

Unique common coupled fixed point theorem for four maps in S_b -metric spaces

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Abstract. In this paper we prove a unique common coupled fixed point theorem for two pairs of w -compatible mappings in S_b -metric spaces satisfying a contractive type condition. We furnish an example to support our main theorem. We also give a corollary for Jungck type maps.

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1. Introduction

In 2012, Sedghi et al. [10] introduced the notion of S -metric space and proved several results, for example, refer [7, 11]. On the other hand, the concept of b -metric space was introduced by Czerwik [2].

Recently, Sedghi et al. [8] defined S_b -metric spaces by using the concepts of S and b -metric spaces and proved common fixed point theorem for four maps in S_b -metric spaces.

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Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point and proved some coupled fixed point results. Several authors proved coupled fixed point theorems in various spaces, for example, see the references [5, 6] and the references therein.

The aim of this paper is to prove a unique common coupled fixed point theorem for four mappings in S_b -metric spaces. Throughout this paper \mathcal{R}^+ and \mathcal{N} denote the set of all non-negative real numbers and positive integers respectively. First we recall some definitions, lemmas and examples.

Definition 1.1 ([10]) Let X be a non-empty set. A S -metric on X is a function $S : X^3 \rightarrow \mathcal{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$,

- (S1) : $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
- (S2) : $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
- (S3) : $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then the pair (X, S) is called a S -metric space.

Definition 1.2 ([2]) Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathcal{R}^+$ is called a b -metric if the following axioms are satisfied for all $x, y, z \in X$,

- (b₁) $d(x, y) = 0$ if and only if $x = y$,
- (b₂) $d(x, y) = d(y, x)$,
- (b₃) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b -metric space.

Definition 1.3 ([8]) Let X be a non-empty set and $b \geq 1$ be given real number. Suppose that a mapping $S_b : X^3 \rightarrow \mathcal{R}^+$ be a function satisfying the following properties:

- (S_b1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
- (S_b2) $S_b(x, y, z) = 0 \Leftrightarrow x = y = z$,
- (S_b3) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S_b is called a S_b -metric on X and the pair (X, S_b) is called a S_b -metric space.

Remark 1 ([8]) It should be noted that, the class of S_b -metric spaces is effectively larger than that of S -metric spaces. Indeed each S -metric space is a S_b -metric space with $b = 1$.

Following example shows that a S_b -metric on X need not be a S -metric on X .

Example 1.4 ([8]) Let (X, S_b) be a S_b -metric space and $S_b(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_b is a S_b -metric with $b = 2^{2(p-1)}$. Also, (X, S_b) is not necessarily a S -metric space.

Definition 1.5 ([8]) Let (X, S_b) be a S_b -metric space. Then, for $x \in X$ and $r > 0$, we defined the open ball $B_{S_b}(x, r)$ and closed ball $B_{S_b}[x, r]$ with center x and radius r as follows respectively:

$$\begin{aligned} B_{S_b}(x, r) &= \{y \in X : S_b(y, y, x) < r\}, \\ B_{S_b}[x, r] &= \{y \in X : S_b(y, y, x) \leq r\}. \end{aligned}$$

Lemma 1.6 ([8]) In a S_b -metric space, we have

$$S_b(x, x, y) \leq bS_b(y, y, x)$$

and

$$S_b(y, y, x) \leq bS_b(x, x, y).$$

Lemma 1.7 ([8]) In a S_b -metric space, we have

$$S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2S_b(y, y, z).$$

Definition 1.8 ([8]) If (X, S_b) be a S_b -metric space. A sequence $\{x_n\}$ in X is said to be:

- (1) S_b -Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.
- (2) S_b -convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, x, x_n) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{n \rightarrow \infty} x_n = x$.

Definition 1.9 ([8]) A S_b -metric space (X, S_b) is called complete if every S_b -Cauchy sequence is S_b -convergent in X .

Lemma 1.10 ([9]) If (X, S_b) be a S_b -metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b -convergent to x , then we have

$$(i) \frac{1}{2b}S_b(y, x, x) \leq \liminf_{n \rightarrow \infty} S_b(y, y, x_n) \leq \limsup_{n \rightarrow \infty} S_b(y, y, x_n) \leq 2bS_b(y, y, x)$$

and

$$(ii) \frac{1}{b^2}S_b(x, x, y) \leq \liminf_{n \rightarrow \infty} S_b(x_n, x_n, y) \leq \limsup_{n \rightarrow \infty} S_b(x_n, x_n, y) \leq b^2S_b(x, x, y) \text{ for all } y \in X.$$

In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} S_b(x_n, x_n, y) = 0$.

