

## On the irreducibility of the complex specialization of the representation of the Hecke algebra of the complex reflection group $G_7$

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**Abstract.** We consider a 2-dimensional representation of the Hecke algebra  $\mathcal{H}(G_7, u)$ , where  $G_7$  is the complex reflection group and  $u$  is the set of indeterminates

$$u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3).$$

After specializing the indeterminates to non zero complex numbers, we then determine a necessary and sufficient condition that guarantees the irreducibility of the complex specialization of the representation of the Hecke algebra  $\mathcal{H}(G_7, u)$ .

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### 1. Introduction

Let  $V$  be a complex vector space and  $W$  a finite irreducible subgroup of  $GL(V)$  generated by complex reflections. Let  $R$  be the set of reflections in  $W$ . For any element  $s$  of  $R$ , denote by  $H_s$  its pointwise fixed hyperplane. We define the set  $V^{reg} = V - \cup_{s \in R} H_s$  and denote by  $\bar{V}$  the quotient  $V^{reg}/W$ . The braid group associated to  $(W, V)$  is the fundamental group  $B(W) = \pi_1(\bar{V}, \bar{x}_0)$  of  $\bar{V}$  with respect to any point  $\bar{x}_0 \in \bar{V}$ . We choose the set of indeterminates,  $u = (u_{s,j})_{s, 0 \leq j \leq o(s)-1}$ , where  $s$  runs over the generators of  $W$  and

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$u_{s,j} = u_{t,j}$  if  $s$  and  $t$  are conjugate in  $W$ . Here  $o(s)$  denotes the order of  $s$ . The cyclotomic Hecke algebra associated to  $W$  is the quotient of the group algebra  $\mathbb{Z}[u, u^{-1}]BW$  by the ideal generated by the relations  $\prod_{j=0}^{o(s)-1} (s - u_{s,j})$ .

In [7], G. Malle and J. Michel constructed on the cyclotomic hecke algebra  $\mathcal{H}(G_7, u)$  of the complex reflexion group,  $G_7$ , an irreducible representation

$$\phi : \mathcal{H}(G_7, u) \rightarrow M_2(\mathbb{C}(u^{\frac{1}{2}}, u^{-\frac{1}{2}})),$$

where  $u$  is the set of indeterminates  $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$ . In our work, we specialize the indeterminates  $x_1, x_2, y_1, y_2, y_3, z_1, z_2$  and  $z_3$  to nonzero complex numbers  $\rho e^{i\alpha}$ , where  $\alpha \in (-\pi, \pi]$  and  $\rho$  a positive real number. We then get a representation

$$\varphi : \mathcal{H}(G_7, u) \rightarrow GL_2(\mathbb{C}).$$

In section 3, we consider the case when  $x_1 = x_2$  and we show that  $\varphi$  is irreducible if and only if  $z_1 \neq \frac{y_1 z_2}{y_2}$  and  $z_1 \neq \frac{y_2 z_2}{y_1}$  (Theorem 3.4). In section 4, we assume that  $x_1 \neq x_2$  and we show that  $\varphi$  is irreducible if and only if  $x_1 y_2 z_2 \neq x_2 y_1 z_1$ ,  $x_1 y_1 z_2 \neq x_2 y_2 z_1$ ,  $x_1 y_2 z_1 \neq x_2 y_1 z_2$  and  $x_1 y_1 z_1 \neq x_2 y_2 z_2$  (Theorem 4.5).

## 2. Preliminaries

**Definition 2.1** [6] Let  $V$  be a complex vector space of dimension  $n$ . A complex reflection of  $GL(V)$  is a non-trivial element of  $GL(V)$  which acts trivially on a hyperplane.

**Definition 2.2** [6] Let  $V$  be a complex vector space of dimension  $n$ . A complex reflection group is the subgroup of  $GL(V)$  generated by complex reflections.

Examples of complex reflection groups include dihedral groups and symmetric groups. For  $n \geq 3$ , the dihedral group,  $D_n$ , is the group of the isometries of the plane preserving a regular polygon, with the operation being composition.

