Normalized Laplacian spectrum of two new types of join graphs

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Abstract. Let $G$ be a graph without an isolated vertex, the normalized Laplacian matrix $\tilde{L}(G)$ is defined as $\tilde{L}(G) = D^{-\frac{1}{2}}L(G)D^{-\frac{1}{2}}$, where $D$ is a diagonal matrix whose entries are degree of vertices of $G$. The eigenvalues of $\tilde{L}(G)$ are called as the normalized Laplacian eigenvalues of $G$. In this paper, we obtain the normalized Laplacian spectrum of two new types of join graphs. In continuing, we determine the integrality of normalized Laplacian eigenvalues of graphs. Finally, the normalized Laplacian energy and degree Kirchhoff index of these new graph products are derived.

Keywords: Join of graphs, normalized Laplacian eigenvalue, integral eigenvalue.

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1. Introduction

Let $G = (V, E)$ be a simple graph (namely a graph without loop and multiple edges) on $n$ vertices and $m$ edges and the degree of vertex $v \in V(G)$ is denoted by $\deg(v)$. The eigenvalues of the adjacency matrix $A = A(G)$ of $G$ are called the eigenvalues of $G$ denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be the distinct eigenvalues of $G$ with multiplicity $t_1, t_2, \ldots, t_n$, respectively. The multiset $\{[\lambda_1(G)]^{t_1}, [\lambda_2(G)]^{t_2}, \ldots, [\lambda_n(G)]^{t_n}\}$ of eigenvalues of $A$ is called the spectrum of $G$. 

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The energy of graph $G$ is a graph invariant introduced by Ivan Gutman as

$$
\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|,
$$

see for more details [8, 10, 17]. Let $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ be the Laplacian matrix of graph $G$, where $\mathcal{D}(G) = [d_{ij}]$ is the diagonal matrix whose entries are degree of vertices, namely, $d_{ii} = \deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$. The matrix $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is called the signless Laplacian matrix of $G$. The signless Laplacian eigenvalues of $G$ are denoted by $q_1(G), q_2(G), \ldots, q_n(G)$. If $G$ is regular of degree $r$, then the eigenvalues of $\mathcal{Q}(G)$ are $2r, r + \lambda_2, \ldots, r + \lambda_n$, see [1, p.14]. In [15], Jooyandeh et al. introduced the concept of incidence energy of a graph as

$$
\mathcal{RE}(G) = \sum_{i=1}^{n} \sqrt{q_i(G)}.
$$

Let $G$ be a graph without an isolated vertex, the normalized Laplacian matrix $\tilde{\mathcal{L}}(G)$ is defined as $\tilde{\mathcal{L}}(G) = \mathcal{D}^{-\frac{1}{2}} \mathcal{L}(G) \mathcal{D}^{-\frac{1}{2}}$ which implies that its $(i, j)$–entry is 1 if $i = j$, and it is $-1/\sqrt{\deg(v_i)\deg(v_j)}$, if $i \neq j$ and the vertices $v_i, v_j$ are adjacent, and is zero otherwise. The eigenvalues of $\tilde{\mathcal{L}}(G)$ are the roots of $g(x) = \det(\delta I - \tilde{\mathcal{L}}(G))$ and we call them as normalized Laplacian eigenvalues of $G$, denoted by $\delta_1(G), \delta_2(G), \ldots, \delta_n(G)$, where $\delta_n(G) = 0$ for all graphs, see for more details [2, 3, 6].

The complement of graph $G$ is denoted by $\overline{G}$ and a complete graph on $n$ vertices is denoted by $K_n$. Also a cycle graph with $n$ vertices is denoted by $C_n$. Clearly, $K_n$ is empty graph. The subdivision graph $S(G)$ of a graph $G$ is obtained by inserting an additional vertex in the middle of each edge of $G$. Equivalently, each edge of $G$ is replaced by a path of length 2, see Figure 1. It is a well-known fact that $P_{S(G)}(x) = x^{m-n}Q_G(x^2)$, where $P_G(x)$ and $Q_G(x)$ denote the characteristic polynomial and the signless Laplacian characteristic polynomial of $G$, respectively [7, p. 63]. From the definition, one can see that $\pm \sqrt{q_1(G)}, \pm \sqrt{q_2(G)}, \ldots, \pm \sqrt{q_n(G)}$ and $\{0\}^{m-n}$ are all eigenvalues of $S(G)$.

