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Normalized Laplacian spectrum of two new types of join graphs

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Abstract. Let G be a graph without an isolated vertex, the normalized Laplacian matrix $\tilde{\mathcal{L}}(G)$ is defined as $\tilde{\mathcal{L}}(G) = \mathcal{D}^{-\frac{1}{2}}\mathcal{L}(G)\mathcal{D}^{-\frac{1}{2}}$, where $\mathcal{D}$ is a diagonal matrix whose entries are degree of vertices of G. The eigenvalues of $\tilde{\mathcal{L}}(G)$ are called as the normalized Laplacian eigenvalues of G. In this paper, we obtain the normalized Laplacian spectrum of two new types of join graphs. In continuing, we determine the integrality of normalized Laplacian eigenvalues of graphs. Finally, the normalized Laplacian energy and degree Kirchhoff index of these new graph products are derived.

Keywords: Join of graphs, normalized Laplacian eigenvalue, integral eigenvalue.

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1. Introduction

Let $G = (V, E)$ be a simple graph (namely a graph without loop and multiple edges) on n vertices and m edges and the degree of vertex $v \in V(G)$ is denoted by $deg(v)$. The eigenvalues of the adjacency matrix $\mathcal{A} = \mathcal{A}(G)$ of G are called the eigenvalues of G denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_s(G)$ be the distinct eigenvalues of G with multiplicity $t_1, t_2, \ldots, t_s$, respectively. The multiset $\{[\lambda_1(G)]^{t_1}, [\lambda_2(G)]^{t_2}, \ldots, [\lambda_s(G)]^{t_s}\}$ of eigenvalues of $\mathcal{A}$ is called the spectrum of G.

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The energy of graph G is a graph invariant introduced by Ivan Gutman as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

see for more details [8, 10, 17]. Let $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ be the Laplacian matrix of graph G, where $\mathcal{D}(G) = [d_{ij}]$ is the diagonal matrix whose entries are degree of vertices, namely, $d_{ii} = deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$. The matrix $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is called the signless Laplacian matrix of G. The signless Laplacian eigenvalues of G are denoted by $q_1(G), q_2(G), \ldots, q_n(G)$. If G is regular of degree r, then the eigenvalues of $\mathcal{Q}(G)$ are $2r, r + \lambda_2, \ldots, r + \lambda_n$, see [1, p.14]. In [15], Jooyandeh et al. introduced the concept of incidence energy of a graph as

$$\mathcal{RE}(G) = \sum_{i=1}^{n} \sqrt{q_i(G)}. \tag{1}$$

Let G be a graph without an isolated vertex, the normalized Laplacian matrix $\tilde{\mathcal{L}}(G)$ is defined as $\tilde{\mathcal{L}}(G) = \mathcal{D}^{-\frac{1}{2}}\mathcal{L}(G)\mathcal{D}^{-\frac{1}{2}}$ which implies that its $(i,j)-$entry is 1 if $i = j$, and it is $-1/\sqrt{deg(v_i)deg(v_j)}$, if $i \neq j$ and the vertices v_i, v_j are adjacent, and is zero otherwise. The eigenvalues of $\tilde{\mathcal{L}}(G)$ are the roots of $g(x) = det(\delta I - \tilde{\mathcal{L}}(G))$ and we call them as normalized Laplacian eigenvalues of G, denoted by $\delta_1(G), \delta_2(G), \ldots, \delta_n(G)$, where $\delta_n(G) = 0$ for all graphs, see for more details [2, 3, 6].