A classification of all irreducible reflection groups shows that there are 34 primitive irreducible reflection groups [8]. The starting point was with A. Cohen, who provided a data for those irreducible complex reflection groups of rank 2 [5].

**Definition 2.3** [3] The complex reflection group,  $G_7$ , is an abstract group defined by the presentation

$$G_7 = \langle t, u, s \mid t^2 = u^3 = s^3 = 1, tus = ust = stu \rangle .$$

**Theorem 2.4** [1] The braid group of  $G_7$  is isomorphic to the group

$$B = \langle s_1, s_2, s_3 \mid s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle .$$

Definitions and properties of braid groups are found in [2].

**Definition 2.5** [7] Let  $u$  be the set of indeterminates  $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$ . The cyclotomic Hecke algebra  $\mathcal{H}(G_7, u)$  of  $G_7$  is the quotient of the group algebra of  $B$  over  $\mathbb{Z}[u, u^{-1}]$  by the relations

$$(s_1 - x_1)(s_1 - x_2) = 0, \quad \prod_{i=1}^3 (s_2 - y_i) = 0, \quad \prod_{i=1}^3 (s_3 - z_i) = 0.$$

For more details about the Hecke algebra of  $G_7$ , see [4].

**Definition 2.6** [7] Let  $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$ . The representation  $\phi$  is defined as follows:

$$\phi : \mathcal{H}(G_7, u) \rightarrow M_2(\mathbb{C}(u^{\pm\frac{1}{2}}))$$

$$s_1 = \begin{pmatrix} x_1 \frac{y_1+y_2}{y_1y_2} - \frac{(z_1+z_2)x_2}{r} & \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_1} \\ -y_1y_2x_1 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & -r \\ r & z_1 + z_2 \end{pmatrix},$$

where  $r = \sqrt{x_1x_2y_1y_2z_1z_2}$ .

We specialize the indeterminates  $x_1, x_2, y_1, y_2, z_1, z_2$  and  $z_3$  to nonzero complex numbers,  $\rho e^{i\alpha}$ , where  $\alpha \in (-\pi, \pi]$  and  $\rho$  a positive real number. We then get a representation  $\varphi : \mathcal{H}(G_7, u) \rightarrow GL_2(\mathbb{C})$ .

**Definition 2.7** Principal square root function is defined as follows:

$$z \in \mathbb{C}, \quad z = (\rho, \alpha), \quad \rho \geq 0 \quad \text{and} \quad \sqrt{z} = \sqrt{\rho}e^{i\frac{\alpha}{2}} \quad \text{where} \quad -\pi < \alpha \leq \pi.$$

Since  $\alpha \in (-\pi, \pi]$ , it follows that  $\sqrt{z^2} = z$  for any complex number  $z$ .

### 3. Irreducibility of the representation $\varphi$ for $x_1 = x_2$

We assume that  $x_1 = x_2$  and we find a necessary and sufficient condition that guarantees the irreducibility of the representation  $\varphi : \mathcal{H}(G_7, u) \rightarrow GL_2(\mathbb{C})$ . Under this assumption, we have that the images of the generators of  $\mathcal{H}(G_7, u)$  are

$$s_1 = \begin{pmatrix} x_2 \frac{y_1+y_2}{y_1y_2} - \frac{(z_1+z_2)x_2}{\sqrt{x_2^2y_1y_2z_1z_2}} & \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_2} \\ -y_1y_2x_2 & 0 \end{pmatrix}$$

and

$$s_3 = \begin{pmatrix} 0 & -\frac{\sqrt{x_2^2y_1y_2z_1z_2}}{x_2y_1y_2} \\ r & z_1 + z_2 \end{pmatrix}.$$

For the matrix  $s_1$ , we denote by  $s_1(i, j)$  the term of the matrix  $s_1$  which lies in the  $i$ th row and in the  $j$ th column.