![Figure 1. Two graphs $G$ and subdivision $S(G)$.](image)

Indulal in [13] defined two classes of join graphs as follows:

**Definition 1.1** The $S_{\text{vertex}}$ join of two graphs $G_1$ and $G_2$ denoted by $G_1 \check{\vee} G_2$ is obtained from $S(G_1)$ and $G_2$ by joining all vertices of $G_1$ with all vertices of $G_2$, see Figure 2.
Definition 1.2 The $S_{edge}$ join of two graphs $G_1$ and $G_2$ denoted by $G_1 \sqcup G_2$ is obtained from $S(G_1)$ and $G_2$ by joining all vertices of $S(G_1)$ corresponding to the edges of $G_1$ with all vertices of $G_2$, see Figure 3.

The following results are crucial throughout this paper.

Lemma 1.3 [7] Let $G$ be a $r$-regular $(n, m)$-graph with adjacency matrix $A(G)$ and incidence matrix $R$. Let $L(G)$ be its line graph. Then

$$RR^T = A(G) + rI$$

and

$$R^T R = A(L(G)) + 2I.$$ 

Moreover, if $G$ has an eigenvalue equals to $-r$ with an eigenvector $V$, then $G$ is bipartite and $R^T V = 0$.

Lemma 1.4 [7] Let $G$ be an $r$-regular $(n, m)$-graph with eigenvalues $r, \lambda_2, \ldots, \lambda_n$. Then $\{2r - 2^1, [\lambda_i + r - 2]^1, [-2]^{m-n}\}, (2 \leq i \leq n)$ consist the spectrum of line graph. Also $V$ is an eigenvector belonging to the eigenvalue $-2$ if and only if $RV = 0$.

2. The normalized Laplacian spectra of two new join graphs

In this section, we determine the normalized Laplacian spectrum of $G_1 \hat{\sqcup} G_2$ and $G_1 \triangledown G_2$, where $G_i$ is $r_i$-regular ($i = 1, 2$). Consider two graphs $G_1, G_2$ with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$, $E(G_1) = \{e_1, e_2, \ldots, e_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Then $I_k$ and $0_k$ are two vectors of order $k$ with all elements equal to 1 and 0, respectively. Moreover $0_{i \times j}$ denotes an $i \times j$ matrix whose entries are zero, $J_{i \times j}$ is one whose entries are 1 and $I_n$ is the indentity matrix of order $n$.

Theorem 2.1 For $i = 1, 2$, let $G_i$ be $r_i$-regular graph with $n_i$ vertices with incidence matrix $R$ and eigenvalues $r_i = \lambda_1(G_i) \geq \lambda_2(G_i) \geq \cdots \geq \lambda_n(G_i)$. Then the normalized
Laplacian spectrum of \( G_1 \sqrt{V}G_2 \) is \( \{(0)^1, [1 \pm \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}}]^1, [1 - \frac{\lambda_j(G_2)}{n_1 + r_2}]^1, [1]^m_{i - n_1}\}, (2 \leq j \leq n_1), (2 \leq k \leq n_2) \) together with the roots of

\[
x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2} x + \frac{2n_1 n_2 + 2r_1 n_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = 0.
\]

