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m-Projections involving Minkowski inverse and Range Symmetric property in Minkowski Space

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Abstract. In this paper we study the impact of Minkowski metric matrix on a projection in the Minkowski Space $\mathcal{M}$ along with their basic algebraic and geometric properties.The relation between the m-projections and the Minkowski inverse of a matrix A in the minkowski space $\mathcal{M}$ is derived. In the remaining portion commutativity of Minkowski inverse in Minkowski Space $\mathcal{M}$ is analyzed in terms of m-projections as an analogous development and extension of the results on EP matrices.

Keywords: Minkowski inverse, m-projections, range symmetric, EP matrix.

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1. Introduction and preliminaries

Let us denote by $M_{(m,n)}(\mathbb{C})$ the set of $m \times n$ matrices and when $m = n$ we write $M_n(\mathbb{C})$ for $M_{(n,n)}(\mathbb{C})$. The symbols $A^*, A^\sim, A^\oplus, \|A\|, A^\dagger$, $R(A)$, and $N(A)$ denote the conjugate transpose, Minkowski adjoint, Minkowski inverse, norm, Moore-Penrose inverse, range space and null space of a matrix A respectively. I_n denote the identity matrix of order $n \times n$ and R^+ denotes the set of positive real numbers. We use $\bar{P} = I_n - P$ (this notation has nothing to do with the complex conjugate of a matrix element). Also we use $\tilde{P}_A = I - AA^\oplus$ and $P_A = AA^\oplus$.

Indefinite inner product is a scalar product defined by

$$[u, v] = \langle u, Mv\rangle = u^*Mv,$$

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where $\langle \ , \rangle$ denotes the conventional Hilbert Space inner product and M is a Hermitian matrix. This hermitian matrix M is referred to as metric matrix. Minkowski space is an indefinite inner product space which is agreed upon to be the most suitable space for the study of Einstein's theory of special relativity and has been recently taken into consideration in a more generalized form by changing the dimension and the signature of the metric associated with its indefinite inner product. Moreover the matrix argument is taken into consideration. In Minkowski Space $\mathcal{M}$ the metric matrix is denoted by G and is defined as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \text{ satisfying } G^2 = I_n \text{ and } G^* = G.$$

G is called the Minkowski metric matrix.
In case $u \in \mathbb{C}^n$, indexed as $u = (u_0, u_1, ..., u_{n-1})$, G is called the Minkowski metric tensor and is defined as $Gu = (u_0, -u_1, ..., -u_{n-1})$. For detailed study of indefinite linear algebra refer to [13]

An idempotent matrix is called a projection if it is a hermitian i,e. $A^2 = A = A^*$. Projections are widely used in the literature e.g. see [1, 7, 15, 16, 20, 23–27, 34]. There are many extensions and generalizations of the projections like generalized projections, k-generalized projections, hypergeneralized projections etc. see [5, 11, 14, 18]. Projections are also studied in indefinite inner product spaces see [10, and refrences therein]. In this paper we first take into account the impact of Minkowski metric matrix on a projection, giving rise to a new class of projections called m-projections.

Another part of this paper is devoted to study the commutativity of Minkowski inverse of a matrix A in terms of the m-projections. A matrix $A \in M_n(\mathbb{C})$ is said to be range hermitian or EP matrix if $N(A) = N(A^*)$ or equivalently $AA^\dagger = A^\dagger A$ [1, p.157]. For detailed study of EP matrices see [1, 6, 9, 12, 17, 28, 30–33]. The EP property was further extended to the elements of Banach Spaces, operators, Banach Algebras and in particular to C^* -Algebras see [8, and references therein]. Meenakshi in [2] introduced the concept of range symmetric matrices in the Minkowski space analogous to the EP matrices in the unitary space. $A \in M_n(\mathbb{C})$ is said to be range symmetric in Minkowski space $\mathcal{M}$ if and only if $N(A) = N(A^\sim)$ [2], where $A^\sim$ is the Minkowski adjoint of the matrix A. The Minkowski adjoint of a matrix $A \in M_{(m,n)}(\mathbb{C})$, denoted by $A^\sim$, is defined as $A^\sim = G_1 A^* G_2$, where $G_1 \in M_m(\mathbb{C})$ and $G_2 \in M_n(\mathbb{C})$ are the Minkowski metric matrices of respective size and A is said to be m-symmetric if $A^\sim = A$. We will show that the commutativity of a matrix A with its Minkowski inverse $A^\oplus$ is also related to the EP property in the Minkowski Space $\mathcal{M}$.

The Minkowski inverse of a matrix $A \in M_{(m,n)}(\mathbb{C})$ in the Minkowski Space $\mathcal{M}$ is defined as the unique matrix X satisfying the following four conditions

$$(1)\ AXA = A \ (2)\ XAX = X \ (3)\ (AX)^\sim = AX \text{ and } (4)\ (XA)^\sim = XA$$

In [22] the author has investigated certain properties of the Minkowski Inverse in the indefinite inner product space by generalizing the signature of the Minkowski metric matrix G. Unlike the Moore-Penrose inverse of matrices the Minkowski inverse of a matrix does not exist always. The following result of [3] gives the necessary and sufficient condition for the existence of the Minkowski inverse of a matrix A in the Minkowski Space $\mathcal{M}$

Lemma 1.1 For $A \in M_{(m,n)}(\mathbb{C})$, the Minkowski inverse $A^\oplus$ exists if and only if $rank(A) = rank(AA^\sim) = rank(A^\sim A)$. if $A^\oplus$ exists , then it is unique.