**Lemma 3.1**  $s_1(1, 2) = 0$  if and only if  $z_1 = \frac{y_1z_2}{y_2}$  or  $z_1 = \frac{y_2z_2}{y_1}$ .

**Proof.** We show that if  $s_1(1, 2) = 0$  then  $z_1 = \frac{y_1z_2}{y_2}$  or  $z_1 = \frac{y_2z_2}{y_1}$ . Assume that  $s_1(1, 2) = 0$ . This implies that  $\frac{y_1+y_2}{y_1y_2} = \frac{(z_1+z_2)x_2}{x_2\sqrt{y_1y_2z_1z_2}}$ . This implies that  $(y_1 + y_2)\sqrt{y_1y_2z_1z_2} = (z_1 + z_2)y_1y_2$ . Using  $y_1y_2 = (\sqrt{y_1y_2})^2$ , we get  $(y_1 + y_2)\sqrt{z_1z_2} = (z_1 + z_2)\sqrt{y_1y_2}$ . Squaring both sides, we obtain  $(y_1 + y_2)^2z_1z_2 = (z_1 + z_2)^2y_1y_2$ . This implies that  $z_1 = \frac{y_1z_2}{y_2}$  or  $z_1 = \frac{y_2z_2}{y_1}$ . On the other hand, direct computations show that if  $z_1 = \frac{y_1z_2}{y_2}$  or  $z_1 = \frac{y_2z_2}{y_1}$  then  $s_1(1, 2) = 0$ . ■

We now determine a sufficient condition for irreducibility.

**Proposition 3.2** The representation  $\varphi$  is irreducible if  $z_1 \neq \frac{y_1 z_2}{y_2}$  and  $z_1 \neq \frac{y_2 z_2}{y_1}$ .

**Proof.** Using the hypothesis and Lemma 3.1, we get  $s_1(1, 2) \neq 0$ . Let  $S$  be a non trivial proper invariant subspace of  $\mathbb{C}^2$ . The eigenspace of  $s_1$  is generated by  $e_1$ . This implies that  $S$  is of the form  $\langle v \rangle$ , where  $v = ae_1$  for some non-zero complex number  $a$ .  $S$  is invariant implies that  $s_2 v = (a(y_1 + y_2), -ax_2 y_1 y_2) \in S$ , which is a contradiction. Therefore  $S$  is irreducible. ■

We determine a necessary condition for irreducibility.

**Proposition 3.3** The representation  $\varphi$  is reducible if  $z_1 = \frac{y_1 z_2}{y_2}$  or  $z_1 = \frac{y_2 z_2}{y_1}$ .

**Proof.** In each case, we show that the 1-dimensional subspace  $M$  generated by the vector  $u = (-\frac{1}{x_2 y_2}, 1)$  is invariant.

**Case1.**  $z_1 = \frac{y_1 z_2}{y_2}$ . Substituting in Definition 2.6, we get

$$s_1 = \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_2} \\ -x_2 y_1 y_2 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & -\frac{z_2}{x_2 y_2} \\ x_2 y_1 z_2 & z_2 + \frac{y_1 z_2}{y_2} \end{pmatrix}.$$

It is easy to see that  $s_2 u = y_1 u$  and  $s_3 u = z_2 u$ . This implies that  $M$  is invariant.

**Case2.**  $z_1 = \frac{y_2 z_2}{y_1}$ . Substituting in Definition 2.6, we get

$$s_1 = \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_2} \\ -x_2 y_1 y_2 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & -\frac{z_2}{x_2 y_1} \\ x_2 y_2 z_2 & z_2 + \frac{y_2 z_2}{y_1} \end{pmatrix}.$$

It is also easy to see that  $s_2 u = y_1 u$  and  $s_3 u = z_2 \frac{y_2}{y_1} u$ . This implies that  $M$  is invariant. ■

Here we have proved the following theorem:

**Theorem 3.4** The representation  $\varphi$  is irreducible if and only if  $z_1 \neq \frac{y_1 z_2}{y_2}$  and  $z_1 \neq \frac{y_2 z_2}{y_1}$ .