**Proof.** Let \( G_i \) be an \( r_i \)-regular graph with \( n_i \) vertices and an adjacency matrix \( A(G_i) \), \( i = 1, 2 \). Let \( R \) be the incidence matrix of \( G_1 \). Then by a proper labeling of vertices, the normalized Laplacian matrix of \( G_1 \sqrt{V}G_2 \) can be written as

\[
\tilde{L}(G_1 \sqrt{V}G_2) = D^{-\frac{1}{2}} L(G_1 \sqrt{V}G_2)D^{-\frac{1}{2}}
\]

where

\[
D = \begin{bmatrix}
(n_2 + r_1)I_{n_1} & 0_{n_1 \times n_2} & 0_{n_1 \times n_2} \\
0_{n_2 \times n_1} & 2I_{m_1} & 0_{m_1 \times n_2} \\
0_{n_2 \times n_1} & 0_{n_2 \times n_1} & (n_1 + r_2)I_{n_2}
\end{bmatrix},
\]

and

\[
L(G_1 \sqrt{V}G_2) = \begin{bmatrix}
(n_2 + r_1)I_{n_1} & -R_{n_1 \times n_2} & -J_{n_1 \times n_2} \\
R_{n_2 \times n_1} & 2I_{m_1} & 0_{m_1 \times n_2} \\
-J_{n_2 \times n_1} & 0_{n_2 \times n_1} & (n_1 + r_2)I_{n_2} - A(G)
\end{bmatrix}.
\]

\( \mathbf{1}_{n_1} \) is an eigenvector corresponding to eigenvalue \( r_1 \) of \( G \) and the other eigenvectors are orthogonal to \( \mathbf{1}_{n_1} \). Let \( X \) be an eigenvector of \( G_1 \) corresponding to an eigenvalue \( \lambda_j(G_1) \neq r_1, 2 \leq j \leq n_1 \). Then \( \Phi = [\alpha X \; \mathbb{R}^T X \; 0]^T \) is eigenvector of \( \tilde{L}(G_1 \sqrt{V}G_2) \) corresponding to the normalized eigenvalue \( \delta \) of graph \( G_1 \sqrt{V}G_2 \). Then \( \tilde{L} \Phi = \theta \Phi \). Thus

\[
\begin{bmatrix}
\mathbf{1}_{n_1} & -R_{n_1 \times n_2} & -J_{n_1 \times n_2} \\
\frac{R_{n_1 \times n_1}}{\sqrt{2(n_2 + r_1)}} & I_{m_1} & 0_{m_1 \times n_2} \\
\frac{-J_{n_2 \times n_1}}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} & 0_{n_2 \times m_1} & I_{n_2} - \frac{A(G)}{n_1 + r_2}
\end{bmatrix}
\begin{bmatrix}
\alpha X \\
\mathbb{R}^T X \\
0
\end{bmatrix} = \theta \begin{bmatrix}
\alpha X \\
\mathbb{R}^T X \\
0
\end{bmatrix}.
\]

This implies that

\[
\begin{bmatrix}
\frac{(\alpha - \frac{R \mathbb{R}^T}{\sqrt{2(n_2 + r_1)}})X}{\sqrt{2(n_2 + r_1)}} \\
(1 - \frac{\alpha}{\sqrt{2(n_2 + r_1)}}) \mathbb{R}^T X \\
0
\end{bmatrix} = \begin{bmatrix}
\theta \alpha X \\
\theta \mathbb{R}^T X \\
0
\end{bmatrix}.
\]

The last equality follows from Lemma 1.3. By solving these equations, we get \( \alpha = \ldots \).
Therefore, \( \Phi_k = [\pm \sqrt{\lambda_j(G_1)} + r_1 X R^T X 0]^T \) is an eigenvector of \( \tilde{L}(G_1 \vee G_2) \) corresponding to the normalized eigenvalues \( \theta_k \) of graph \( G_1 \vee G_2 \), where \( \theta_k = 1 \pm \frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)} \) \( (k = 1, 2) \).

