

Computational aspect to the nearest southeast submatrix that makes multiple a prescribed eigenvalue

A. Nazari^{a*}, A. Nezami^a

^aDepartment of Mathematics, Arak University,
P.O. Box 38156-8-8349, Arak, Iran.

Received 1 November 2016; Revised 25 January 2017; Accepted 18 April 2017.

Abstract. Given four complex matrices A, B, C and D where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$ and let the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a normal matrix and assume that λ is a given complex number that is not eigenvalue of matrix A . We present a method to calculate the distance norm (with respect to 2-norm) from D to the set of matrices $X \in \mathbb{C}^{m \times m}$ such that, λ be a multiple eigenvalue of matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$. We also find the nearest matrix X to the matrix D .

© 2017 IAUCTB. All rights reserved.

Keywords: Normal matrix, multiple eigenvalues, singular value, distance matrices.

2010 AMS Subject Classification: 15A18, 15A60, 15A09, 93B10.

1. Introduction

In paper[4], A.N. Malyshev obtained the following formula for the 2-norm distance $\text{rsep}(A)$ from a complex $n \times n$ matrix to a closest matrix with a multiple eigenvalue:

$$\text{rsep}(A) = \min_{\lambda \in \mathbb{C}} \max_{\gamma \geq 0} \sigma_{2n-1}(G(\gamma)),$$

*Corresponding author.

E-mail address: a-nazari@araku.ac.ir (A. Nazari).

where,

$$G(\gamma) = \begin{pmatrix} \lambda I - A & \gamma I \\ 0 & \lambda I - A \end{pmatrix},$$

and $\sigma_{2n-1}(G(\gamma))$ is the penultimate singular value of the matrix G , assuming that the singular values are numbered in decreasing order. Ikramov and Nazari in [2] introduced a correction for Malyshev's formula for a normal matrix. In recent paper [5] Nazari and Rajabi used the same correction to [2] for the paper of Lippert [3] for normal matrices.

In the recent paper [1], Gracia and Velasco obtained the following formula for the 2-norm distance D from a complex $m \times m$ matrix to a closest matrix with a multiple eigenvalue:

$$\min_{X \in C^{m \times m}, m(\lambda, M(\alpha, X)) \geq 2} \|X - D\| = \sup_{t \in \mathbb{R}} \sigma_{2m-1}(S_2(t)), \quad (1)$$

where

$$M(\alpha, X) = \begin{pmatrix} A & B \\ C & X \end{pmatrix}, \quad \text{and} \quad S_2(t) = \begin{pmatrix} M & tN \\ 0 & M \end{pmatrix},$$

that

$$M = (D - \lambda I_m) - C(A - \lambda I_n)^{-1}B, \quad (2)$$

$$N = I_m + C(A - \lambda I_n)^{-2}B, \quad (3)$$

where λ is not eigenvalue of matrix A and $\sigma_{2m-1}(S_2(t))$ is the penultimate singular value of the matrix $S_2(t)$, where $\alpha = (A, B, C) \in L_{n,m} = \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{m \times n}$. In the two Theorems that follows, we briefly describe the article of Gracia and Velasco.

Theorem 1.1 Let $t^* > 0$ be a local optimizer of function $s_2(t) = \sigma_{2m-1}(S_2(t))$. Suppose $\sigma^* = s_2(t^*) > 0$, then there exists a pair of normalized singular vectors associated with the singular value t^* of $s_2(t^*)$, namely a left vector

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1, u_2 \in \mathbb{C}^m$$

and a corresponding right vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}^m$$

such that

$$\operatorname{Re}(u_1^* N v_2) = 0. \quad (4)$$

where the matrix N is defined by (3). Moreover, the matrices

$$U = (u_1 \ u_2), \quad V = (v_1 \ v_2), \quad (5)$$

satisfy the relation

$$V^*V = U^*U \in \mathbb{C}^{2 \times 2}. \tag{6}$$

Theorem 1.2 Let λ is not eigenvalue of matrix A and t^* in Theorem (1.1) is a positive number. The matrix $D + \Delta$, where

$$\Delta = -\sigma^*UV^\dagger, \tag{7}$$

is the closest (with respect to the 2-norm) matrix to matrix X , such that the matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$ having multiple eigenvalue λ and

$$\|\Delta\|_2 = \sigma^*, \tag{8}$$

where denote by V^\dagger the Moore-Penrose inverse matrix of V .