Definition 1.11 ([3]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \rightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.12 ([4]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called

- (i) a coupled coincident point of mappings $F : X \times X \rightarrow X$ and $f : X \rightarrow X$ if $fx = F(x, y)$ and $fy = F(y, x)$,
- (ii) a common coupled fixed point of mappings $F : X \times X \rightarrow X$ and $f : X \rightarrow X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 1.13 ([1]) Let X be a nonempty set and $F : X \times X \rightarrow X$ and $f : X \rightarrow X$. Then $\{F, f\}$ is said to be w -compatible pair if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever there exist $x, y \in X$ with $fx = F(x, y)$ and $fy = F(y, x)$.

2. Main Result

Now we give our main result.

Theorem 2.1 Let (X, S_b) be a S_b -metric space. Suppose that $f, g : X \times X \rightarrow X$ and $F, G : X \rightarrow X$ be satisfying

$$(2.1.1) \quad f(X \times X) \subseteq G(X), g(X \times X) \subseteq F(X),$$

$$(2.1.2) \quad \{f, F\} \text{ and } \{g, G\} \text{ are } w\text{-compatible pairs,}$$

$$(2.1.3) \quad \text{One of } F(X) \text{ or } G(X) \text{ is } S_b\text{-complete subspace of } X,$$

$$(2.1.4) \quad S_b(f(x, y), f(x, y), g(u, v))$$

$$\leq k \max \left\{ \begin{array}{l} S_b(Fx, Fx, Gu), S_b(Fy, Fy, Gv), S_b(f(x, y), f(x, y), Fx), \\ S_b(f(y, x), f(y, x), Fy), S_b(g(u, v), g(u, v), Gu), S_b(g(v, u), g(v, u), Gv), \\ \frac{1}{4b^2} [S_b(f(x, y), f(x, y), Gu) + S_b(g(u, v), g(u, v), Fx)], \\ \frac{1}{4b^2} [S_b(f(y, x), f(y, x), Gv) + S_b(g(v, u), g(v, u), Fy)] \end{array} \right\},$$

for all $x, y, u, v \in X$, where $0 \leq k < \frac{1}{4b^5}$.

Then f, g, F and G have a unique common coupled fixed point in $X \times X$.

Proof. Let $x_0, y_0 \in X$. From (2.1.1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} f(x_{2n}, y_{2n}) &= Gx_{2n+1} = z_{2n}, \\ f(y_{2n}, x_{2n}) &= Gy_{2n+1} = w_{2n}, \\ g(x_{2n+1}, y_{2n+1}) &= Fx_{2n+2} = z_{2n+1}, \\ g(y_{2n+1}, x_{2n+1}) &= Fy_{2n+2} = w_{2n+1}, \quad n = 0, 1, 2, \dots \end{aligned}$$

Case(i): Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m . Put

$$S_{2m} = \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\}.$$

From (2.1.4), we have

$$\begin{aligned} &S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) \\ &= S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), g(x_{2m+1}, y_{2m+1})) \\ &\leq k \max \left\{ \begin{array}{l} S_b(Fx_{2m+2}, Fx_{2m+2}, Gx_{2m+1}), S_b(Fy_{2m+2}, Fy_{2m+2}, Gy_{2m+1}), \\ S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), Fx_{2m+2}), \\ S_b(f(y_{2m+2}, x_{2m+2}), f(y_{2m+2}, x_{2m+2}), Fy_{2m+2}), \\ S_b(g(x_{2m+1}, y_{2m+1}), g(x_{2m+1}, y_{2m+1}), Gx_{2m+1}), \\ S_b(g(y_{2m+1}, x_{2m+1}), g(y_{2m+1}, x_{2m+1}), Gy_{2m+1}), \\ \frac{1}{4b^2} [S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), Gx_{2m+1}) \\ + S_b(g(x_{2m+1}, y_{2m+1}), g(x_{2m+1}, y_{2m+1}), Fx_{2m+2})], \\ \frac{1}{4b^2} [S_b(f(y_{2m+2}, x_{2m+2}), f(y_{2m+2}, x_{2m+2}), Gy_{2m+1}) \\ + S_b(g(y_{2m+1}, x_{2m+1}), g(y_{2m+1}, x_{2m+1}), Fy_{2m+2})] \end{array} \right\} \\ &= k \max \left\{ \begin{array}{l} S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ \frac{1}{4b^2} [S_b(z_{2m+2}, z_{2m+2}, z_{2m}) + S_b(z_{2m+1}, z_{2m+1}, z_{2m+1})], \\ \frac{1}{4b^2} [S_b(w_{2m+2}, w_{2m+2}, w_{2m}) + S_b(w_{2m+1}, w_{2m+1}, w_{2m+1})] \end{array} \right\} \\ &\leq kS_{2m+1}. \end{aligned}$$

Similarly, we can prove

$$S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \leq kS_{2m+1}.$$

It follows that $z_{2m+2} = z_{2m+1}$ and $w_{2m+2} = w_{2m+1}$. Continuing this process we can conclude that $z_{2m+k} = z_{2m}$ and $w_{2m+k} = w_{2m}$ for all $k \geq 0$. It follows that $\{z_m\}$ and

$\{w_m\}$ are Cauchy sequences.