Now consider the \( n_1 - n_2 \) linearly independent eigenvectors \( Z_l \) \( (1 \leq l \leq m_1 - n_1 \) of \( L(G_1) \) corresponding to the eigenvalue \(-2\). Then by Lemma 1.4, \( R Z_l = 0 \). Consequently \( \Omega_l = [0 \ 0 \ 0 \ 0]^T \) is an eigenvector of \( \tilde{L}(G_1 \vee G_2) \) with an eigenvalue 1, since

\[
\begin{bmatrix}
I_{n_1} & -R_{n_1 \times n_2} & -J_{n_1 \times n_2} & 0_{n_1 \times n_2} \\
0_{n_2 \times n_1} & \frac{1}{\sqrt{2(n_2 + r_1)}} & 0_{n_2 \times n_2} & \frac{1}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} \\
0_{n_2 \times n_1} & 0_{n_2 \times n_2} & I_{n_2} - \frac{A(G_2)}{n_1 + r_2} & 0_{n_2 \times n_2}
\end{bmatrix}
= \begin{bmatrix}
0 & Z_l \\
Z_l & 0
\end{bmatrix}.
\]

On the other hand, \( 1_{n_2} \) is an eigenvector of \( G_2 \) corresponding to the eigenvalue \( r_2 \), in which all other eigenvectors are orthogonal to \( 1_{n_2} \). Let \( Y \) be the eigenvector of \( G_2 \) corresponding to the eigenvalue \( \lambda_j(G_2) \neq r_2 \) \( (2 \leq j \leq n_2) \). By a similar arguments we can deduce that \( \Psi = [0 \ 0 \ Y]^T \) is eigenvector of \( \tilde{L}(G_1 \vee G_2) \) corresponding to the eigenvalue \( 1 - \frac{\lambda_j(G_2)}{n_1 + r_2} \) \( (2 \leq j \leq n_2) \). Let \( \Lambda = [\alpha J_{n_1 \times 1} \beta J_{n_1 \times 1} \gamma J_{n_1 \times 1}]^T \) for some vector \( (\alpha, \beta, \gamma) \neq (0, 0, 0) \). The equation \( \tilde{L}(G_1 \vee G_2) \Lambda = x \Lambda \) yields that \( \Lambda \) is an eigenvector of \( \mathcal{A} \) corresponding to eigenvalue \( x \) if and only if \( \Lambda = [\alpha \beta \gamma]^T \) is an eigenvector of the matrix \( M \), where

\[
M = \begin{bmatrix}
1 & -r_1 \frac{1}{\sqrt{2(n_2 + r_1)}} & -r_2 \frac{1}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} \\
-2 \frac{1}{\sqrt{2(n_2 + r_1)}} & 1 & 0 \\
-n_1 \frac{1}{\sqrt{(n_1 + r_2)(n_2 + r_1)}} & 0 & 1 - \frac{A(G_2)}{n_1 + r_2}
\end{bmatrix}.
\]

The characteristic polynomial of \( M \) is \( x^3 - 3n_1 + 2r_2 x^2 + \frac{2n_1 n_2 + 2r_1 n_1 + n_2 - r_2}{(n_1 + r_2)(n_2 + r_1)} x = 0 \) and this completes the proof.

**Corollary 2.2** If \( G_2 \cong K_{n_2} \), then the normalized Laplacian spectrum of \( G_1 \vee G_2 \) is \( \{0\}^1, [1]^{m_1 - n_1 + n_2}, [1 + \sqrt{\lambda_j(G_1) + r_1}]^1, [2]^1 \}, \) \( (2 \leq j \leq n_1) \).

**Theorem 2.3** For \( i = 1, 2 \), let \( G_i \) be an \( r_i \)-regular graph on \( n_i \) vertices with incidence matrix \( R \) and the eigenvalues \( r_i = \lambda_1(G_i) \geq \lambda_2(G_i) \geq \cdots \geq \lambda_{n_i}(G_i) \). Then the normalized Laplacian spectrum of \( G_1 \vee G_2 \) is \( \{0\}^1, \frac{1}{2} \left( \lambda_j(G_1) + r_1 \right) \frac{n_1 + n_2}{n_1 + r_2}, 1 - \frac{\lambda_j(G_2)}{m_1 + r_2}, [1]^{m_1 - n_1} \}, \) \( (2 \leq j \leq n_1) \), \( (2 \leq k \leq n_2) \), together with the roots of \( x^2 - 3n_1 + 2r_2 x + \frac{2n_1 n_2 + 4n_1 + n_2 - r_2}{(n_1 + r_2)(n_2 + r_1)} = 0 \).

