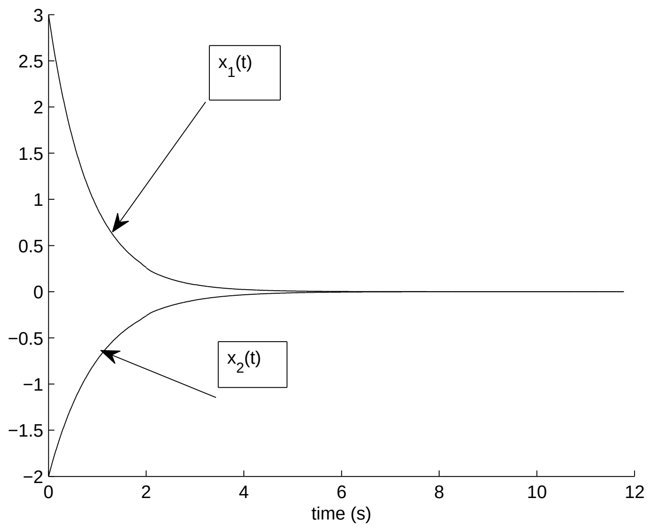

Figure 2. *The trajectories of the system* (7.2)

***Example* 7.3** Let us consider the following system

$$\begin{cases} \dot{x}_1 = -(1+\theta^2)(1+\cos(t))x_1 + \frac{\rho(t)}{1+3\theta^4}x_2 \\ \\ \dot{x}_2 = -(1+\theta^2)(1+\cos(t))x_2 + \frac{\rho(t)}{1+4\theta^2}x_1 \end{cases} \tag{15}$$

where ρ is a continuous, integrable and unbounded function defined in example (7.1). The above system has the same form of (1) where

$$f(t,x,\theta) = \begin{bmatrix} -(1+\theta^2)(1+\cos(t))x_1 \\ -(1+\theta^2)(1+\cos(t))x_2 \end{bmatrix},$$

and

$$g(t,x,\theta) = \rho(t)\begin{bmatrix} \frac{1}{1+3\theta^4}x_2 \\ \frac{1}{1+4\theta^2}x_1 \end{bmatrix}.$$

Let us consider the Lyapunov function

$$W_\theta(t,x) = (1+\theta^2 e^{-\theta^2})(x_1^2 + x_2^2).$$

The derivative of $W_\theta(t,x)$ a long the trajectories of the nominal system is given by,

$$\dot{W}_\theta(t,x) = -2(1+\theta^2)(1+\cos(t))W_\theta(t,x) = -p(t)W_\theta(t,x) \leqslant 0,$$

where $p(t) = 2(1+\cos(t))$. We notice that $\dot{W}_\theta((2k+1)\pi, x) = 0, \forall\, k \in \mathbb{Z}$.
The strict Lyapunov function $U_\theta(t,x)$ is given by

$$U_\theta(t,x) = 2\pi(1+\theta^2 e^{-\theta^2})^2(2+\sin(t)))\|x\|^4,$$

and we have the following conditions:

i) $2\pi\|x\|^4 \leqslant U_\theta(t,x) \leqslant 6\pi(1+e^{-1})\|x\|^4$

ii) The derivative of $U_\theta(t,x)$ a long the trajectories of the nominal system is given by

$$\dot{U}_\theta(t,x) \leqslant -2\pi(1+e^{-1})\|x\|^4.$$

Now, if we choose $\rho(t)$ a function as in example 7.1 and $[x_1,x_2]^T = [3,-2]^T$ as initial condition and $\theta = 0$, then the origin of the system is G.U.E.S (see figure 3).

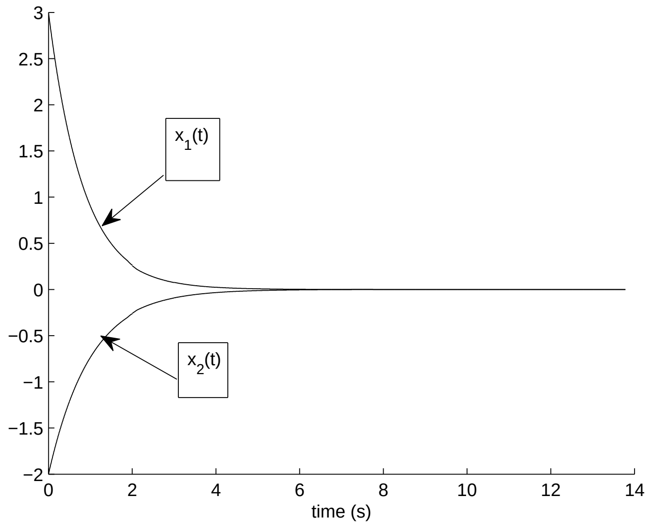

Figure 3. *The trajectories of the system (15)*

8. Conclusion

In this paper, we dealt with the analysis of stability problem of nonlinear time-varying parameterised perturbed systems. We have established the global uniform exponential stability and global uniform asymptotic stability for some classes of perturbed parameterised systems, by using Lyapunov techniques. The restriction about the perturbed term is that the perturbation is bounded by an integrable function not necessary bounded in time under the assumption that the nominal system is globally uniformly exponentially stable. Some illustrative examples in the plane are given showing the importance of this study.

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