Case (ii): Assume that $z_n \neq z_{n+1}$ or $w_n \neq w_{n+1}$ for all n . From (2.1.4), we have

$$\begin{aligned}
 & S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) \\
 &= S_b(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), g(x_{2n+1}, y_{2n+1})) \\
 &\leq k \max \left\{ \begin{array}{l} S_b(Fx_{2n+2}, Fx_{2n+2}, Gx_{2n+1}), S_b(Fy_{2n+2}, Fy_{2n+2}, Gy_{2n+1}), \\ S_b(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), Fx_{2n+2}), \\ S_b(f(y_{2n+2}, x_{2n+2}), f(y_{2n+2}, x_{2n+2}), Fy_{2n+2}), \\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}), \\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}), \\ \frac{1}{4b^2} \left[S_b(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), Gx_{2n+1}) \right. \\ \left. + S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Fx_{2n+2}) \right], \\ \frac{1}{4b^2} \left[S_b(f(y_{2n+2}, x_{2n+2}), f(y_{2n+2}, x_{2n+2}), Gy_{2n+1}) \right. \\ \left. + S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Fy_{2n+2}) \right] \end{array} \right\} \\
 &= k \max \left\{ \begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\ S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2} [S_b(z_{2n+2}, z_{2n+2}, z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})], \\ \frac{1}{4b^2} [S_b(w_{2n+2}, w_{2n+2}, w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})] \end{array} \right\}.
 \end{aligned}$$

But

$$\begin{aligned}
 & \frac{1}{4b^2} [S_b(z_{2n+2}, z_{2n+2}, z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})] \\
 &\leq \frac{1}{4b^2} [2bS_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) + bS_b(z_{2n}, z_{2n}, z_{2n+1})] \\
 &\leq \frac{1}{4b^2} [2bS_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) + b^2S_b(z_{2n+1}, z_{2n+1}, z_{2n})] \\
 &\leq \max\{S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n})\} \\
 &\leq \max\{S_{2n+1}, S_{2n}\}
 \end{aligned}$$

Similarly,

$$\frac{1}{4b^2} [S_b(w_{2n+2}, w_{2n+2}, w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})] \leq \max\{S_{2n+1}, S_{2n}\}.$$

Hence,

$$S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) \leq k \max\{S_{2n+1}, S_{2n}\}.$$

Similarly,

$$S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leq k \max\{S_{2n+1}, S_{2n}\}.$$

Hence, it is clear that

$$S_{2n+1} \leq k \max\{S_{2n+1}, S_{2n}\}.$$

If S_{2n+1} is maximum, then we get contradiction. Hence S_{2n} is maximum. Therefore

$$S_{2n+1} \leq kS_{2n} < S_{2n}. \quad (1)$$

Similarly, we can conclude that $S_{2n} < S_{2n-1}$. Thus, $\{S_n\}$ is non-increasing sequence of non-negative real numbers and hence converges to $r \geq 0$. Suppose $r > 0$. Letting $n \rightarrow \infty$ in (1), we have

$$r \leq kr < r.$$

It is a contradiction. Hence $r = 0$. Thus,

$$\lim_{n \rightarrow \infty} S_b(z_{n+1}, z_{n+1}, z_n) = 0 \quad (2)$$

and

$$\lim_{n \rightarrow \infty} S_b(w_{n+1}, w_{n+1}, w_n) = 0. \quad (3)$$

Now, we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X, S_b) . On contrary suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$.

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \geq \epsilon \quad (4)$$

and

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k-2}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k-2})\} < \epsilon. \quad (5)$$

From (4) and (5), we have that

$$\begin{aligned} \epsilon &\leq \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\ &\leq 2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\ &\quad + b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+2}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k+2})\} \\ &\leq 4b^2 \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\} \\ &\quad + 2b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\} \\ &\quad + 2b^2 \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k+1})\} \\ &\quad + b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}. \end{aligned} \quad (6)$$