**Proof.** It is not difficult to see that the normalized Laplacian matrix of \( G_1 \vee G_2 \) can be written as follows:

\[
\tilde{L}(G_1 \vee G_2) = D^{-\frac{1}{2}} L(G_1 \vee G_2) D^{-\frac{1}{2}}
= \begin{bmatrix}
I_{n_1} & -R_{n_1 \times n_2} & -J_{n_1 \times n_2} & 0_{n_1 \times n_2} \\
0_{n_2 \times n_1} & \frac{1}{\sqrt{r_1(n_2 + 2)}} & \frac{1}{\sqrt{(n_1 + r_2)(n_2 + 2)}} & 0_{n_2 \times n_2} \\
0_{n_2 \times n_1} & -J_{n_2 \times n_1} & I_{n_2} - \frac{A(G_2)}{r_1} & 0_{n_2 \times n_2}
\end{bmatrix},
\]
where

\[
D = \begin{pmatrix}
(n_2 + 2)I_{n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times n_2} \\
0_{m_1 \times n_1} & r_1I_{m_1} & 0_{m_1 \times n_2} \\
0_{m_2 \times n_1} & 0_{n_2 \times m_1} & (m_1 + r_2)I_{n_2}
\end{pmatrix},
\]

and

\[
\mathcal{L}(G_1 \vee G_2) = \begin{pmatrix}
(n_2 + 2)I_{n_1} - \mathcal{R}_{m_1 \times n_1} & 0_{n_1 \times n_2} \\
\mathcal{R}_{m_2 \times n_1} & r_1I_{m_1} - \mathcal{J}_{m_1 \times n_2} \\
0_{n_2 \times n_1} & -\mathcal{J}_{n_2 \times m_1} (m_1 + r_2)I_{n_2} - \mathcal{A}(G)
\end{pmatrix}.
\]

Continuing of the proof is similar to that of Theorem 2.1.

Corollary 2.4 If \(G_2 \cong \overline{K}_{n_2}\), then the normalized Laplacian spectrum of \(G_1 \vee G_2\) is \([0]^1, [1]^{m_1-n_1+n_2}, [1 \pm \sqrt{\frac{\lambda_j(G_1)+r_2}{r_1(n_2+2)}}]^1, [2]^1\), \(2 \leq j \leq n_1\).

3. Integrality of two new join graphs

The graph \(G\) is \(\mathcal{L}\)-integral if all its \(\mathcal{L}\)-eigenvalues are integral. Since all \(\mathcal{L}\)-eigenvalues of \(G\) are in interval \([0, 2]\), we can deduced that \(G\) is \(\mathcal{L}\)-integral if and only if its eigenvalues are 0, 1, or 2. The complete bipartite graphs are such graphs; in fact, no other connected graphs are \(\mathcal{L}\)-integral, see [14].

Proposition 3.1 [14] Let \(G\) be a connected graph. Then the following statements are equivalent.

1) \(G\) is bipartite with three \(\mathcal{L}\)-eigenvalues,
2) \(G\) is \(\mathcal{L}\)-integral,
3) \(G\) is complete bipartite.

The following propositions give a necessary and sufficient conditions in which \(S_{\text{vertex}}\) and \(S_{\text{edge}}\) joins of graphs are \(\mathcal{L}\)-integral.

Proposition 3.2 If \(G_i\) is \(r_i\)-regular connected graph \((i = 1, 2)\), then \(G_1 \vee G_2\) and \(G_1 \vee G_2\) are \(\mathcal{L}\)-integral if and only if \(G_1 \cong K_2\) and \(G_2 \cong \overline{K}_{n_2}\).

Proof.