Definition 1.2 A matrix $A \in M_n(\mathbb{C})$ is said to be G-unitary if $AA^\sim = A^\sim A = I$.

Following result from [2] will also be used in the forth coming sections

Theorem 1.3 Let $A \in M_n(\mathbb{C})$, if $A^{\oplus}$ exists in $\mathcal{M}$ then $AA^{\oplus}$ and $A^{\oplus}A$ are projections on $R(A)$ and $R(A^{\sim})$ respectively.

Theorem 1.4 For $A \in M_n(\mathbb{C})$, the following statements are equivalent:

(i) A is range symmetric in $\mathcal{M}$
(ii) GA and AG are EP
(iii) $N(A^*) = N(AG)$.
(iv) $R(A) = R(A^{\sim})$

2. m-projections and their algebraic properties

The concept of m-projections roots from the Minkowski adjoint of a projection P satisfying the condition of being m-symmetric. We thus have a class of projections which satisfy the condition of being m-symmetric and we call such projections as m-projections (Minkowski projections). Hence we have the following definition

Definition 2.1 A Projection P is said to be m-projection if $P^2 = P = P^{\sim}$, where $P^{\sim} = GP^*G$ is the Minkowski adjoint of the projection P.

For the sake of completion we prove the following two results, concluding the basic algebraic perspective of the m-projection.

Lemma 2.2 If P is a m-projection then $I - P$ is also an m-projection

Proof. The proof is obvious from definition of the m - projection. ■

Theorem 2.3 Let S be the set of all m-projections in a Minkowski space M.Then for $P_1, P_2 \in S$ we have

(i) $P_1 + P_2 \in S$ if and only if $P_1P_2 = 0 = P_2P_1$
(ii) $P_1 - P_2 \in S$ if and only if $P_1P_2 = P_2P_1 = P_2$
(iii) $P_1P_2 \in S$ if and only if $P_1P_2 = P_2P_1$

Proof. (i) Clearly we have $(P_1 + P_2)^{\sim} = P_1 + P_2$ and

$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2 \Leftrightarrow P_1P_2 + P_2P_1 = 0 \tag{1}$$

Premultiplying (1) by P_1 we get

$$P_1^2P_2 + P_1P_2P_1 = 0 \Rightarrow P_1^{\sim}P_2 + P_1P_2P_1 = 0 \tag{2}$$

Now postmultipling (1) by P_1 we get

$$P_1P_2P_1 + P_2P_1^{\sim} = 0 \tag{3}$$

Thus we have $P_1^{\sim}P_2 = P_2P_1^{\sim}$ which again on pre and post multiplying by $P_1^{\sim}$ gives $P_1P_2 = P_2P_1$ and the result follows. (ii) and (iii) are obvious. ■

Remark 1 *Since the Minkowski inverse of a matrix A exits if and only if $rank(AA^{\sim}) = rank(A^{\sim}A) = rank(A)$ and this condition is trivially satisfied by a* m-*projection. For a* m-*projection P we have $PP^{\sim} = P^2 = P = P^2 = P^{\sim}P$ therefore* *we have rank*$(PP^{\sim}) = $ *rank*$(P^{\sim}P) = $ *rank* (P) *and hence* $P^{\oplus} = P$. *Given any matrix* $A \in \mathbb{C}^{m\times n}$, $AA^{\oplus}$ *is a* m*-projection*

The following result will be used as definition for the Range Symmetric matrix in $\mathcal{M}$

Theorem 2.4 Let $A \in M_{m,n}(\mathbb{C})$ and $A^{\oplus}$ exists in $\mathcal{M}$, then the following statements are equivalent:

(a) A is Range Symmetric (EP) in $\mathcal{M}$
(b) $AA^{\oplus} = A^{\oplus}A$

Proof. Assume that $A^{\oplus}$ exists and A is EP in $\mathcal{M}$. Using Theorem (1.3), we have $AA^{\oplus}$ is the projection on $R(A)$ and $A^{\oplus}A$ is the projection on $R(A^{\sim})$. Also from Theorem (1.4), statement (iv) we have $R(A) = R(A^{\sim})$, the result follows. ■

We recover the following corollary from [4]

Corollary 2.5 Let $A \in M_n(\mathbb{C})$ be an EP matrix of rank r, Then, there exists a unitary matrix P and a nonsingular matrix D such that $A = P(D \oplus 0)P^{\sim}$ in the Minkowski Space $\mathcal{M}$