Let $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. By a similar method that introduced in [2], we discuss some issues related to the computer implementation of this method. It turns out that the case of a general matrix G is substantially different from that of a normal matrix G .

2. Normal matrix

Let G be a normal matrix. Let

$$A = \begin{pmatrix} 12 & 7 & 7 \\ 7 & 16 & 10 \\ 7 & 10 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 5 \\ 3 & 3 \\ 11 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 & 3 & 11 \\ 5 & 3 & 11 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix},$$

then it is easy to see that the matrix $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is normal matrix. Assume that $\lambda = 0$. By MATLAB software with long format for computation, we found the values $t^* = 7.60093750000001$, $\sigma^* = 7.60093621516323$. The singular value σ_{2m-2} equals 7.60093750000000. These two values are approximately the same, namely

$$\sigma_{2m-1}(S_2(t^*)) \simeq \sigma_{2m-2}(S_2(t^*)).$$

Thus, in the optimal matrix $S_2(t^*)$, the value σ^* is iterated. Let $u^{(2m-1)}$, $v^{(2m-1)}$ and $u^{(2m-2)}$, $v^{(2m-2)}$ be the pairs of singular vectors of $S_2(t^*)$ associated with σ_{2m-1} and σ_{2m-2} , respectively, that MATLAB gives us. An attempt to use any of these pairs for implementing the construction described in Theorem (1.2) leads to catastrophic results. Namely, for the matrix

$$\Delta^{2m-1} = -\sigma^*U^{(2m-1)}V^{(2m-1)\dagger},$$

we obtain

$$\| \Delta^{2m-1} \| = 2.345558000417827 \times 10^{16},$$

while $\Delta^{2m-2} = \sigma^* U^{(2m-2)} V^{(2m-2)\dagger}$ has the norm

$$\| \Delta^{2m-2} \| = 2.345558396903085 \times 10^{16}.$$

It is easy to find the reason why equality (8) is violated in both cases. The value of $u_1^{*2m-1} N v_2^{2m-1}$, for two vectors $u^{(2m-1)} = \begin{pmatrix} u_1^{2m-1} \\ u_2^{2m-1} \end{pmatrix}$ and $v^{(2m-1)} = \begin{pmatrix} v_1^{2m-1} \\ u_2^{2m-1} \end{pmatrix}$, is

$$-0.78923059741806$$

and for the pair vectors $u^{(2m-2)} = \begin{pmatrix} u_1^{2m-2} \\ u_2^{2m-2} \end{pmatrix}$ and $v^{(2m-1)} = \begin{pmatrix} v_1^{2m-2} \\ u_2^{2m-2} \end{pmatrix}$, is

$$1.000000000000000.$$

In any case above, equality (4), even approximately does not hold. It follows that equality (6) is violated.i.e.

$$U^*U = \begin{pmatrix} 0.10538463458652 & 0.30704838931280 \\ 0.30704838931280 & 0.89461536541348 \end{pmatrix},$$

$$V^*V = \begin{pmatrix} 0.89461536541348 & 0.30704838931280 \\ 0.30704838931280 & 0.10538463458652 \end{pmatrix}.$$

If we calculate the eigenvalues of matrix G , we see that

$$10^7 \times \left\{ \begin{array}{l} \text{The eigenvalues of matrix } G = \\ 0.00000371178081, \\ 0.00000065233548, \\ -0.00000139525368, \\ 0.00000069856511 + 4.06720386782124i, \\ 0.00000069856511 - 4.06720386782124i \end{array} \right\}.$$

Since $\lambda = 0$, by Theorem (1.2) we must have a multiple eigenvalue zero in matrix G , and all of eigenvalue that calculated above far from zero.

The situation can be rectified as follows. Consider the number

$$\sigma^* = \sigma_{2m-1}(S_2(t))$$

as a double singular value of $S_2(t)$ and the vectors $u^{(2m-1)}$ and $u^{(2m-2)}$ as an orthonormal basis in the left singular subspace associated with σ^* . In this subspace, we look for a normalized vector

$$u = \alpha u^{(2m-1)} + \beta u^{(2m-2)}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{9}$$

and combined with the associated right singular vector

$$v = \alpha v^{(2m-1)} + \beta v^{(2m-2)} \tag{10}$$

in order to satisfy relation (4). From (4) we have

$$\operatorname{Re}(u_1^* N v_2) = 0. \tag{11}$$

Substituting (9) and (10) into (11), we achieve the relation

$$(\bar{\alpha} \ \bar{\beta}) \operatorname{Re}W \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{12}$$