#### 4. Irreducibility of the representation $\varphi$ for $x_1 \neq x_2$

We assume that  $x_1 \neq x_2$  and we find a necessary and sufficient condition that guarantees the irreducibility of the representation  $\varphi: \mathcal{H}(G_7, u) \rightarrow GL_2(\mathbb{C})$ . For simplicity, we denote by  $w$  the term

$$(x_1 - x_2)^2 y_1^2 y_2^2 z_1 z_2 + [(y_1 + y_2)r - x_1 y_1 y_2 (z_1 + z_2)][(y_1 + y_2)r - x_2 y_1 y_2 (z_1 + z_2)] \quad (1)$$

**Lemma 4.1** The complex number  $w$ , defined in (1), is different from zero if and only if  $x_1 y_2 z_2 \neq x_2 y_1 z_1$ ,  $x_1 y_1 z_2 \neq x_2 y_2 z_1$ ,  $x_1 y_2 z_1 \neq x_2 y_1 z_2$  and  $x_1 y_1 z_1 \neq x_2 y_2 z_2$ .

**Proof.** Simple calculations show that  $w = \alpha\beta$ , where

$$\begin{aligned} \alpha &= x_2 y_1 y_2 z_1 + x_1 y_1 y_2 z_2 - (y_1 + y_2)r \\ \beta &= x_1 y_1 y_2 z_1 + x_2 y_1 y_2 z_2 - (y_1 + y_2)r. \end{aligned}$$

Assume that  $w = 0$ . This implies that  $\alpha = 0$  or  $\beta = 0$ . If  $\alpha = 0$ , then

$$x_2y_1y_2z_1 + x_1y_1y_2z_2 = (y_1 + y_2)r.$$

Squaring both sides, we get

$$y_1y_2(-x_2y_2z_1 + x_1y_1z_2)(-x_2y_1z_1 + x_1y_2z_2) = 0.$$

This implies that  $x_1y_1z_2 = x_2y_2z_1$  or  $x_1y_2z_2 = x_2y_1z_1$ . If  $\beta = 0$ , then

$$x_1y_1y_2z_1 + x_2y_1y_2z_2 = (y_1 + y_2)r.$$

Squaring both sides, we get

$$y_1y_2(x_1y_2z_1 - x_2y_1z_2)(x_1y_1z_1 - x_2y_2z_2) = 0.$$

This implies that  $x_1y_2z_1 = x_2y_1z_2$  or  $x_1y_1z_1 = x_2y_2z_2$ . On the other hand, we assume that any of the following conditions holds true.

$$x_1y_2z_2 = x_2y_1z_1, \quad x_1y_1z_2 = x_2y_2z_1, \quad x_1y_2z_1 = x_2y_1z_2 \quad \text{or} \quad x_1y_1z_1 = x_2y_2z_2$$

Under direct computations, we easily verify that  $w = 0$ . ■

We now give a sufficient condition for the irreducibility of the representation  $\varphi$ .

**Proposition 4.2** The representation  $\phi$  is irreducible if  $x_1y_2z_2 \neq x_2y_1z_1$ ,  $x_1y_1z_2 \neq x_2y_2z_1$ ,  $x_1y_2z_1 \neq x_2y_1z_2$  and  $x_1y_1z_1 \neq x_2y_2z_2$ .

**Proof.** If the term  $s_1(1, 2) = \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r}$  equals zero, then neither  $e_1$  nor  $e_2$  is a common eigenvector for  $s_2$  and  $s_3$ . This implies that the representation is irreducible. We note that under this case, we have that the complex number  $w$  is not zero and hence, by Lemma 4.1, we also have that  $x_1y_2z_2 \neq x_2y_1z_1$ ,  $x_1y_1z_2 \neq x_2y_2z_1$ ,  $x_1y_2z_1 \neq x_2y_1z_2$  and  $x_1y_1z_1 \neq x_2y_2z_2$ . If  $s_1(1, 2) = \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r}$  is not zero, we diagonalize the matrix  $S_1$  by the invertible matrix

$$T = \begin{pmatrix} 1 & \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r} \\ 0 & x_2 - x_1 \\ & 1 \end{pmatrix}.$$