From (2.1.4), we have

$$\begin{aligned}
 & S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}) \\
 &= S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), g(x_{2n_k+1}, y_{2n_k+1})) \\
 &\leq k \max \left\{ \begin{array}{l} S_b(Fx_{2m_k+2}, Fx_{2m_k+2}, Gx_{2n_k+1}), S_b(Fy_{2m_k+2}, Fy_{2m_k+2}, Gy_{2n_k+1}), \\ S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), Fx_{2m_k+2}), \\ S_b(f(y_{2m_k+2}, x_{2m_k+2}), f(y_{2m_k+2}, x_{2m_k+2}), Fy_{2m_k+2}), \\ S_b(g(x_{2n_k+1}, y_{2n_k+1}), g(x_{2n_k+1}, y_{2n_k+1}), Gx_{2n_k+1}), \\ S_b(g(y_{2n_k+1}, x_{2n_k+1}), g(y_{2n_k+1}, x_{2n_k+1}), Gy_{2n_k+1}), \\ \frac{1}{4b^2} \left[S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), Gx_{2n_k+1}) \right. \\ \left. + S_b(g(x_{2n_k+1}, y_{2n_k+1}), g(x_{2n_k+1}, y_{2n_k+1}), Fx_{2m_k+2}) \right], \\ \frac{1}{4b^2} \left[S_b(f(y_{2m_k+2}, x_{2m_k+2}), f(y_{2m_k+2}, x_{2m_k+2}), Gy_{2n_k+1}) \right. \\ \left. + S_b(g(y_{2n_k+1}, x_{2n_k+1}), g(y_{2n_k+1}, x_{2n_k+1}), Fy_{2m_k+2}) \right] \end{array} \right\}, \\
 &= k \max \left\{ \begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2} \left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\ \frac{1}{4b^2} \left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right] \end{array} \right\}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \\
 &\leq k \max \left\{ \begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2} \left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\ \frac{1}{4b^2} \left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right] \end{array} \right\}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \max \{ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \} \\
 &\leq k \max \left\{ \begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2} \left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\ \frac{1}{4b^2} \left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right] \end{array} \right\}. \tag{7}
 \end{aligned}$$

But

$$\begin{aligned}
& \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \\
& \leq 2b \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& \quad + b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k})\} \\
& \leq 2b \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& \quad + b^2 \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
& \leq 2b \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& \quad + b^2 (2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}) \\
& \quad + b^2 (b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-2}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-2}})\}) \\
& < 2b \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& \quad + 2b^3 \epsilon + b^3 (2b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-1}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-1}})\}) \\
& \quad + b^3 (b \max\{S_b(z_{2n_{k-2}}, z_{2n_{k-2}}, z_{2n_{k-1}}), S_b(w_{2n_{k-2}}, w_{2n_{k-2}}, w_{2n_{k-1}})\}) \\
& \leq 2b \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& \quad + 2b^3 \epsilon + 2b^4 \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-1}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-1}})\} \\
& \quad + b^5 \max\{S_b(z_{2n_{k-1}}, z_{2n_{k-1}}, z_{2n_{k-2}}), S_b(w_{2n_{k-1}}, w_{2n_{k-1}}, w_{2n_{k-2}})\}.
\end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \leq 2b^3 \epsilon.$$

Also

$$\begin{aligned}
& \frac{1}{4b^2} (S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})) \\
& \leq \frac{1}{4b^2} \left(2b S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + b^2 S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \right) \\
& \quad + 2b S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + b S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \\
& \leq \frac{1}{b^2} \max \left\{ \begin{array}{l} 2b S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), b^2 S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ 2b S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), b S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \end{array} \right\} \\
& \leq \max \left\{ \begin{array}{l} 2 S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ 2 S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ S_b(w_{2m_k+1}, w_{2m_k+1}, z_{2n_k}) \end{array} \right\}.
\end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{4b^2} (S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})) \leq 2b\epsilon.$$

Similarly,

$$\lim_{k \rightarrow \infty} \frac{1}{4b^2} (S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})) \leq 2b\epsilon.$$

Now, letting $k \rightarrow \infty$ in (7), we have

$$\lim_{k \rightarrow \infty} \max \left\{ \begin{array}{l} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), \\ S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \end{array} \right\} \leq k \max \{ 2b^3\epsilon, 0, 0, 2b\epsilon, 2b\epsilon \} \\ = k 2b^3\epsilon.$$

Hence, letting $k \rightarrow \infty$ in (6), we have

$$\epsilon \leq 0 + 0 + 0 + b^2 k 2b^3\epsilon < \epsilon,$$

it is a contradiction. Hence, $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b -Cauchy sequences. In addition

$$\begin{aligned} & \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2m+1})\} \\ & \leq 2b \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\ & \quad + b \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2n}), S_b(w_{2m+1}, w_{2m+1}, w_{2n})\} \\ & \leq 2b \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\ & \quad + 2b^2 \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\} \\ & \quad + b^2 \max\{S_b(z_{2n}, z_{2n}, z_{2m}), S_b(w_{2n}, w_{2n}, w_{2m})\}. \end{aligned}$$

From(2), (3) and since $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b -Cauchy sequences, it follows that $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b -Cauchy sequences in (X, S_b) . Thus, $\{z_n\}$ and $\{w_n\}$ are S_b -Cauchy sequences in (X, S_b) . Suppose $F(X)$ is complete subspace of X . Then it follows that $\{z_n\}$ and $\{w_n\}$ converges to α and β respectively in $F(X)$. Thus, there exist u and v in $F(X)$ such that

$$\lim_{n \rightarrow \infty} z_n = \alpha = Fu \text{ and } \lim_{n \rightarrow \infty} w_n = \beta = Fv. \tag{8}$$