Proof. Using Theorem (1.4) we have AG is range symmetric in $\mathcal{M}$. Therefore by [21, Theorem 1] AG has the following representation

$$AG = U \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{4}$$

where U is a unitary matrix. This gives

$$A = UG \begin{bmatrix} G_1A_1 & 0 \\ 0 & 0 \end{bmatrix} U^*G \tag{5}$$

Taking $G_1A_1 = D$ and $P = UG$ we get $P^{\sim} = U^*G$. Therefore we have

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{\sim}$$

■

The representation obtained in the corollary (2.5) eases to formulate the Minkowski inverse for the range symmetric matrices and in particular for the m-projections in the Minkowski space. From the definition of the G-unitary matrix it follows at once that a unitary matrix U is G-unitary if and only if $UG = GU$. We will use this assumption in formulating the Minkowski inverse of the m-projections in the forth coming results. Under this assumption the above corollary can be extended to the following equivalent statement:

Corollary 2.6 Let $A \in M_n(\mathbb{C})$ be a matrix of rank r. Then following conditions are equivalent:

(i) $AA^{\oplus} = A^{\oplus}A$
(ii) There exists a unitary matrix P and a nonsingular matrix D such that

$$A = P(D \oplus 0)P^{\sim}, \text{where } P \text{ is G-unitary.}$$

Proof. $(i) \Rightarrow (ii)$ is obvious from Theorem (2.4) and Corollary (2.5). Also $(ii) \Rightarrow (i)$ follows from direct verification by noting that the Minkowski inverse of the A is $A^{\oplus} = P(D^{-1} \oplus 0)P^{\sim}$, where P is G-unitary. ■

Let P be an m-projection in the Minkowski space $\mathcal{M}$. Taking into account Remark (1) and the observations made in the corollaries (2.5) and (2.6) of the Theorem (2.4), there exists a G-unitary matrix U such that

$$P = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^{\sim} \tag{6}$$

The representation (6) can be used to determine the partitioning of any other m- projection say Q with the use of the same G-unitary matrix U such that

$$Q = U \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} U^{\sim}. \tag{7}$$

Using the later part of the definition of a m-projection i.e. $Q = Q^{\sim}$, the representation (7) becomes

$$Q = U \begin{bmatrix} W & X \\ -G_1X^{\sim} & Z \end{bmatrix} U^{\sim} \tag{8}$$

Where $W \in M_r(\mathbb{C})$ is m-symmetric, $Z \in M_{n-r}(\mathbb{C})$ is hermitian and G_1 is the Minkowski metric matrix of order $r \times r$. The following results give the relation between the submatrices W, X and Z of the matrix Q given in (8).

Lemma 2.7 Let Q be a m-projection as given in (8). Then:

(i) $W = W^2 - XG_1X^{\sim}$ or equivalently $W\bar{W} = -XG_1X^{\sim}$
(ii) $X = WX + XZ$ or equivalently $X^{\sim} = X^{\sim}W + G_1ZG_1X^{\sim}$
(iii) $G_1X^{\sim} = G_1X^{\sim}W + ZG_1X^{\sim}$ or equivalently $G_1X^{\sim}\bar{W} = ZG_1X^{\sim}$
(iv) $Z = Z^2 - G_1X^{\sim}X$ or equivalently $Z\bar{Z} = -G_1X^{\sim}X$

Proof. The proof of these relationships is a straightforward consequence of the condition $Q^2 = Q$ ■

Lemma 2.8 Let Q be a m-projection as given in (8). Then:

(i) $\bar{W} = \bar{W}^2 - XG_1X^{\sim}$
(ii) $WX = X\bar{Z}$
(iii) $XZ = \bar{W}X$
(iv) $G_1X^{\sim} = G_1X^{\sim}\bar{W} + \bar{Z}G_1X^{\sim}$
(v) $\bar{Z} = \bar{Z}^2 - G_1X^{\sim}X$

Proof. The proof follows by using Lemma (2.2) and the condition $\bar{Q}^2 = \bar{Q}$ ■

Theorem 2.9 Let Q be a m-projection as given in (8). Then:

(i) $WW^{\oplus}X = X$
(ii) $\bar{W}\bar{W}^{\oplus}X = X$
(iii) $ZZ^{\oplus}G_1X^{\sim} = G_1X^{\sim}$
(iv) $\bar{Z}\bar{Z}^{\oplus}G_1X^{\sim} = G_1X^{\sim}$
(v) $W^{\oplus}X = X\bar{Z}^{\oplus}$
(vi) $XZ^{\oplus} = \bar{W}^{\oplus}X$

Proof. The condition (i) follows on account of the condition (i) of the Lemma (2.7) and the fact that

$$R(W) = R(WW^* + XX^*) = R(WW^*) + R(XX^*) = R(W) + R(X) \tag{9}$$

Thus $R(X) \subseteq R(W)$, which can be expressed equivalently as $WW^{\oplus}X = X$. Analogously we can obtain the next three conditions. Also from the condition (ii) of the Lemma (2.7) we have $W^{\oplus}X = W^{\oplus}(WX + XZ)$. Using condition (i) of the theorem we get $W^{\oplus}X\bar{Z} = X$. Postmultiplying this equation by $\bar{Z}^{\oplus}$ and utilizing condition (iv), we obtain condition (v). The condition (vi) can be established similarly. ∎

Theorem 2.10 Let Q be a m-projection as given in (8). Then:

(i) $P_w = W - X\bar{Z}^{\oplus}G_1X^{\sim}$ and $\tilde{P}_w = \bar{W} + X\bar{Z}^{\oplus}G_1X^{\sim}$
(ii) $P_{\bar{w}} = \bar{W} - XZ^{\oplus}G_1X^{\sim}$ and $\tilde{P}_{\bar{w}} = W + XZ^{\oplus}G_1X^{\sim}$
(iii) $P_z = Z - G_1X^{\sim}\bar{W}^{\oplus}X$ and $\tilde{P}_z = \bar{Z} + G_1X^{\sim}\bar{W}^{\oplus}X$
(iv) $P_{\bar{z}} = \bar{Z} - G_1X^{\sim}W^{\oplus}X$ and $\tilde{P}_{\bar{z}} = Z + G_1X^{\sim}W^{\oplus}X$

Proof. From the condition (i) of lemma (2.7) we have $W = W^2 - XG_1X^{\sim}$. Premultiplying on both sides by $\bar{W}^{\oplus}$ and using condition (vi) of Theorem (2.9) we get the later part of the point (i) and subtracting the obtained expression from I_r on both sides we get the remaining part. On the same lines the other points can be established. ∎

3. m-projections onto certain subspaces

In this section we develop the representations of m-projectors onto certain subspace including their sum and intersection. The second lemma gives a powerful tool for constructing the m-projectors onto given spaces which is obtained as an analogous result from orthogonal projectors.

Lemma 3.1 Let Q be partitioned as in (6) . Then:

(i) $rk(\bar{W}) = r - rk(W) + rk(X)$
(ii) $rk(\bar{Z}) = n - r + rk(X) - rk(Z)$

Proof. Using (i) of Lemma (2.7) and the fact that $rk(W\bar{W}) = rk(W) + rk(\bar{W}) - r$, from (2.12) in [35] we get $rk(\bar{W}) = r + rk(X) - rk(W)$.
Condition (ii) follows analogously. ∎

Lemma 3.2 Let $P,\ Q \in C_n^{mp}$. Then:

(i) $P + \bar{P}(\bar{P}Q)^{\oplus}$ is the m-projector onto $R(P) + R(Q)$
(ii) $P - P(P\bar{Q})^{\oplus}$ is the m-projector onto $R(P) \cap R(Q)$

Proof. The proof follows analogously from the equivalent conditions of Theorems (3) and (4) in [29]. ∎

Using Lemma (3.2) we obtain the following representations of the m-projectors onto the sums and intersection of certain subspaces , including their dimensions.

Lemma 3.3 Let $P, Q \in C_n^{mp}$ and let $\mathbf{Q}$ be partitioned as in (8). Then:

(i) $P_{R(P)+R(Q)} = U \begin{bmatrix} I_r & 0 \\ 0 & P_z \end{bmatrix} U^{\sim}$, where $dim[R(P) + R(Q)] = r + rk(Z)$

(ii) $P_{R(P)+NQ)} = U \begin{bmatrix} I_r & 0 \\ 0 & P_{\bar{z}} \end{bmatrix} U^{\sim}$, where $dim[R(P) + N(Q)] = r + rk(X) + rk(Z)$

(iii) $P_{N(P)+R(Q)} = U \begin{bmatrix} P_w & 0 \\ 0 & I_{n-r} \end{bmatrix} U^{\sim}$, where $dim[N(P) + R(Q)] = n - r + rk(W)$

(iv) $P_{N(P)+N(Q)} = U \begin{bmatrix} P_{\bar{w}} & 0 \\ 0 & I_{n-r} \end{bmatrix} U^{\sim}$, where $dim[N(P) + N(Q)] = n - rk(W) + rk(X)$

Proof. For $P = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{\sim}$ and $Q = U \begin{bmatrix} W & X \\ -G_1X^{\sim} & Z \end{bmatrix} U^{\sim}$. We have $(\bar{P}Q) = U \begin{bmatrix} 0 & 0 \\ -G_1X^{\sim} & Z \end{bmatrix} U^{\sim}$. Now utilising the conditions (iv) of Lemma (2.7) and (iii) of Theorem (2.9), it can be easily verified that the minkowski inverse of $\bar{P}Q$ is

$$(\bar{P}Q)^{\oplus} = U \begin{bmatrix} 0 & XZ^{\oplus} \\ 0 & P_z \end{bmatrix} U^{\sim} \tag{10}$$

Hence, using the statement (i) of Lemma (3.2) and substituting (10), we obtain the m-projector as claimed in point (i). The remaining part is obvious from the representation of the projector. The remaining points can be obtained in a similar fashion. ■

Lemma 3.4 Let $P,\ Q \in C_n^{mp}$ and let Q be partitioned as in (8). Then:

(i) $P_{R(P)\cap R(Q)} = U \begin{bmatrix} \tilde{P}_{\bar{w}} & 0 \\ 0 & 0 \end{bmatrix} U^{\sim}$, where $dim[R(P) \cap R(Q)] = rk(W) - rk(X)$