Since \(K_2 \vee \overline{K}_{n_2} = K_{2,n_2}\) and \(K_2 \vee \overline{K}_{n_2} = K_{1,n_2+2}\) are two graphs of order \(n\) with the normalized Laplacian eigenvalues \([0]^1, [1]^{n-1}, [2]^1\), these graphs are \(\mathcal{L}\)-integral. Conversely, suppose that \(G_1 \vee G_2\) is \(\mathcal{L}\)-integral, \(G_1 \not\cong K_2\) and \(G_2 \not\cong \overline{K}_{n_2}\). By Theorem 2.1, the normalized Laplacian spectrum of \(G_1 \vee G_2\) is \([0]^1, [1 \pm \sqrt{\frac{\lambda_j(G_1)+r_2}{r_1(n_2+2)}}]^1, [1 - \frac{\lambda_j(G_2)}{n_1+r_2}]^1, [1]^1, \), \(2 < j < n_1\), \(2 < k < n_2\) together with the roots of

\[
x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2} x + \frac{2n_1 n_2 + 2r_1 n_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = 0.
\]

Since \(G_1\) is connected graph, \(G_1 \vee G_2\) is connected. Hence, \([0]^1\) is an eigenvalue of \(G_1 \vee G_2\). Thus, the other eigenvalues must be equal to \([1]^{n-1}, [2]^1\). Since \(1 - \frac{\lambda_j(G_2)}{n_1+r_2} < 2, 2 < j < n_2\), we have \(\frac{\lambda_j(G_2)}{n_1+r_2} = 0\). So, \(\lambda_j(G_2) = 0, 2 < j < n_2\) and it implies that \(G_2 \not\cong \overline{K}_{n_2}\).
a contradiction. Since 1 − \( \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}} < 2 \), 1 \( \leq j \leq n_1 \), we achieve \( \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}} = 0 \) and thus \( \lambda_j(G_1) = -r_1, 2 \leq j \leq n_1 \), a contradiction. On the other hand, 1 < 1 + \( \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}} \) \( \leq 2 \), 2 \( \leq j \leq n_1 \), yields that \( \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}} = 1 \). So, \( \lambda_j(G_1) = -r_1 + 2n_2 > 0 \), 2 \( \leq j \leq n_1 \), a contradiction. As well as, for two eigenvalues that are the roots of

\[
x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2} x + \frac{2n_1 n_2 + 2r_1 n_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = 0.
\]

Since, \( G \) is connected and the normalized Laplacian eigenvalues are integral, one can deduce that Eq. (2) is equal to \((x - 1)^2 = x^2 - 2x + 1\) or \((x - 1)(x - 2) = x^2 - 3x + 2\). In the first case, one can conclude easily that \( \frac{2n_1 n_2 + 2r_1 n_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = 1 \), which yields that \( n_1 n_2 = r_2 - n_1 \), for \( r_1 \geq 1 \) which comes to a contradiction, because for example, there is no an 8-regular graph of order 5. Thus, the second case holds, namely

\[
x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2} x + \frac{2n_1 n_2 + 2r_1 n_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = x^2 - 3x + 2
\]

and the proof is completed. 

**Proposition 3.3** Let \( G_1 \) be an empty graph of order \( n_1 \) and \( G_2 \) be \( r \)-regular graph of order \( n_2 \), then \( G_1 \dot{\cup} G_2 \) and \( G_1 \uplus G_2 \) are \( \mathcal{L} \)-integral if and only if \( G_2 \cong \overline{K}_{n_2} \) or \( G_2 \) is complete bipartite graph.

**Proof.** By using Proposition 3.1, Theorems 2.1 and 2.3, the proof is stright forward.

The normalized Laplacian energy of \( G \) is defined as \( \mathcal{LE}(G) = \sum_{i=1}^{n} |\delta_i(G) - 1| \), see [4, 12] for details.

**Proposition 3.4** Let \( G_1 \) is an \( r_1 \)-regular of graph of order \( n_1 \) and \( G_2 \cong \overline{K}_{n_2} \). Then

1) \( \mathcal{LE}(G_1 \dot{\cup} G_2) = 2 \left( 1 + \frac{\mathcal{RE}(G_1) - \sqrt{2r_1}}{\sqrt{2(n_2 + r_1)}} \right) \),

2) \( \mathcal{LE}(G_1 \uplus G_2) = 2 \left( 1 + \frac{\mathcal{RE}(G_1) - \sqrt{2r_1}}{\sqrt{r_1(n_2 + 2)}} \right) \).