Now, we have to prove that $\alpha = f(u, v)$ and $\beta = f(v, u)$. Using (2.1.4) and Lemma

(1.10), we obtain that

$$\begin{aligned}
& \frac{1}{2b} S_b(f(u, v), f(u, v), \alpha) \\
& \leq \limsup_{n \rightarrow \infty} S_b(f(u, v), f(u, v), g(x_{2n+1}, y_{2n+1})) \\
& \leq \limsup_{n \rightarrow \infty} k \max \left\{ \begin{array}{l} S_b(Fu, Fu, Gx_{2n+1}), S_b(Fv, Fv, Gy_{2n+1}), \\ S_b(f(u, v), f(u, v), Fu), S_b(f(v, u), f(v, u), Fv), \\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}), \\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}), \\ \frac{1}{4b^2} \left[S_b(f(u, v), f(u, v), Gx_{2n+1}) \right. \\ \left. + S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Fu) \right], \\ \frac{1}{4b^2} \left[S_b(f(v, u), f(v, u), Gy_{2n+1}) \right. \\ \left. + S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Fv) \right] \end{array} \right\} \\
& = \limsup_{n \rightarrow \infty} k \max \left\{ \begin{array}{l} S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n}), S_b(f(u, v), f(u, v), \alpha), \\ S_b(f(v, u), f(v, u), \beta), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2} [S_b(f(u, v), f(u, v), z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, \alpha)], \\ \frac{1}{4b^2} [S_b(f(v, u), f(v, u), w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, \beta)] \end{array} \right\} \\
& \leq k \max \left\{ 0, 0, S_b(f(u, v), f(u, v), \alpha), S_b(f(v, u), f(v, u), \beta), 0, 0, \right. \\
& \quad \left. \frac{1}{4b^2} [2b S_b(f(u, v), f(u, v), \alpha) + 0], \frac{1}{4b^2} [2b S_b(f(v, u), f(v, u), \beta) + 0] \right\} \\
& = k \max \{ S_b(f(u, v), f(u, v), \alpha), S_b(f(v, u), f(v, u), \beta) \}.
\end{aligned}$$

Similarly,

$$\frac{1}{2b} S_b(f(v, u), f(v, u), \beta) \leq k \max \{ S_b(f(u, v), f(u, v), \alpha), S_b(f(v, u), f(v, u), \beta) \}.$$

Thus, we have

$$\frac{1}{2b} \max \left\{ \begin{array}{l} S_b(f(u, v), f(u, v), \alpha), \\ S_b(f(v, u), f(v, u), \beta) \end{array} \right\} \leq k \max \left\{ \begin{array}{l} S_b(f(u, v), f(u, v), \alpha), \\ S_b(f(v, u), f(v, u), \beta) \end{array} \right\}.$$

It follows that $f(u, v) = \alpha$ and $f(v, u) = \beta$. Thus, (α, β) is a coupled coincidence point of f and F . Since $\{f, F\}$ is a w -compatible pair, we have $F\alpha = f(\alpha, \beta)$ and $F\beta = f(\beta, \alpha)$.

From (2.1.4) and Lemma (1.10), we obtain that

$$\begin{aligned}
 & \frac{1}{2b} S_b(f(\alpha, \beta), f(\alpha, \beta), \alpha) \\
 & \leq \limsup_{n \rightarrow \infty} S_b(f(\alpha, \beta), f(\alpha, \beta), g(x_{2n+1}, y_{2n+1})) \\
 & \leq \limsup_{n \rightarrow \infty} k \max \left\{ \begin{array}{l} S_b(F\alpha, F\alpha, Gx_{2n+1}), S_b(F\beta, F\beta, Gy_{2n+1}), \\ S_b(f(\alpha, \beta), f(\alpha, \beta), F\alpha), S_b(f(\beta, \alpha), f(\beta, \alpha), F\beta), \\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}), \\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}), \\ \frac{1}{4b^2} \left[S_b(f(\alpha, \beta), f(\alpha, \beta), Gx_{2n+1}) \right. \\ \left. + S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), F\alpha) \right], \\ \frac{1}{4b^2} \left[S_b(f(\beta, \alpha), f(\beta, \alpha), Gy_{2n+1}) \right. \\ \left. + S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), F\beta) \right] \end{array} \right\} \\
 & = \limsup_{n \rightarrow \infty} k \max \left\{ \begin{array}{l} S_b(F\alpha, F\alpha, z_{2n}), S_b(F\beta, F\beta, w_{2n}), \\ S_b(f(\alpha, \beta), f(\alpha, \beta), F\alpha), S_b(f(\beta, \alpha), f(\beta, \alpha), F\beta), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2} \left[S_b(f(\alpha, \beta), f(\alpha, \beta), z_{2n}) \right. \\ \left. + S_b(z_{2n+1}, z_{2n+1}, f(\alpha, \beta)) \right], \\ \frac{1}{4b^2} \left[S_b(f(\beta, \alpha), f(\beta, \alpha), w_{2n}) \right. \\ \left. + S_b(w_{2n+1}, w_{2n+1}, f(\beta, \alpha)) \right] \end{array} \right\} \\
 & \leq k \max \left\{ \begin{array}{l} 2bS_b(f(\alpha, \beta), f(\alpha, \beta), \alpha), 2bS_b(f(\beta, \alpha), f(\beta, \alpha), \beta), 0, 0, 0, 0, \\ \frac{1}{4b^2} [2bS_b(f(\alpha, \beta), f(\alpha, \beta), \alpha) + b^2S_b(\alpha, \alpha, f(\alpha, \beta))], \\ \frac{1}{4b^2} [2bS_b(f(\beta, \alpha), f(\beta, \alpha), \beta) + b^2S_b(\beta, \beta, f(\beta, \alpha))] \end{array} \right\} \\
 & = 2b k \max \{ S_b(f(\alpha, \beta), f(\alpha, \beta), \alpha), S_b(f(\beta, \alpha), f(\beta, \alpha), \beta) \}.
 \end{aligned}$$