(ii) $P_{R(P)\cap N(Q)} = U \begin{bmatrix} \tilde{P}_w & 0 \\ 0 & 0 \end{bmatrix} U^{\sim}$, where $dim[R(P) \cap N(Q)] = r - rk(W)$

(iii) $P_{N(P)\cap R(Q)} = U \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_{\bar{z}} \end{bmatrix} U^{\sim}$, where $dim[R(P) \cap R(Q] = rk(Z) - rk(X)$

(iv) $P_{N(P)\cap R(Q)} = U \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_z \end{bmatrix} U^{\sim}$, where $dim[R(P) \cap R(Q)] = n - r - rk(Z)$

Proof. With given P and Q we have $P\bar{Q} = U \begin{bmatrix} \bar{W} & -X \\ 0 & 0 \end{bmatrix} U^{\sim}$. Using the conditions (i) and (ii) of Lemma (2.7) and Theorem (2.9) respectively direct verification shows that the Minkowski inverse $(P\bar{Q})^{\oplus}$ of $P\bar{Q}$ is

$$(P\bar{Q})^{\oplus} = U \begin{bmatrix} \bar{W}\bar{W}^{\oplus} & 0 \\ G_1X^{\sim}\bar{W}^{\oplus} & 0 \end{bmatrix} U^{\sim} \tag{11}$$

Now using the statement (ii) of the Lemma (3.2) and substituting (11) we obtain the representation claimed in point (i) of the lemma. Also $dim(P_{R(P)\cap R(Q)}) = rk(\tilde{P}_{\bar{w}}) = r - rk(\bar{W})$. Whereupon utilizing the rank equality obtained in statement (i) of the Lemma (3.1) we get the remaining part of the result. The remaining representations can be obtained similarly. ■

4. Minkowski inverse and m-projections

In this section we characterize the relation between the m-projections and the Minkowski inverse of a matrix in the Minkowski Space $\mathcal{M}$. We denote by $\mathcal{M}^{-1}$ the set of all invertible elements in Minkowski space $\mathcal{M}$

Theorem 4.1 Let $A \in M_n(\mathbb{C})$ in the Minkowski Space $\mathcal{M}$, then the following conditions are equivalent:

(i) There exists a unique m-projection P such that $A + P \in \mathcal{M}^{-1}$ and $AP = PA = 0$
(ii) A is range symmetric in $\mathcal{M}$.

Proof. $(i) \Rightarrow (ii)$
Let P be an m-projection. Since $AP = 0$ and $P^2 = P$, we have $AP + P^2 = P$ which implies $(A + P)P = P$ Also $(A + P)^{-1}$ exists so we have $P = P(A + P)^{-1}$. Similarly we get $P = (A + P)^{-1}P$. we claim that

$$A\{(A + P)^{-1} - P\} = I - P \tag{12}$$

we have

$$\begin{aligned} A\{(A + P)^{-1} - P\} &= (A + P - P)\{(A + P)^{-1} - P\} \\ &= (A + P)\{(A + P)^{-1} - P\} - P\{(A + P)^{-1} - P\} \\ &= I - P. \end{aligned} \tag{13}$$

Similarly $\{(A + P)^{-1} - P\}A = I - P$. Now we prove that $A^{\oplus} = X = \{(A + P)^{-1} - P\}$ In order to prove this we show that X satisfies the conditions of the definition of the Minkowski inverse of A. Using (12) and the given condition that $AP = PA = 0$ we have

$$AXA = A\{(A + P)^{-1} - P\}A = (I - P)A = A - PA = A \tag{14}$$

Also

$$XAX = \{(A + P)^{-1} - P\}A\{(A + P)^{-1} - P\} = \{(A + P)^{-1} - P\} \tag{15}$$

We now prove the m-symmetric conditions i,e. $(AX)^{\sim} = AX$ and $(XA)^{\sim} = XA$
Again using (12) we have $(AX)^{\sim} = G[\{(A + P)^{-1} - P\}A]^*G = I - P = AX$
Analogously we can prove that $(XA)^{\sim} = XA$
Therefore $X = A^{\oplus}$. Also $AA^{\oplus} = I - P = A^{\oplus}A$ i,e. $AA^{\oplus} = A^{\oplus}A$. Therefore by Theorem (2.4) A is Range Symmetric i,e. A is EP in $\mathcal{M}$
$(ii) \Rightarrow (i)$ Let P be the m-projection defined by $P = I - AA^{\oplus}$.
Evidently we have $AP = 0$ and $PA = 0$. Also

$$(A + P)(A^{\oplus} + P) = AA^{\oplus} + AP + PA^{\oplus} + P = AA^{\oplus} + P = I \tag{16}$$

