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The method of radial basis functions for the solution of nonlinear Fredholm integral equations system

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Abstract. In this paper, an effective simple method is proposed for approximating the solution of Fredholm integral equations systems using radial basis functions (RBFs). We present an algorithm based on interpolation by radial basis functions including multiquadratics (MQs), using Legendre-Gauss-Lobatto nodes and weights. A theorem is presented to prove the convergence of the algorithm as well. Some numerical examples are illustrated and the results compared with those obtained by triangular functions (TFs), delta basis functions (DFs), block-pulse functions (BPFs), Sinc fucntion, Adomian decomposition, Haar wavelet and direct methods demonstrate the validity, accuracy and applicability of the RBF method.

Keywords: Radial basis functions, Multiquadric, Nonlinear Fredholm integral equations system.

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1. Introduction

In general form, a nonlinear integral equations system can be written as:

$$f_i(s) = g_i(s) + \lambda \sum_{j=1}^{n} \int_{\Gamma} k_{ij}(s,t)\, u_{ij}(f_j(t))\, dt, \quad 0 \leqslant s,t \leqslant 1,\ i = 1,2,\ldots,n, \tag{1}$$

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where $f_j(s), j = 1, 2 \ldots, n$ are the unknown functions $k_{i,j}(s,t) \in L^2([0,1] \times [0,1])$ and $g_i(s) \in L^2[0,1], i, j = 1, 2, \ldots, n$ are kernels and unknown functions, respectively. The functions $u_{ij}(f_j(t))$ are polynomials of $f_j(t)$ with constant coefficients [9]-[5]. For convenience, let $u_{ij}(f_j(t)) = [f_j(t)]^{p_{ij}}$, where $p_{ij},\ i, j = 1, 2, \ldots, n$ are positive integers. Integration interval defined by notation Γ can be considered as either $[0,1]$ or $[0,s]$ in Fredholm or Volterra integral equations system, respectively. In this paper, we approximate the solution of Fredholm integral equations system. Systems of integral equations can describe many different events in science and engineering. There are some numerical methods to obtain the approximate solutions of linear and nonlinear integral equations systems [1–4, 7, 11, 14, 15]. Recently Almasieh *et. al* in [3] have applied multiquadric radial basis functions to obtain the solution of a class of mixed two dimensional nonlinear Volterra-Fredholm integral equations. Their method is based on a powerful technique for the scattered data interpolation problem which is well known as RBFs method. The main advantage of this technique is the meshless characteristic of them [13]. The stability interval of shape parameter affects the obtained solution [8]. In this paper, we use multiquadratic RBFs method to obtain the numerical solution of Fredholm integral equations system. MQs have shape parameter and stability interval of them, is $[0,2]$. A theorem is proved for convergence analysis as well and it is shown that as N and c increase, error decreases drastically. We compare the results obtained by presented method with those obtained by available nine various methods for showing the accuracy and efficiency of the presented method using an optimal value of c for all examples.

This paper is organized as follows: In Section 2, we describe MQs interpolation. In Section 3, we introduce the Legendre-Gauss-Lobatto nodes and weights. In Section 4, the RBFs method is implemented to find the numerical solution of Fredholm integral equations system. We discuss a convergence analysis for recent systems in Section 5. Finally, some numerical examples illustrate the efficiency and accuracy of the proposed method. Section 7 conclused the paper.

2. MQs interpolation

Let $\phi(r)$ be an MQ function, we can approximate a function $f(s)$ with interpolation by the function $\phi(r)$, i.e.,

$$f_j(s) \simeq \sum_{i=0}^{N} c_i^j \phi_i(s) = {C^j}^T \mathbf{\Phi}(s), \quad j = 1, 2, \ldots, n, \tag{2}$$