We get

$$T^{-1}s_1T = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}.$$

We then conjugate  $s_2$  by the matrix  $T$ . We get

$$T^{-1}s_2T = \begin{pmatrix} M & w \\ -x_1y_1y_2 & P \end{pmatrix},$$

where

$$M = -\frac{x_2(-x_1y_1y_2z_1 - x_1y_1y_2z_2 + y_1r + y_2r)}{(x_1 - x_2)r},$$

$$P = -\frac{x_1(-x_2y_1y_2z_1 - x_2y_1y_2z_2 - y_1r - y_2r)}{(x_1 - x_2)r}.$$

By conjugating  $s_3$  by  $T$ , we get

$$T^{-1}s_3T = \begin{pmatrix} A & B \\ r & C \end{pmatrix},$$

where

$$A = \frac{(y_1 + y_2)r - x_2y_1y_2(z_1 + z_2)}{(x_1 - x_2)y_1y_2}$$

and

$$B = \frac{1}{(x_1 - x_2)^2r^3}x_1x_2z_1z_2(-x_1x_2y_1y_2z_1^2 - x_1x_2y_1^2z_1z_2 - x_1^2y_1y_2z_1z_2$$

$$- 2x_1x_2y_1y_2z_1z_2 - x_2^2y_1y_2z_1z_2 - x_1x_2y_2^2z_1z_2$$

$$- x_1x_2y_1y_2z_2^2 + x_1y_1z_1r + x_2y_1z_1r$$

$$+ x_1y_2z_1r + x_2y_2z_1r + x_1y_1z_2r + x_2y_1z_2r$$

$$+ x_1y_2z_2r + x_2y_2z_2r)$$

and

$$C = \frac{-r(y_1 + y_2) + x_1y_1y_2(z_1 + z_2)}{(x_1 - x_2)y_1y_2}.$$

For simplicity, we denote  $T^{-1}s_iT$  by  $b_i$  for  $1 \leq i \leq 3$ . Suppose, to get contradiction, that the representation is reducible. That is, there exists a non trivial proper invariant subspace  $M$  of  $\mathbb{C}^2$  of dimension 1. The subspace  $M$  has to be one of the following subspaces  $\langle e_1 \rangle$  or  $\langle e_2 \rangle$ .

**Case1**  $S = \langle e_1 \rangle$ . Since  $e_1 \in M$ , it follows that  $b_3e_1 = (A, r) \in M$ . This implies that  $r = 0$ , a contradiction.

**Case2**  $S = \langle e_2 \rangle$ . Since  $e_2 \in M$ , it follows that  $b_2e_2 = (w, P) \in M$ . By Lemma 4.1, we have that  $w$  is different from zero, which is a contradiction. Therefore the representation is irreducible. ■

We now present a lemma concerning the number  $B$  used in defining  $T^{-1}s_3T$  in Proposition 4.2.

**Lemma 4.3** The complex number  $B$  equals zero in each of the following cases:

- (1)  $x_1y_2z_2 = x_2y_1z_1$
- (2)  $x_1y_1z_2 = x_2y_2z_1$
- (3)  $x_1y_2z_1 = x_2y_1z_2$
- (4)  $x_1y_1z_1 = x_2y_2z_2$