Similarly,

$$\frac{1}{2b} S_b(f(\beta, \alpha), f(\beta, \alpha), \beta) \leq 2b k \max \left\{ \begin{array}{l} S_b(f(\alpha, \beta), f(\alpha, \beta), \alpha), \\ S_b(f(\beta, \alpha), f(\beta, \alpha), \beta) \end{array} \right\}.$$

Hence,

$$\frac{1}{2b} \max \left\{ \begin{array}{l} S_b(f(\alpha, \beta), f(\alpha, \beta), \alpha), \\ S_b(f(\beta, \alpha), f(\beta, \alpha), \beta) \end{array} \right\} \leq 2b k \max \left\{ \begin{array}{l} S_b(f(\alpha, \beta), f(\alpha, \beta), \alpha), \\ S_b(f(\beta, \alpha), f(\beta, \alpha), \beta) \end{array} \right\}.$$

It follows that $\alpha = F\alpha = f(\alpha, \beta)$, and $\beta = F\beta = f(\beta, \alpha)$. Therefore (α, β) is common coupled fixed point of (f, F) . Since $f(X \times X) \subseteq G(X)$, there exist $a, b \in X$ such that

$f(\alpha, \beta) = \alpha = Ga$ and $f(\beta, \alpha) = \beta = Gb$. From (2.1.4), by Lemma 1.7 and $b \geq 1$ we have

$$\begin{aligned} S_b(\alpha, \alpha, g(a, b)) &= S_b(f(\alpha, \beta), f(\alpha, \beta), g(a, b)) \\ &\leq k \max \left\{ \begin{array}{l} S_b(F\alpha, F\alpha, Ga), S_b(F\beta, F\beta, Gb), S_b(f(\alpha, \beta), f(\alpha, \beta), F\alpha), \\ S_b(f(\beta, \alpha), f(\beta, \alpha), F\beta), S_b(g(a, b), g(a, b), Ga), S_b(g(b, a), g(b, a), Gb), \\ \frac{1}{4b^2} [S_b(f(\alpha, \beta), f(\alpha, \beta), Ga) + S_b(g(a, b), g(a, b), F\alpha)], \\ \frac{1}{4b^2} [S_b(f(\beta, \alpha), f(\beta, \alpha), Gb) + S_b(g(b, a), g(b, a), F\beta)] \end{array} \right\} \\ &= k \max \left\{ \begin{array}{l} 0, 0, 0, 0, S_b(g(a, b), g(a, b), \alpha), S_b(g(b, a), g(b, a), \beta), \\ \frac{1}{4b^2} [0 + S_b(g(a, b), g(a, b), F\alpha)], \frac{1}{4b^2} [0 + S_b(g(b, a), g(b, a), F\beta)] \end{array} \right\} \\ &\leq b k \max \{ S_b(\alpha, \alpha, g(a, b)), S_b(\beta, \beta, g(b, a)) \}. \end{aligned}$$

Similarly,

$$S_b(\beta, \beta, g(b, a)) \leq b k \max \{ S_b(\alpha, \alpha, g(a, b)), S_b(\beta, \beta, g(b, a)) \}.$$

Thus,

$$\max \{ S_b(\alpha, \alpha, g(a, b)), S_b(\beta, \beta, g(b, a)) \} \leq bk \max \{ S_b(\alpha, \alpha, g(a, b)), S_b(\beta, \beta, g(b, a)) \}.$$