The complement of graph G is denoted by $\overline{G}$ and a complete graph on n vertices is denoted by K_n. Also a cycle graph with n vertices is denoted by C_n. Clearly, $\overline{K_n}$ is empty graph. The subdivision graph $S(G)$ of a graph G is obtained by inserting an additional vertex in the middle of each edge of G. Equivalently, each edge of G is replaced by a path of length 2, see Figure 1. It is a well-known fact that $P_{S(G)}(x) = x^{m-n}\mathcal{Q}_G(x^2)$, where $P_G(x)$ and $\mathcal{Q}_G(x)$ denote the characteristic polynomial and the signless Laplacian characteristic polynomial of G, respectively [7, p. 63]. From the definition, one can see that $\pm\sqrt{q_1(G)}, \pm\sqrt{q_2(G)}, \ldots, \pm\sqrt{q_n(G)}$ and $[0]^{m-n}$ are all eigenvalues of $S(G)$.

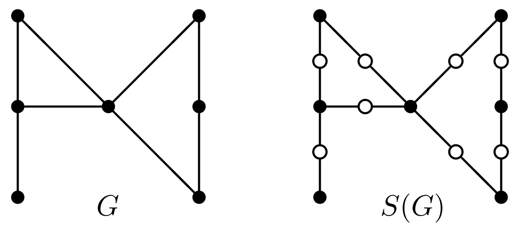

Figure 1. Two graphs G and subdivision $S(G)$.

Indulal in [13] defined two classes of join graphs as follows:

Definition 1.1 The S_{vertex} join of two graphs G_1 and G_2 denoted by $G_1 \dot{\vee} G_2$ is obtained from $S(G_1)$ and G_2 by joining all vertices of G_1 with all vertices of G_2, see Figure 2.

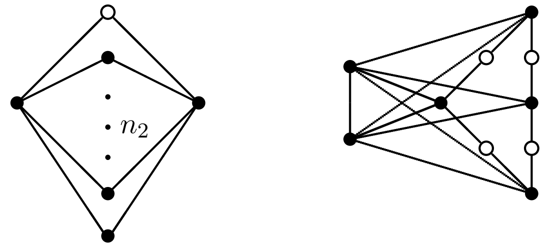

Figure 2. Graphs $K_2\dot{\vee}\overline{K}_{n_2}$ and $C_4\dot{\vee}K_2$.

Definition 1.2 The S_{edge} join of two graphs G_1 and G_2 denoted by $G_1 \veebar G_2$ is obtained from $S(G_1)$ and G_2 by joining all vertices of $S(G_1)$ corresponding to the edges of G_1 with all vertices of G_2, see Figure 3.

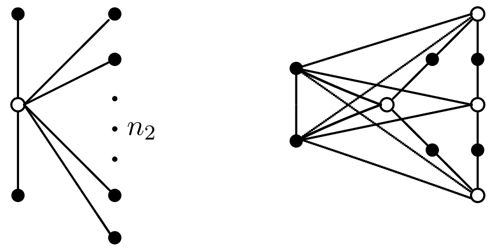

Figure 3. Graphs $K_2 \veebar \overline{K}_{n_2}$ and $C_4 \veebar K_2$.

The following results are crucial throughout this paper.

Lemma 1.3 [7] Let G be a $r-$regular $(n, m)-$graph with adjacency matrix $\mathcal{A}(G)$ and incidence matrix $\mathcal{R}$. Let $L(G)$ be its line graph. Then

$$\mathcal{R}\mathcal{R}^T = \mathcal{A}(G) + rI \quad \text{and} \quad \mathcal{R}^T\mathcal{R} = \mathcal{A}(\mathcal{L}(G)) + 2I.$$

Moreover, if G has an eigenvalue equals to $-r$ with an eigenvector $\mathcal{V}$, then G is bipartite and $\mathcal{R}^T\mathcal{V} = 0$.

Lemma 1.4 [7] Let G be an $r-$regular $(n, m)-$graph with eigenvalues $r, \lambda_2, \ldots, \lambda_n$. Then $\{[2r-2]^1, [\lambda_i + r - 2]^1, [-2]^{m-n}\}$, $(2 \leqslant i \leqslant n)$ consist the spectrum of line grah. Also $\mathcal{V}$ is an eigenvector belonging to the eigenvalue -2 if and only if $\mathcal{R}\mathcal{V} = 0$.