This shows that $(A + P)$ is invertible and $(A + P)^{-1} = (A^{\oplus} + P)$
Finally we show that the m-projection P is unique.
Assume that Q is another m-projection such that $AQ = 0 = QA$ and $A + Q \in \mathcal{M}^{-1}$ then,

$$A^{\oplus} = (A + P)^{-1} - P = (A + Q)^{-1} - Q \tag{17}$$

Premultiplying (10) by A we get

$$AA^{\oplus} = A(A + P)^{-1} - AP = A(A + Q)^{-1} - AQ \tag{18}$$

$$\Rightarrow A(A+P)^{-1} = A(A+Q)^{-1} \Rightarrow P = Q \tag{19}$$

This shows that the m-projection P is unique. ■

Remark 2 *For $A, B \in M_n(\mathbb{C})$ we have*

$$(A+B)^{\sim} = A^{\sim} + B^{\sim},\ (\lambda A)^{\sim} = \bar{\lambda}A^{\sim},\ (AB)^{\sim} = B^{\sim}A^{\sim} \textit{and}\ \ (A^{\sim})^{\sim} = A. \tag{20}$$

It is interesting to observe from (20) that the mapping $A \mapsto A^{\sim}$ satisfies the conditions of being an involution. Under this involution the similar results will hold in the C^*-algebra.

Following the usual notation we also denote the unique m-projection $P = I - AA^{\oplus}$ by $\bar{P}_A$ and we have the following corollary

Corollary 4.2 Let $A \in M_{m,n}(\mathbb{C})$ and $A^{\oplus}$ exists in $\mathcal{M}$, then the following are equivalent

(i) $A^k = A^{\oplus}$
(ii) A is range symmetric and $A^{k+1} + \bar{P}_A = 1$

Proof. Clearly $A^k = A^{\oplus} \Rightarrow AA^{\oplus} = A^{\oplus}A$. Thus A is EP in $\mathcal{M}$. Now using the facts that $\bar{P}_A = I - AA^{\oplus}$ and $A^{\oplus} = (A+\bar{P}_A)^{-1} - \bar{P}_A$, we have

$$\begin{aligned} A^{\oplus} &= A^k \\ &\Leftrightarrow (A+\bar{P}_A)^{-1} - \bar{P}_A = A^k \\ &\Leftrightarrow (A+\bar{P}_A)^{-1} = A^k + \bar{P}_A \end{aligned} \tag{21}$$

This gives $(A+\bar{P}_A)(A^k+\bar{P}_A) = I_n \Leftrightarrow A^{k+1} + \bar{P}_A = I_n$. ■

Corollary 4.3 Let $A \in \mathcal{M}$ be EP, then

(i) $A^{\oplus} = (A+\bar{P}_A)^{-1}(I-\bar{P}_A)$
(ii) $\bar{P}_A$ is idempotent
(iii) $A\bar{P}_A = \bar{P}_A A = A^{\oplus}\bar{P}_A = \bar{P}_A A^{\oplus} = 0$
(iv) $\bar{P}_A = 0$ if and only if A is nonsingular
(v) $A = P(D \oplus 0)P^{\sim}$ then $\bar{P}_A = P(0 \oplus I_{n-r})P^{\sim}$
(vi) $rank(\bar{P}_A) = n - rank(A)$

Proof. Noting that $\bar{P}_A = I - AA^{\oplus}$, A is EP in $\mathcal{M}$ and using Theorem (2.4) we have

$$\begin{aligned} (A+\bar{P}_A)^{-1}(I-\bar{P}_A) &= (A+\bar{P}_A)^{-1} - (A+\bar{P}_A)^{-1}\bar{P}_A \\ &= (A+\bar{P}_A)^{-1} - (A+\bar{P}_A)^{-1}(I - AA^{\oplus}) \\ &= (A+\bar{P}_A)^{-1}(A+\bar{P}_A-\bar{P}_A)A^{\oplus} \\ &= (A+\bar{P}_A)^{-1}(A+\bar{P}_A)A^{\oplus} - (A+\bar{P}_A)^{-1}\bar{P}_A A^{\oplus} \\ &= A^{\oplus} - (A+\bar{P}_A)^{-1}(I-AA^{\oplus})A^{\oplus} \\ &= A^{\oplus} \end{aligned}$$

The remaining statements are obvious. ■

Remark 3 *If A is range symmetric in $\mathcal{M}$, then using Theorem (2.4), the definition of the Minkowski inverse X of a matrix A reduces to $AXA = A$, $XAX = X$, and $(AX)^{\sim} =$*

$(XA)^{\sim}$. *which is analogous to the definition of the Group inverse in the Minkowski space* $\mathcal{M}$. *Also from [2, Theorem 2.9] we have if* A *is range symmetric in* $\mathcal{M}$ *and* $rank(A^2) = rank(A)$ *then* $A^{\oplus}$ *exists. But* $rank(A^2) = rank(A)$ *is the necessary and sufficient condition for the existence of the group inverse* $A^{\sharp}$ *of a matrix* A *[1, page156]. Thus in this case we have* $A^{\oplus} = A^{\sharp}$

5. Commutativity with an range symmetric element

In this section some characterizations of the matrices which commute with their Minkowski inverse are obtained in terms of m-projections. Using the involution defined in the Remark (*2*) and the fact that every m-projection is also a projection we have an analogous results in the settings of a C^*-algebra. The following result of [19] is a very useful representation of the Weighted Minkowski inverse of a matrix $A \in M_{m,n}(\mathbb{C})$ and is used in this section.