It follows that $g(a, b) = \alpha = Ga$ and $g(b, a) = \beta = Gb$. Since the pair $\{g, G\}$ is w -compatible, we have $G\alpha = g(\alpha, \beta)$ and $G\beta = g(\beta, \alpha)$. Using (2.1.4), we obtain

$$\begin{aligned} S_b(\alpha, \alpha, g(\alpha, \beta)) &= S_b(f(\alpha, \beta), f(\alpha, \beta), g(\alpha, \beta)) \\ &\leq k \max \left\{ \begin{array}{l} S_b(F\alpha, F\alpha, G\alpha), S_b(F\beta, F\beta, G\beta), \\ S_b(f(\alpha, \beta), f(\alpha, \beta), F\alpha), S_b(f(\beta, \alpha), f(\beta, \alpha), F\beta), \\ S_b(g(\alpha, \beta), g(\alpha, \beta), G\alpha), S_b(g(\beta, \alpha), g(\beta, \alpha), G\beta), \\ \frac{1}{4b^2} [S_b(f(\alpha, \beta), f(\alpha, \beta), G\alpha) + S_b(g(\alpha, \beta), g(\alpha, \beta), F\alpha)], \\ \frac{1}{4b^2} [S_b(f(\beta, \alpha), f(\beta, \alpha), G\beta) + S_b(g(\beta, \alpha), g(\beta, \alpha), F\beta)] \end{array} \right\} \\ &\leq k \max \left\{ \begin{array}{l} S_b(\alpha, \alpha, g(\alpha, \beta)), S_b(\beta, \beta, g(\beta, \alpha)), 0, 0, 0, 0, \\ \frac{1}{4b^2} [S_b(\alpha, \alpha, g(\alpha, \beta)) + S_b(g(\alpha, \beta), g(\alpha, \beta), \alpha)], \\ \frac{1}{4b^2} [S_b(\beta, \beta, g(\beta, \alpha)) + S_b(g(\beta, \alpha), g(\beta, \alpha), \beta)] \end{array} \right\} \\ &= k \max \{ S_b(\alpha, \alpha, g(\alpha, \beta)), S_b(\beta, \beta, g(\beta, \alpha)) \}. \end{aligned}$$

Similarly,

$$S_b(\beta, \beta, g(\beta, \alpha)) \leq k \max \{ S_b(\alpha, \alpha, g(\alpha, \beta)), S_b(\beta, \beta, g(\beta, \alpha)) \}.$$

Thus,

$$\max \{ S_b(\alpha, \alpha, g(\alpha, \beta)), S_b(\beta, \beta, g(\beta, \alpha)) \} \leq k \max \{ S_b(\alpha, \alpha, g(\alpha, \beta)), S_b(\beta, \beta, g(\beta, \alpha)) \}.$$

It follows that $\alpha = g(\alpha, \beta)$ and $\beta = g(\beta, \alpha)$. Thus, $\alpha = G\alpha = g(\alpha, \beta)$ and $\beta = G\beta = g(\beta, \alpha)$. Hence, (α, β) is a common coupled fixed point of f, g, F and G . To prove uniqueness let us suppose $(\alpha^1, \beta^1) \in X \times X$ is another common coupled fixed point of f, g, F

and G such that $\alpha \neq \alpha^1$ and $\beta \neq \beta^1$. From (2.1.4), we have that

$$\begin{aligned} S_b(\alpha, \alpha, \alpha^1) &= S_b(f(\alpha, \beta), f(\alpha, \beta), g(\alpha^1, \beta^1)) \\ &\leq k \max \left\{ \begin{array}{l} S_b(F\alpha, F\alpha, G\alpha^1), S_b(F\beta, F\beta, G\beta^1), \\ S_b(f(\alpha, \beta), f(\alpha, \beta), F\alpha), S_b(f(\beta, \alpha), f(\beta, \alpha), F\beta), \\ S_b(g(\alpha^1, \beta^1), g(\alpha^1, \beta^1), G\alpha^1), S_b(g(\beta^1, \alpha^1), g(\beta^1, \alpha^1), G\beta^1), \\ \frac{1}{4b^2} [S_b(f(\alpha, \beta), f(\alpha, \beta), G\alpha^1) + S_b(g(\alpha^1, \beta^1), g(\alpha^1, \beta^1), F\alpha)], \\ \frac{1}{4b^2} [S_b(f(\beta, \alpha), f(\beta, \alpha), G\beta^1) + S_b(g(\beta^1, \alpha^1), g(\beta^1, \alpha^1), F\beta)] \end{array} \right\} \\ &= k \max \left\{ \begin{array}{l} S_b(\alpha, \alpha, \alpha^1), S_b(\beta, \beta, \beta^1), 0, 0, 0, 0, \\ \frac{1}{4b^2} [S_b(\alpha, \alpha, \alpha^1) + S_b(\alpha^1, \alpha^1, \alpha)], \frac{1}{4b^2} [S_b(\beta, \beta, \beta^1) + S_b(\beta^1, \beta^1, \beta)] \end{array} \right\} \\ &= k \max \{ S_b(\alpha, \alpha, \alpha^1), S_b(\beta, \beta, \beta^1) \}. \end{aligned}$$