2. The normalized Laplacian spectra of two new join graphs

In this section, we determine the normalized Laplacian spectrum of $G_1\dot{\vee}G_2$ and $G_1 \veebar G_2$, where G_i is r_i-regular $(i = 1, 2)$. Consider two graphs G_1, G_2 with $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Then $\mathbf{1}_k$ and $\mathbf{0}_k$ are two vectors of order k with all elements equal to 1 and 0, respectively. Moreover $\mathbf{0}_{i\times j}$ denotes an $i \times j$ matrix whose entries are zero, $\mathbf{J}_{i\times j}$ is one whose entries are 1 and $\mathbf{I}_n$ is the identity matrix of order n.

Theorem 2.1 For $i = 1, 2$, let G_i be r_i-regular graph with n_i vertices with incidence matrix $\mathcal{R}$ and eigenvalues $r_i = \lambda_1(G_i) \geqslant \lambda_2(G_i) \geqslant \cdots \geqslant \lambda_n(G_i)$. Then the normalized

Laplacian spectrum of $G_1\dot{\vee}G_2$ is $\{[0]^1, \big[1 \pm \sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}\big]^1, \big[1-\frac{\lambda_k(G_2)}{n_1+r_2}\big]^1, [1]^{m_1-n_1}\}$, $(2 \leqslant j \leqslant n_1)$, $(2 \leqslant k \leqslant n_2)$ together with the roots of

$$x^2 - \frac{3n_1+2r_2}{n_1+r_2}x + \frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)} = 0.$$

Proof. Let G_i be an r_i-regular graph with n_i vertices and an adjacency matrix $\mathcal{A}(G_i)$, $i=1,2$. Let $\mathcal{R}$ be the incidence matrix of G_1. Then by a proper labeling of vertices, the normalized Laplacian matrix of $G_1\dot{\vee}G_2$ can be written as

$$\begin{aligned}\tilde{\mathcal{L}}(G_1\dot{\vee}G_2) &= \mathcal{D}^{-\frac{1}{2}}\mathcal{L}(G_1\dot{\vee}G_2)\mathcal{D}^{-\frac{1}{2}}\\ &= \begin{bmatrix} \mathbf{I}_{n_1} & \frac{-\mathcal{R}_{n_1\times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{-\mathbf{J}_{n_1\times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-\mathcal{R}^T_{m_1\times n_1}}{\sqrt{2(n_2+r_1)}} & \mathbf{I}_{m_1} & \mathbf{0}_{m_1\times n_2} \\ \frac{-\mathbf{J}_{n_2\times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & \mathbf{0}_{n_2\times m_1} & \mathbf{I}_{n_2} - \frac{\mathcal{A}(G_2)}{n_1+r_2} \end{bmatrix},\end{aligned}$$

where

$$\mathcal{D} = \begin{bmatrix} (n_2+r_1)\mathbf{I}_{n_1} & \mathbf{0}_{n_1\times m_1} & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{m_1\times n_1} & 2\mathbf{I}_{m_1} & \mathbf{0}_{m_1\times n_2} \\ \mathbf{0}_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1} & (n_1+r_2)\mathbf{I}_{n_2} \end{bmatrix},$$

and

$$\mathcal{L}(G_1\dot{\vee}G_2) = \begin{bmatrix} (n_2+r_1)\mathbf{I}_{n_1} & -\mathcal{R}_{n_1\times m_1} & -\mathbf{J}_{n_1\times n_2} \\ \mathcal{R}^T_{m_1\times n_1} & 2I_{m_1} & \mathbf{0}_{m_1\times n_2} \\ -\mathbf{J}_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1} & (n_1+r_2)\mathbf{I}_{n_2} - \mathcal{A}(G) \end{bmatrix}.$$