in which

$$W = \begin{pmatrix} u_1^{(2m-1)H} * N * v_2^{(2m-1)} & u_1^{(2m-1)H} * N * v_2^{(2m-2)} \\ u_1^{(2m-2)H} * N * v_2^{(2m-1)} & u_1^{(2m-2)H} * N * v_2^{(2m-2)} \end{pmatrix} \tag{13}$$

and

$$W_r = \operatorname{Re}(W) = \begin{pmatrix} \operatorname{Re}(W_{11}) & \frac{(\bar{W}_{21} + W_{12})}{2} \\ \frac{(\bar{W}_{12} + W_{21})}{2} & \operatorname{Re}(W_{22}) \end{pmatrix} \tag{14}$$

The existence of a nontrivial solution for Eq. (12) is ensured by the fact that the Hermitian matrix (12) is indefinite. In fact, let us call $g(t) = \sigma_{2m-2}(S_2(t))$. Let $\mu_1 \geq \mu_2$ be the eigenvalues of the matrix $\operatorname{Re}W$. Then the right derivatives of the functions S_2 and g at t^* are equal to μ_2 and μ_1

$$S_2'(t^{*+}) = \mu_2, \quad g'(t^{*+}) = \mu_1$$

respectively. Since S_2 is decreasing and g is increasing at right of t^* , we deduce that

$$\mu_2 < 0 \text{ and } \mu_1 > 0.$$

The numbers α and β can be found, for example, in following manner. Let

$$W_r = PMP^*, \quad M = \operatorname{diag}(\mu_1, \mu_2),$$

be the spectral decomposition of W . Set

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} T \\ K \end{pmatrix}, \tag{15}$$

and recast (12) as

$$\mu_1 |t|^2 + \mu_2 |\sigma|^2 = 0, \quad |t|^2 + |\sigma|^2 = 1. \tag{16}$$

The pair

$$\left(\frac{|\mu_2|}{|\mu_1| + |\mu_2|} \right)^{\frac{1}{2}}, \quad \left(\frac{|\mu_1|}{|\mu_1| + |\mu_2|} \right)^{\frac{1}{2}}$$

is a solution to system (16). (Recall again that μ_1 and μ_2 are numbers of different signs.) Using (15), we obtain the corresponding pair α, β . In the example above with matrix G , this technique yields

$$\alpha = -0.74759578100550, \quad \beta = 0.66415400941556$$

For the corresponding singular vectors (9) and (10), we have

$$u_1^* N v_2 = -2.238877486182567 \times 10^{-17}$$

The matrix Δ constructed from these vectors has the norm

$$7.60093681855830$$

which is in very good agreement with σ^* . Finally, we found

$$U^*U = \begin{pmatrix} 0.50000003728108 & 0.17160917645601 \\ 0.17160917645601 & 0.49999996271892 \end{pmatrix},$$

$$V^*V = \begin{pmatrix} 0.49999996271892 & 0.17160917645601 \\ 0.17160917645601 & 0.50000003728108 \end{pmatrix}$$

it follows that $U^*U \simeq V^*V$ and the eigenvalues of matrix G as follows:

$$\text{The eigenvalues of matrix } G = \begin{Bmatrix} 38.23444569157454, \\ 8.77515531971971, \\ 6.20796134531682, \\ 0.0000000420761, \\ -0.00000113313609 \end{Bmatrix}.$$

References

- [1] Juan-Miguel Gracia, Francisco E. Velasco, Nearest Southeast Submatrix that makes multiple a prescribed eigenvalue. Part 1, *Linear Algebra Appl.* 430 (2009) 1196-1215.
- [2] Kh.D. Ikramov, A.M. Nazari, Computational aspects of the use of Malyshev's formula, *Zh. Vychisl. Mat. Mat. Fiz.* 44 (1) (2004), 3-7.
- [3] Ross A. Lippert, Fixing two eigenvalues by a minimal perturbation, *Linear Algebra Appl.* 406 (2005), 177-200.
- [4] A.N. Malyshev, A formula for the 2-norm distance from a matrix to the set of matrices with multiple eigenvalues, *Numer. Math.*, 83 (1999), 443-454.
- [5] A.M. Nazari, D. Rajabi, Computational aspect to the nearest matrix with two prescribed eigenvalues, *Linear Algebra Appl.* 432 (2010), 1-4.