Theorem 5.1 Let $A \in M_{m,n}(\mathbb{C}$ in $\mathcal{M}$ and $A = N^{-1}G_1A^*G_2M$ where $M \in M_m(\mathbb{C})$ and $N \in M_n(\mathbb{C})$ are positive definite matrices and G_1 and G_2 are Minkowski metric matrices of order $n \times n$ and $m \times m$ respectively such that $\sigma(A^{\alpha}) \subset R^+$. Then

$$A^{\oplus}_{M,N} = \lim_{t\to 0}(tI + A^{\sim}A)^{-1}A^{\sim}$$

In particular when $M = I_n$ and $N = I_n$, then $A^{\oplus}_{M,N}$ reduces to the Minkowski inverse $A^{\oplus}$ of A in $\mathcal{M}$

Lemma 5.2 Let P be an m-projection and A be Minkowski invertible i,e. $A^{\oplus}$ exists such that $PA = AP$ then

(i) $A^{\oplus}P = PA^{\oplus}$
(ii) PAP is Minkowski Invertible and $(PAP)^{\oplus} = PA^{\oplus}P$

Proof. (*i*) From $AP = PA$ and $P^{\sim} = P$ we have $A^{\sim}P = PA^{\sim}$. This gives

$$A^{\sim}PA = PA^{\sim}A \Rightarrow A^{\sim}AP = PA^{\sim}A.$$

Thus for some $t \in R^+$ we have

$$\Rightarrow P(A^{\sim}A + tI_n)^{-1} = (A^{\sim}A + tI_n)^{-1}P \tag{22}$$

Now using the Theorem (5.1) we deduce that $A^{\oplus}P = PA^{\oplus}$
(*ii*) It follows from the direct verification. ■

Theorem 5.3 Let $A, B \in M_n(\mathbb{C})$ such that A is EP in $\mathcal{M}$ and $AB = BA = 0$. Then

(i) $\bar{P}_AB = B = B\bar{P}_A$
(ii) $A^{\oplus}B = BA^{\oplus} = 0$
(iii) $B^{\oplus}$ exists implies $AB^{\oplus} = B^{\oplus}A = 0$
(iv) $B^{\oplus}$ exists implies $(A + B)^{\oplus}$ exists and $(A + B)^{\oplus} = A^{\oplus} + B^{\oplus}$
(v) B is EP in $\mathcal{M}$ implies $A + B$ is EP in $\mathcal{M}$ and $\bar{P}_{A+B} = \bar{P}_A + \bar{P}_B - I$

Proof. (i) we have

$$\begin{aligned}(A+\bar{P}_A)(I-\bar{P}_A)B &= (A+\bar{P}_A-A\bar{P}_A-\bar{P}_A)B\\ &= AB+\bar{P}_AB-A\bar{P}_AB-\bar{P}_AB\\ &= 0.\end{aligned}\tag{23}$$

This gives $(A+\bar{P}_A)(I-\bar{P}_A)B=0$. But $(A+\bar{P}_A)\in\mathcal{M}^{-1}$. Therefore $(I-\bar{P}_A)B=0$ implies $\bar{P}_AB=B$. In a similar fashion, by using $B(I-\bar{P}_A)(A+\bar{P}_A)=0$ we get $B\bar{P}_A=B$. The equality (i) can also be obtained directly by using the give condition that A is EP in $\mathcal{M}$ and $\bar{P}_A=I-AA^{\oplus}$.

(ii) Recall that in Theorem (4.1) we have proved that $A^{\oplus}=(\bar{P}_A+A)^{-1}-\bar{P}_A$. Also $(\bar{P}_A+A)B=B$ implies $B=(\bar{P}_A+A)^{-1}B$. Therefore

$$\begin{aligned}A^{\oplus}B &= \{(\bar{P}_A+A)^{-1}-\bar{P}_A\}B\\ &= (\bar{P}_A+A)^{-1}B-\bar{P}_AB\\ &= 0\end{aligned}\tag{24}$$

This can also be proved by using Corollary (4.3).

(iii) From lemma (5.2), statement (ii) we have $(\bar{P}_AB\bar{P}_A)^{\oplus}=\bar{P}_AB^{\oplus}\bar{P}_A$. Also $\bar{P}_AB\bar{P}_A=B$. Therefore $B^{\oplus}=\bar{P}_AB^{\oplus}\bar{P}_A$ implies $AB^{\oplus}=A\bar{P}_AB^{\oplus}\bar{P}_A=0$. Similarly $B^{\oplus}A=0$.

(iv) This follows from direct verification.

(v) B is EP in $\mathcal{M}$ implies $BB^{\oplus}=B^{\oplus}B$. Also $(A+B)^{\oplus}=(A^{\oplus}+B^{\oplus})$. Using Theorem (2.4) we have $A+B$ is EP in $\mathcal{M}$. Also $\bar{P}_{A+B}=\bar{P}_A+\bar{P}_B-I$ follows by doing simple algebra. ∎

Let us define the norm of a matrix A as given in [34, page 49]. For $A\in M_{m,n}(\mathbb{C})$ lets define the norm of A as $\|A\|=\{tr(A^TA)\}^{1/2}=\sqrt{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}a_{ij}^2}$, then we have the following result used in next theorem from [34, page 49].