$\mathbf{1}_{n_1}$ is an eigenvector corresponding to eigenvalue r_1 of G and the other eigenvectors are orthogonal to $\mathbf{1}_{n_1}$. Let X be an eigenvector of G_1 corresponding to an eigenvalue $\lambda_j(G_1) \neq r_1$, $2 \leqslant j \leqslant n_1$. Then $\Phi = [\alpha X\ \mathcal{R}^T X\ 0]^T$ is eigenvector of $\tilde{\mathcal{L}}(G_1\dot{\vee}G_2)$ corresponding to the normalized eigenvalue δ of graph $G_1\dot{\vee}G_2$. Then $\tilde{\mathcal{L}}.\Phi = \theta\Phi$. Thus

$$\begin{bmatrix} \mathbf{I}_{n_1} & \frac{-\mathcal{R}_{n_1\times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{-\mathbf{J}_{n_1\times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-\mathcal{R}^T_{m_1\times n_1}}{\sqrt{2(n_2+r_1)}} & \mathbf{I}_{m_1} & \mathbf{0}_{m_1\times n_2} \\ \frac{-\mathbf{J}_{n_2\times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & \mathbf{0}_{n_2\times m_1} & \mathbf{I}_{n_2} - \frac{\mathcal{A}(G_2)}{n_1+r_2} \end{bmatrix} \begin{bmatrix} \alpha X \\ \mathcal{R}^T X \\ 0 \end{bmatrix} = \theta \begin{bmatrix} \alpha X \\ \mathcal{R}^T X \\ 0 \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} (\alpha - \frac{\mathcal{R}\mathcal{R}^T}{\sqrt{2(n_2+r_1)}})X \\ (1-\frac{\alpha}{\sqrt{2(n_2+r_1)}})\mathcal{R}^T X \\ 0 \end{bmatrix} = \begin{bmatrix} \theta\alpha X \\ \theta\mathcal{R}^T X \\ 0 \end{bmatrix}.$$

The last equality follows from Lemma 1.3. By solving these equations, we get $\alpha =$

$\pm\sqrt{\lambda_j(G_1)+r_1}$, $2 \leqslant j \leqslant n_1$. Therefore, $\Phi_k = [\pm\sqrt{\lambda_j(G_1)+r_1}X\ \mathcal{R}^T X\ 0]^T$ is an eigenvector of $\tilde{\mathcal{L}}(G_1\dot{\vee}G_2)$ corresponding to the normalized eigenvalues θ_k of graph $G_1\dot{\vee}G_2$, where $\theta_k = 1 \pm \sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}$ $(k = 1,2)$.

Now consider the $m_1 - n_1$ linearly independent eigenvectors $\mathcal{Z}_l$ $(1 \leqslant l \leqslant m_1 - n_1)$ of $L(G_1)$ corresponding to the eigenvalue -2. Then by Lemma 1.4, $\mathcal{R}\mathcal{Z}_l = 0$. Consequently $\Omega_l = [0\ \mathcal{Z}_l\ 0]^T$ is an eigenvector of $\tilde{\mathcal{L}}(G_1\dot{\vee}G_2)$ with an eigenvalue 1, since

$$\tilde{\mathcal{L}}.\Omega = \begin{bmatrix} \mathbf{I}_{n_1} & \frac{-\mathcal{R}_{n_1\times m_1}}{\sqrt{2(n_2+r_1)}} & \frac{-\mathbf{J}_{n_1\times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-\mathcal{R}^T_{m_1\times n_2}}{\sqrt{2(n_2+r_1)}} & \mathbf{I}_{m_1} & \mathbf{0}_{m_1\times n_2} \\ \frac{-\mathbf{J}_{n_2\times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & \mathbf{0}_{n_2\times m_1} & \mathbf{I}_{n_2} - \frac{\mathcal{A}(G_2)}{n_1+r_2} \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{Z}_l \\ 0 \end{bmatrix} = \mathbf{I}_{n_1\times m_1} \begin{bmatrix} 0 \\ \mathcal{Z}_l \\ 0 \end{bmatrix}.$$