**Proof.** By Corollary 2.2, Corollary 2.4, respectively, we have

\[
\mathcal{LE}(G_1 \dot{\cup} G_2) = \sum_{i=1}^{n} |\delta_i(G_1 \dot{\cup} G_2) - 1|
\]

\[
= 1 + \sum_{j=2}^{n_1} \pm \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}} + 1
\]

\[
= 2 + 2 \sum_{j=2}^{n_1} \sqrt{\frac{\lambda_j(G_1) + r_1}{2(n_2 + r_1)}}
\]

\[
= 2 \left( 1 + \frac{\mathcal{RE}(G_1) - \sqrt{2r_1}}{\sqrt{r_1(n_2 + 2)}} \right).
\]
\[ \mathcal{LE}(G_1 \uplus G_2) = \sum_{i=1}^{\infty} |\delta_i(G_1 \uplus G_2) - 1| \]
\[ = 1 + \sum_{j=2}^{n_1} \left| \pm \frac{\lambda_j(G_1) + r_1}{r_1(n_2 + 2)} \right| + 1 \]
\[ = 2 + 2 \sum_{j=2}^{n_1} \frac{\lambda_j(G_1) + r_1}{r_1(n_2 + 2)} \]
\[ = 2 \left( 1 + \frac{\mathcal{RE}(G_1) - \sqrt{2}r_1}{\sqrt{r_1(n_2 + 2)}} \right), \]

where the last equality follows from \( \mathcal{RE}(G_1) - \sqrt{2}r_1 = \sum_{j=2}^{n_1} \sqrt{\lambda_j(G_1) + r_1}. \)

The degree Kirchhoff index of graph \( G \) is defined by Chen et al. in [5] as
\[ Kf^*(G) = \sum_{i<j} d_i r_{ij} d_j, \]

where \( r_{ij} \) denotes the resistance-distance between vertices \( v_i \) and \( v_j \) in a graph \( G \). They proved that \( Kf^*(G) = 2m \sum_{i=2}^{n} \frac{1}{\delta_i(G)} \), where \( \delta_i(G) \) is normalized Laplacian eigenvalues of \( G \) of order \( n (2 \leq i \leq n) \), see [11, 18] for details.

**Proposition 3.5** Let \( G_1 \) is an \( r_1 \)-regular graph of order \( n_1 \) with \( m_1 \) edges and \( G_2 \cong K_{n_2} \). Then
\[ Kf^*(G_1 \uplus G_2) = 2(2m_1 + n_1 n_2) \left( \frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1 - 1)\sqrt{2(n_2 + r_1)}}{(n_1 - 1)\sqrt{2(n_2 + r_1)} + (\mathcal{RE}(G_1) - \sqrt{2}r_1)} \right), \]
\[ Kf^*(G_1 \uplus G_2) = 2m_1 (2 + n_2) \left( \frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1 - 1)\sqrt{2(n_2 + 2)}}{(n_1 - 1)\sqrt{2(n_2 + 2)} + (\mathcal{RE}(G_1) - \sqrt{2}r_1)} \right). \]

**Proof.** It is not difficult to see that the number of edges of \( G_1 \uplus G_2 \) and \( G_1 \uplus G_2 \) are \( E(G_1 \uplus G_2) = 2m_1 + n_1 n_2 \) and \( E(G_1 \uplus G_2) = m_1 (2 + n_2) \), respectively. On the other hand, since \( G_1 \) is \( r_1 \)-regular by using Eq.(1),
\[ \mathcal{RE}(G_1) = \sum_{j=1}^{n_1} \sqrt{\lambda_j(G_1) + r_1} \]

and so
\[ \mathcal{RE}(G_1) - \sqrt{2}r_1 = \sum_{j=2}^{n_1} \sqrt{\lambda_j(G_1) + r_1}. \]

Now, Corollaries 2.2, 2.4 complete the proof. \( \blacksquare \)
References