Lemma 5.4 $\|AB\|\leqslant\|A\|.\|B\|$ and $\|A+B\|\leqslant\|A\|+\|B\|$. Also if P is a projection then $\|AP\|\leqslant\|A\|$ with equality if $PA=A$.

Theorem 5.5 Let $A,B\in M_n(\mathbb{C})$ such that A is EP in $\mathcal{M}$, then $\|(I-\bar{P}_A)B\bar{P}_A\|\leqslant\|A^{\oplus}\|.\|AB-BA\|$. and $\|\bar{P}_AB(I-\bar{P}_A)\|\leqslant\|A^{\oplus}\|.\|AB-BA\|$

Proof. Since $\bar{P}_A$ and $(I-\bar{P}_A)$ are idempotent projections, we have $\|\bar{P}_A\|=\|(I-\bar{P}_A)\|=1$. Also $(I-\bar{P}_A)(AB-BA)\bar{P}_A=AB\bar{P}_A-\bar{P}_AAB\bar{P}_A+\bar{P}_ABA\bar{P}_A=AB\bar{P}_A$. This gives $\|(I-\bar{P}_A)(AB-BA)\bar{P}_A\|=\|AB\bar{P}_A\|$.

Thus we have

$$\begin{aligned}\|(I-\bar{P}_A)B\bar{P}_A\| &= \|\{I-(I-AA^{\oplus})\}B\bar{P}_A\|\\ &\leqslant \|A^{\oplus}\|.\|AB\bar{P}_A\|\\ &= \|A^{\oplus}\|.\|(I-\bar{P}_A)(AB-BA)\bar{P}_A\|\\ &= \|A^{\oplus}\|.\|AB-BA\|\end{aligned}$$

Analogously we can prove the second inequality. ∎

Corollary 5.6 Let $A, B \in M_n(\mathbb{C})$ such that A is EP in $\mathcal{M}$. If $AB = BA$ then $B\bar{P}_A = \bar{P}_A B$

Proof. The proof follows by using $AB = BA$ in the above proved inequalities. ■

Theorem 5.7 Let $A, B \in M_n(\mathbb{C})$ such that A and B are EP in $\mathcal{M}$ and $AB = BA$. Then

(i) $\bar{P}_A \bar{P}_B = \bar{P}_B \bar{P}_A$
(ii) $AB^{\oplus} = B^{\oplus} A$ and $A^{\oplus} B = BA^{\oplus}$
(iii) $A^{\oplus} B^{\oplus} = B^{\oplus} A^{\oplus} = (AB)^{\oplus}$

Proof. (i) Since A and B are both EP in $\mathcal{M}$. Therefore from Corollary (5.6) we have $\bar{P}_A B = B\bar{P}_A$ and $\bar{P}_B A = A\bar{P}_B$. Also from lemma (5.2), point (i) we have $B^{\oplus}\bar{P}_A = \bar{P}_A B^{\oplus}$. Now $\bar{P}_A \bar{P}_B = \bar{P}_A(I - BB^{\oplus}) = \bar{P}_A - \bar{P}_A BB^{\oplus} = \bar{P}_B \bar{P}_A$

(ii) Again using corollary (5.6) we have

$$AB + A\bar{P}_B = BA + \bar{P}_B A \tag{25}$$

which implies

$$A(B + \bar{P}_B) = (B + \bar{P}_B)A \tag{26}$$

Thus invertibility of $(B + \bar{P}_B)$ implies

$$(B + \bar{P}_B)^{-1} A = A(B + \bar{P}_B)^{-1} \tag{27}$$

Hence we have

$$\begin{aligned} AB^{\oplus} &= A\{(B + \bar{P}_B)^{-1} - \bar{P}_B\} \\ &= A(B + \bar{P}_B)^{-1} - A\bar{P}_B \\ &= B^{\oplus} A \end{aligned} \tag{28}$$

Similarly, $A^{\oplus} B = BA^{\oplus}$

(iii) we have

$$(B + \bar{P}_B)^{-1} A = A(B + \bar{P}_B)^{-1} \tag{29}$$

Similarly,

$$(B + \bar{P}_B)^{-1} \bar{P}_A = \bar{P}_A (B + \bar{P}_B)^{-1} \tag{30}$$

Adding equations (29) and (30), we get

$$(B + \bar{P}_B)^{-1}(A + \bar{P}_A)^{-1} = (A + \bar{P}_A)^{-1}(B + \bar{P}_B)^{-1} \tag{31}$$

Now

$$A^{\oplus} B^{\oplus} = \{(A + \bar{P}_A)^{-1} - \bar{P}_A\}\{(B + \bar{P}_B)^{-1} - \bar{P}_B\}$$

Using the equation (31) we get the result. ■

6. Conclusion

In this paper we have studied the algebraic and the geometric behavior of m-projections in Minkowski Space, and have established a relation between an m-projection and the Minkowski inverse associated with a matrix. Further we have characterized some Range Symmetric elements in Minkowski Space by using m-projections.The next step naturally will be directed towards establishing the full characterization of the Range Symmetric elements in the Minkowski Space.

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