On the other hand, 1_{n_2} is an eigenvector of G_2 corresponding to the eigenvalue r_2, in which all other eigenvectors are orthogonal to 1_{n_2}. Let Y be the eigenvector of G_2 corresponding to the eigenvalue $\lambda_j(G_2) \neq r_2$ $(2 \leqslant j \leqslant n_2)$. By a similar arguments we can deduce that $\Psi = [0\ 0\ Y]^T$ is eigenvector of $\tilde{\mathcal{L}}(G_1\dot{\vee}G_2)$ corresponding to the eigenvalue $1 - \frac{\lambda_j(G_2)}{n_1+r_2}$ $(2 \leqslant j \leqslant n_2)$. Let $\Lambda = [\alpha\mathbf{J}_{n_1\times 1}\ \beta\mathbf{J}_{n_1\times 1}\ \gamma\mathbf{J}_{n_1\times 1}]^T$ for some vector $(\alpha,\beta,\gamma) \neq (0,0,0)$. The equation $\tilde{\mathcal{L}}(G_1\dot{\vee}G_2).\Lambda = x\Lambda$ yields that Λ is an eigenvector of $\mathcal{A}$ corresponding to eigenvalue x if and only if $\Lambda = [\alpha\ \beta\ \gamma]^T$ is an eigenvector of the matrix M, where

$$M = \begin{bmatrix} 1 & \frac{-r_1}{\sqrt{2(n_2+r_1)}} & \frac{-n_2}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \frac{-2}{\sqrt{2(n_2+r_1)}} & 1 & 0 \\ \frac{-n_1}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0 & 1 - \frac{\mathcal{A}(G_2)}{n_1+r_2} \end{bmatrix}.$$

The characteristic polynomial of M is $x^3 - \frac{3n_1+2r_2}{n_1+r_2}x^2 + \frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}x = 0$ and this completes the proof. ■

Corollary 2.2 If $G_2 \cong \overline{K}_{n_2}$ then the normalized Laplacian spectrum of $G_1\dot{\vee}G_2$ is $\{[0]^1$, $[1]^{m_1-n_1+n_2}$, $\left[1 \pm \sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}\right]^1$, $[2]^1\}$, $(2 \leqslant j \leqslant n_1)$.

Theorem 2.3 For $i = 1,2$, let G_i be an r_i-regular graph on n_i vertices with incidence matrix $\mathcal{R}$ and the eigenvalues $r_i = \lambda_1(G_i) \geqslant \lambda_2(G_i) \geqslant \cdots \geqslant \lambda_n(G_i)$. Then the normalized Laplacian spectrum of $G_1\veebar G_2$ is $\{[0]^1$, $1\pm\sqrt{\frac{\lambda_j(G_1)+r_1}{r_1(n_2+2)}}$, $1-\frac{\lambda_k(G_2)}{m_1+r_2}$, $[1]^{m_1-n_1}\}$, $(2 \leqslant j \leqslant n_1)$, $(2 \leqslant k \leqslant n_2)$, together with the roots of $x^2 - \frac{3m_1+2r_2}{m_1+r_2}x + \frac{2m_1n_2+4m_1+n_2r_2}{(m_1+r_2)(2+n_2)} = 0$.

Proof. It is not difficult to see that the normalized Laplacian matrix of $G_1 \veebar G_2$ can be written as follows:

$$\begin{aligned} \tilde{\mathcal{L}}(G_1 \veebar G_2) &= \mathcal{D}^{-\frac{1}{2}}\mathcal{L}(G_1 \veebar G_2)\mathcal{D}^{-\frac{1}{2}} \\ &= \begin{bmatrix} \mathbf{I}_{n_1} & \frac{-\mathcal{R}_{n_1\times m_1}}{\sqrt{r_1(n_2+2)}} & \mathbf{0}_{n_1\times n_2} \\ \frac{-\mathcal{R}^T_{m_1\times n_1}}{\sqrt{r_1(n_2+2)}} & \mathbf{I}_{m_1} & \frac{-\mathbf{J}_{m_1\times n_2}}{\sqrt{(m_1+r_2)(n_2+2)}} \\ \mathbf{0}_{n_2\times n_1} & \frac{-\mathbf{J}_{n_2\times m_1}}{\sqrt{(m_1+r_2)(n_2+2)}} & \mathbf{I}_{n_2} - \frac{\mathcal{A}(G_2)}{m_1+r_2} \end{bmatrix}, \end{aligned}$$