**Proof.** We verify that  $B = 0$  in case (1). Suppose that  $x_1y_2z_2 = x_2y_1z_1$  then

$$\begin{aligned}
 B &= \frac{1}{(x_1 - x_2)^2 r^3} x_1 x_2 z_1 z_2 \left( -\frac{x_2 y_1 z_1}{y_2 z_2} x_2 y_1 y_2 z_1^2 - \frac{x_2 y_1 z_1}{y_2 z_2} x_2 y_1^2 z_1 z_2 - \frac{x_2^2 y_1^2 z_1^2}{y_2^2 z_2^2} y_1 y_2 z_1 z_2 \right. \\
 &\quad - 2 \frac{x_2 y_1 z_1}{y_2 z_2} x_2 y_1 y_2 z_1 z_2 - x_2^2 y_1 y_2 z_1 z_2 - \frac{x_2 y_1 z_1}{y_2 z_2} x_2 y_2^2 z_1 z_2 \\
 &\quad - \frac{x_2 y_1 z_1}{y_2 z_2} x_2 y_1 y_2 z_2^2 + \frac{x_2 y_1 z_1}{y_2 z_2} y_1 z_1 x_2 y_1 z_1 + x_2 y_1 z_1 x_2 y_1 z_1 \\
 &\quad + \frac{x_2 y_1 z_1}{y_2 z_2} y_2 z_1 x_2 y_1 z_1 + x_2 y_2 z_1 x_2 y_1 z_1 + \frac{x_2 y_1 z_1}{y_2 z_2} y_1 z_2 x_2 y_1 z_1 \\
 &\quad \left. + x_2 y_1 z_2 x_2 y_1 z_1 + \frac{x_2 y_1 z_1}{y_2 z_2} y_2 z_2 x_2 y_1 z_1 + x_2 y_2 z_2 x_2 y_1 z_1 \right) \\
 &= \frac{1}{(x_1 - x_2)^2 r^3} x_1 x_2 z_1 z_2 \left( -\frac{x_2^2 y_1^2 z_1^3}{z_2} - \frac{x_2^2 y_1^3 z_1^2}{y_2} - \frac{x_2^2 y_1^3 z_1^3}{y_2 z_2} \right. \\
 &\quad - 2x_2^2 y_1^2 z_1^2 - x_2^2 y_1 y_2 z_1 z_2 - x_2^2 y_1 y_2 z_1^2 - x_2^2 y_1^2 z_1 z_2 + \frac{x_2^2 y_1^3 z_1^3}{y_2 z_2} \\
 &\quad + x_2^2 y_1^2 z_1^2 + \frac{x_2^2 y_1^2 z_1^3}{z_2} + x_2^2 y_1 y_2 z_1^2 + \frac{x_2^2 y_1^3 z_1^2}{y_2} + x_2^2 y_1^2 z_1 z_2 \\
 &\quad \left. + x_2^2 y_1^2 z_1^2 + x_2^2 y_1 y_2 z_1 z_2 \right) = 0.
 \end{aligned}$$

Likewise, we show that  $B = 0$  under each of the other conditions. ■

We now present a necessary condition for irreducibility.

**Proposition 4.4** The representation is reducible in each of the following cases:

- (1)  $x_1 y_2 z_2 = x_2 y_1 z_1$
- (2)  $x_1 y_1 z_2 = x_2 y_2 z_1$
- (3)  $x_1 y_2 z_1 = x_2 y_1 z_2$
- (4)  $x_1 y_1 z_1 = x_2 y_2 z_2$

**Proof.** Assume that we have either one of the following conditions holds true:

$$x_1 y_2 z_2 = x_2 y_1 z_1, \quad x_1 y_1 z_2 = x_2 y_2 z_1, \quad x_1 y_2 z_1 = x_2 y_1 z_2 \text{ or } x_1 y_1 z_1 = x_2 y_2 z_2.$$

Let  $S$  be the one dimensional subspace generated by  $e_2$ . If

$$s_1(1, 2) = \frac{y_1 + y_2}{y_1 y_2} - \frac{x_2(z_1 + z_2)}{r} = 0,$$

then  $w$ , as defined in section 4, equals  $(x_1 - x_2)^2 y_1^2 y_2^2 z_1 z_2$ . This implies that  $w \neq 0$ . By Lemma 4.1, we get a contradiction. Therefore, without loss of generality, we assume that  $s_1(1, 2) \neq 0$ . We then conjugate the representation by the invertible matrix  $T$ . Recall that  $b_i = T^{-1} s_i T$  ( $i = 1, 2, 3$ ). We then have that  $b_2 e_2 = (w, P) = (0, P)$  by Lemma 4.1, and  $b_3 e_2 = (B, C) = (0, C)$  by Lemma 4.3. It follows that  $S$  is invariant under this representation. ■

This leads us to state a necessary and sufficient condition for the irreducibility of the representation.

**Theorem 4.5** The representation is irreducible if and only if  $x_1y_2z_2 \neq x_2y_1z_1$ ,  $x_1y_1z_2 \neq x_2y_2z_1$ ,  $x_1y_2z_1 \neq x_2y_1z_2$  and  $x_1y_1z_1 \neq x_2y_2z_2$ .

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