Similarly,

$$S_b(\beta, \beta, \beta^1) \leq k \max \{ S_b(\alpha, \alpha, \alpha^1), S_b(\beta, \beta, \beta^1) \}.$$

Thus,

$$\max \{ S_b(\alpha, \alpha, \alpha^1), S_b(\beta, \beta, \beta^1) \} \leq k \max \{ S_b(\alpha, \alpha, \alpha^1), S_b(\beta, \beta, \beta^1) \}.$$

It is a contradiction. Hence, (α, β) is the unique common coupled fixed point of f, g, F and G . ■

Example 2.2 Let $X = [0, 1]$ and

$$S_b : X \times X \times X \rightarrow \mathbb{R}^+ \quad \text{by} \quad S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2,$$

then S_b is S_b -metric space with $b = 4$. Define $f, g : X \times X \rightarrow X$ and $F, G : X \rightarrow X$ by

$$f(x, y) = \frac{x^2 + y^2}{46}, \quad g(x, y) = \frac{x^2 + y^2}{47}, \quad F(x) = \frac{x^2}{4} \quad \text{and} \quad G(x) = \frac{x^2}{16}$$

also put $k = \frac{1}{4^7}$. Then

$$\begin{aligned}
& S_b(f(x, y), f(x, y), g(u, v)) \\
&= (|f(x, y) + g(u, v) - 2f(x, y)| + |f(x, y) - g(u, v)|)^2 \\
&= 4 (|f(x, y) - g(u, v)|)^2 \\
&= 4 \left| \frac{x^2 + y^2}{4^6} - \frac{u^2 + v^2}{4^7} \right|^2 \\
&= 4 \left| \frac{4x^2 - u^2}{4^7} + \frac{4y^2 - v^2}{4^7} \right|^2 \\
&= 4 \frac{1}{(4^4)^2} \left(\frac{1}{4} \left\{ \left| \frac{4x^2 - u^2}{16} \right| + \left| \frac{4y^2 - v^2}{16} \right| \right\} \right)^2 \\
&\leq 4 \frac{1}{(4^4)^2 4} \left(\frac{1}{2} \left\{ \left| \frac{4x^2 - u^2}{16} \right| + \left| \frac{4y^2 - v^2}{16} \right| \right\} \right)^2 \\
&\leq 4 \frac{1}{4^9} \left(\max \left\{ \left| \frac{4x^2 - u^2}{16} \right|, \left| \frac{4y^2 - v^2}{16} \right| \right\} \right)^2 \\
&= \frac{1}{4^9} \max \{ S_b(Fx, Fx, Gu), S_b(Fy, Fy, Gv) \} \\
&\leq k \max \left\{ \begin{array}{l} S_b(Fx, Fx, Gu), S_b(Fy, Fy, Gv), S_b(f(x, y), f(x, y), Fx), \\ S_b(f(y, x), f(y, x), Fy), S_b(g(u, v), g(u, v), Gu), S_b(g(v, u), g(v, u), Gv), \\ \frac{1}{4b^2} [S_b(f(x, y), f(x, y), Gu) + S_b(g(u, v), g(u, v), Fx)], \\ \frac{1}{4b^2} [S_b(f(y, x), f(y, x), Gv) + S_b(g(v, u), g(v, u), Fy)] \end{array} \right\}.
\end{aligned}$$

It is clear that all conditions of Theorem 2.1 satisfied and $(0, 0)$ is unique common coupled fixed point of f, g, F and G .

From Theorem 2.1, we have the following corollary.

Corollary 2.3 Let (X, S_b) be a S_b -metric space. Suppose that $f : X \times X \rightarrow X$ and $F : X \rightarrow X$ be satisfying

- (2.3.1) $f(X \times X) \subseteq F(X)$,
- (2.3.2) (f, F) are weakly compatible pairs,
- (2.3.3) $F(X)$ is S_b -complete subspace of X ,
- (2.3.4) $S_b(f(x, y), f(x, y), f(u, v))$

$$\leq k \max \left\{ \begin{array}{l} S_b(Fx, Fx, Fu), S_b(Fy, Fy, Fv), S_b(f(x, y), f(x, y), Fx), \\ S_b(f(y, x), f(y, x), Fy), S_b(f(u, v), f(u, v), Fu), S_b(f(v, u), f(v, u), Fv), \\ \frac{1}{4b^2} [S_b(f(x, y), f(x, y), Fu) + S_b(f(u, v), f(u, v), Fx)], \\ \frac{1}{4b^2} [S_b(f(y, x), f(y, x), Fv) + S_b(f(v, u), f(v, u), Fy)] \end{array} \right\}$$

for all $x, y, u, v \in X$, where $0 \leq k < \frac{1}{4b^2}$.

Then f and F have a unique common coupled fixed point in $X \times X$.

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