where

$$\mathcal{D} = \begin{bmatrix} (n_2+2)\mathbf{I}_{n_1} & \mathbf{0}_{n_1\times m_1} & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{m_1\times n_1} & r_1\mathbf{I}_{m_1} & \mathbf{0}_{m_1\times n_2} \\ \mathbf{0}_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1} & (m_1+r_2)\mathbf{I}_{n_2} \end{bmatrix},$$

and

$$\mathcal{L}(G_1 \veebar G_2) = \begin{bmatrix} (n_2+2)\mathbf{I}_{n_1} & -\mathcal{R}_{n_1\times m_1} & \mathbf{0}_{n_1\times n_2} \\ \mathcal{R}^T_{m_1\times n_1} & r_1 I_{m_1} & -\mathbf{J}_{m_1\times n_2} \\ \mathbf{0}_{n_2\times n_1} & -\mathbf{J}_{n_2\times m_1} & (m_1+r_2)\mathbf{I}_{n_2} - \mathcal{A}(G) \end{bmatrix}.$$

Continuing of the proof is similar to that of Theorem 2.1. ■

Corollary 2.4 If $G_2 \cong \overline{K}_{n_2}$, then the normalized Laplacian spectrum of $G_1 \veebar G_2$ is $\{[0]^1$, $[1]^{m_1-n_1+n_2}$, $\left[1 \pm \sqrt{\frac{\lambda_j(G_1)+r_1}{r_1(n_2+2)}}\right]^1$, $[2]^1\}$, $(2 \leqslant j \leqslant n_1)$.

3. Integrality of two new join graphs

The graph G is $\tilde{\mathcal{L}}-$integral if all its $\tilde{\mathcal{L}}-$eigenvalues are integral. Since all $\tilde{\mathcal{L}}-$eigenvalues of G are in interval $[0, 2]$, we can deduced that G is $\tilde{\mathcal{L}}-$integral if and only if its eigenvalues are 0, 1, or 2. The complete bipartite graphs are such graphs; in fact, no other connected graphs are $\tilde{\mathcal{L}}-$integral, see [14].

Proposition 3.1 [14] Let G be a connected graph. Then the following statements are equivalent.

1) G is bipartite with three $\tilde{\mathcal{L}}-$eigenvalues,
2) G is $\tilde{\mathcal{L}}-$integral,
3) G is complete bipartite.

The following propositions give a necessary and sufficient conditions in which S_{vertex} and S_{edge} joins of graphs are $\tilde{\mathcal{L}}-$integral.

Proposition 3.2 If G_i is r_i-regular connected graph $(i = 1, 2)$, then $G_1 \dot{\vee} G_2$ and $G_1 \veebar G_2$ are $\tilde{\mathcal{L}}-$integral if and only if $G_1 \cong K_2$ and $G_2 \cong \overline{K}_{n_2}$.

Proof.

Since $K_2 \dot{\vee} \overline{K}_{n_2} = K_{2,n_2}$ and $K_2 \veebar \overline{K}_{n_2} = K_{1,n_2+2}$ are two graphs of order n with the normalized Laplacian eigenvalues $\{[0]^1$, $[1]^{n-1}$, $[2]^1\}$, these graphs are $\tilde{\mathcal{L}}-$integral. Conversely, suppose that $G_1 \dot{\vee} G_2$ is $\tilde{\mathcal{L}}-$integral, $G_1 \ncong K_2$ and $G_2 \ncong \overline{K}_{n_2}$. By Theorem 2.1, the normalized Laplacian spectrum of $G_1 \dot{\vee} G_2$ is $\{[0]^1$, $\left[1 \pm \sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}\right]^1$, $\left[1 - \frac{\lambda_k(G_2)}{n_1+r_2}\right]^1$, $[1]^{m_1-n_1}\}$, $(2 \leqslant j \leqslant n_1)$, $(2 \leqslant k \leqslant n_2)$ together with the roots of

$$x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2}x + \frac{2n_1n_2 + 2r_1n_1 + n_2r_2}{(n_1 + r_2)(n_2 + r_1)} = 0.$$

Since G_1 is connected graph, $G_1 \dot{\vee} G_2$ is connected. Hence, $[0]^1$ is an eigenvalue of $G_1 \dot{\vee} G_2$. Thus, the other eigenvalues must be equal to $[1]^{n-1}$, $[2]^1$. Since $1 - \frac{\lambda_j(G_2)}{n_1+r_2} < 2$, $2 \leqslant j \leqslant n_2$, we have $\frac{\lambda_j(G_2)}{n_1+r_2} = 0$. So, $\lambda_j(G_2) = 0$, $2 \leqslant j \leqslant n_2$ and it implies that $G_2 \ncong \overline{K}_{n_2}$, a contradiction. Since $1-\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}<2$, $2\leqslant j\leqslant n_1$, we achieve $\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}=0$ and thus $\lambda_j(G_1)=-r_1$, $2\leqslant j\leqslant n_1$, a contradiction. On the other hand, $1<1+\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}\leqslant 2$, $2\leqslant j\leqslant n_1$, yields that $\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}=1$. So, $\lambda_j(G_1)=-r_1+2n_2>0$, $2\leqslant j\leqslant n_1$, a contradiction. As well as, for two eigenvalues that are the roots of

$$x^2-\frac{3n_1+2r_2}{n_1+r_2}x+\frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}=0. \tag{2}$$

Since, G is connected and the normalized Laplacian eigenvalues are integral, one can deduce that Eq.(2) is equal to $(x-1)^2=x^2-2x+1$ or $(x-1)(x-2)=x^2-3x+2$. In the first case, one can conclude easily that $\frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}=1$ which yields that $n_1n_2=r_2-n_1$, for $r_1\geqslant 1$ which comes to a contradiction, because for example, there is no an 8-regular graph of order 5. Thus, the second case holds, namely

$$x^2-\frac{3n_1+2r_2}{n_1+r_2}x+\frac{2n_1n_2+2r_1n_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}=x^2-3x+2$$

and the proof is completed. ■

Proposition 3.3 Let G_1 be an empty graph of order n_1 and G_2 be $r-$regular graph of order n_2, then $G_1\dot{\vee}G_2$ and $G_1\veebar G_2$ are $\tilde{\mathcal{L}}-$integral if and only if $G_2\cong\overline{K}_{n_2}$ or G_2 is complete bipartite graph.

Proof. By using Proposition 3.1, Theorems 2.1 and 2.3, the proof is straight forward. ■

The normalized Laplacian energy of G is defined as $\tilde{\mathcal{L}}\mathcal{E}(G)=\sum_{i=1}^{n}|\delta_i(G)-1|$, see [4, 12] for details.

Proposition 3.4 Let G_1 is an r_1-regular of graph of order n_1 and $G_2\cong\overline{K}_{n_2}$. Then

1) $\tilde{\mathcal{L}}\mathcal{E}(G_1\dot{\vee}G_2)=2\Big(1+\frac{\mathcal{RE}(G_1)-\sqrt{2r_1}}{\sqrt{2(n_2+r_1)}}\Big)$,
2) $\tilde{\mathcal{L}}\mathcal{E}(G_1\veebar G_2)=2\Big(1+\frac{\mathcal{RE}(G_1)-\sqrt{2r_1}}{\sqrt{r_1(n_2+2)}}\Big)$.

Proof. By Corollary 2.2, Corollary 2.4, respectively, we have

$$\begin{aligned}\tilde{\mathcal{L}}\mathcal{E}(G_1\dot{\vee}G_2)&=\sum_{i=1}^{n}|\delta_i(G_1\dot{\vee}G_2)-1|\\&=1+\sum_{j=2}^{n_1}|\pm\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}|+1\\&=2+2\sum_{j=2}^{n_1}\sqrt{\frac{\lambda_j(G_1)+r_1}{2(n_2+r_1)}}\\&=2\Big(1+\frac{\mathcal{RE}(G_1)-\sqrt{2r_1}}{\sqrt{r_1(n_2+2)}}\Big),\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{LE}}(G_1 \veebar G_2) &= \sum_{i=1}^{n} |\delta_i(G_1 \veebar G_2) - 1| \\
&= 1 + \sum_{j=2}^{n_1} | \pm \sqrt{\frac{\lambda_j(G_1) + r_1}{r_1(n_2+2)}} | + 1 \\
&= 2 + 2\sum_{j=2}^{n_1} \sqrt{\frac{\lambda_j(G_1) + r_1}{r_1(n_2+2)}} \\
&= 2\Big(1 + \frac{\mathcal{RE}(G_1) - \sqrt{2r_1}}{\sqrt{r_1(n_2+2)}}\Big),
\end{aligned}
$$

where the last equality follows from $\mathcal{RE}(G_1) - \sqrt{2r_1} = \sum_{j=2}^{n_1} \sqrt{\lambda_j(G_1) + r_1}$. ■

The degree Kirchhoff index of graph G is defined by Chen et al. in [5] as

$$
Kf^*(G) = \sum_{i<j} d_i r_{ij} d_j,
$$

where r_{ij} denotes the resistance-distance between vertices v_i and v_j in a graph G. They proved that $Kf^*(G) = 2m\sum_{i=2}^{n} \frac{1}{\delta_i(G)}$, where $\delta_i(G)$ is normalized Laplacian eigenvalues of G of order n ($2 \leqslant i \leqslant n$), see [11, 18] for details.

Proposition 3.5 Let G_1 is an r_1-regular graph of order n_1 with m_1 edges and $G_2 \cong \overline{K}_{n_2}$. Then

$$
Kf^*(G_1\dot{\vee}G_2) = 2(2m_1 + n_1n_2)\Big(\frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1-1)\sqrt{2(n_2+r_1)}}{(n_1-1)\sqrt{2(n_2+r_1)} \pm (\mathcal{RE}(G_1) - \sqrt{2r_1})}\Big),
$$

$$
Kf^*(G_1 \veebar G_2) = 2m_1(2+n_2)\Big(\frac{1}{2} + m_1 - n_1 + n_2 + \frac{(n_1-1)\sqrt{r_1(n_2+2)}}{(n_1-1)\sqrt{r_1(n_2+2)} \pm (\mathcal{RE}(G_1) - \sqrt{2r_1})}\Big).
$$

Proof. It is not difficult to see that the number of edges of $G_1\dot{\vee}G_2$ and $G_1 \veebar G_2$ are $E(G_1\dot{\vee}G_2) = 2m_1 + n_1n_2$ and $E(G_1 \veebar G_2) = m_1(2+n_2)$, respectively. On the other hand, since G_1 is r_1-regular by using Eq.(1),

$$
\mathcal{RE}(G_1) = \sum_{j=1}^{n_1} \sqrt{\lambda_j(G_1) + r_1}
$$

and so

$$
\mathcal{RE}(G_1) - \sqrt{2r_1} = \sum_{j=2}^{n_1} \sqrt{\lambda_j(G_1) + r_1}.
$$

Now, Corollaries 2.2, 2.4 complete the proof. ■

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