where

$$\phi_i = \phi_i(s) = \phi(||s - s_i||) = \sqrt{||s - s_i||^2 + c^2}, \tag{3}$$

$$\mathbf{\Phi}(s) = [\phi_0, \phi_1, \ldots, \phi_N]^T, \tag{4}$$

and

$$C^j = [c_0^j, c_1^j, \ldots, c_N^j]^T. \tag{5}$$

3. Legendre-Gauss-Lobatto nodes and weights

Let $\mathcal{H}_N[-1,1]$ denote the space of algebraic polynomials of degree $\leqslant N$, by

$$< p_i, p_j >= \frac{2}{2j+1}\delta_{ij}.$$

Here $< .,. >$ represents the usual $L^2[-1,1]$ inner product and $\{p_i\}_{i\geqslant 0}$ are the well-known Legendre polynomials of order i which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1,1]$ and satisfy the following formula

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x, \\ p_{i+1}(x) &= (\frac{2i+1}{i+1})xp_i(x) - \frac{i}{i+1}p_{i-1}(x), \qquad i = 1,2,3,.... \end{aligned}$$

Next, we choose $\{x_j\}_{j=0}^{N}$ as

$$\begin{aligned} &(1-x_j^2)\dot{p}(x_j) = 0, \\ &-1 = x_0 < x_1 < x_2 < \cdots < x_N = 1, \end{aligned}$$

where $\dot{p}(x)$ is derivative of $p(x)$. No explicit formula for the nodes $\{x_j\}_{j=0}^{N}$ is known. However, they are computed numerically using the existing subroutines. Now, we assume $f \in \mathcal{H}_{2N-1}[-1,1]$, we have

$$\int_{-1}^{1} f(x)dx \simeq \sum_{j=0}^{N} w_j f(x_j),$$

where w_j are the Legendre-Gauss-Lobatto weights [3].

4. Numerical solution of nonlinear Fredholm integral equations system

Consider the system of nonlinear Fredholm integral equations in general form as

$$F(s) = G(s) + \lambda \int_0^1 U(s,t,F(t))\,dt, \qquad s \in [0,1], \tag{6}$$

where

$$\begin{aligned} F(s) &= [f_1(s), f_2(s), \ldots, f_n(s)]^T, \\ G(s) &= [g_1(s), g_2(s), \ldots, g_n(s)]^T, \\ U(s,t,F(t)) &= \begin{bmatrix} u_1(s,t,f_1(s),f_2(s),\ldots,f_n(s)) \\ u_2(s,t,f_1(s),f_2(s),\ldots,f_n(s)) \\ \vdots \\ u_n(s,t,f_1(s),f_2(s),\ldots,f_n(s)) \end{bmatrix}, \end{aligned}$$

and

$$u_i(s,t,f_1(s),f_2(s),\ldots,f_n(s)) = \sum_{j=1}^{N} k_{ij}(s,t)[f_j(t)]^{p_{i,j}}, \quad i=1,2,\ldots,n.$$

We consider the ith equation of Eq. (6) as

$$f_i(s) = g_i(s) + \lambda \sum_{j=1}^{n} k_{i,j}(s,t)\,[f_j(t)]^{p_{i,j}}, i=1,2,\ldots,n, \tag{7}$$

By using Eq. (2), we can approximate $F(s)$ as

$$\begin{aligned} F(s) &= [f_1(s), f_2(s), \ldots, f_n(s)]^T \\ &= [{C^1}^T \mathbf{\Phi}, {C^2}^T \mathbf{\Phi}, \ldots, {C^n}^T \mathbf{\Phi}]^T \\ &= C^T\, \tilde{\mathbf{\Phi}}, \end{aligned} \tag{8}$$

where

$$C^T = \begin{bmatrix} {C^1}^T & \bar{O} & \ldots & \bar{O} \\ \bar{O} & {C^2}^T & \ldots & \bar{O} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{O} & \bar{O} & \ldots & {C^n}^T \end{bmatrix}, \quad \tilde{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{\Phi} \\ \mathbf{\Phi} \\ \vdots \\ \mathbf{\Phi} \end{bmatrix}, \tag{9}$$

in which C^j and $\mathbf{\Phi}$ are $(N+1)-$vectors, $\bar{O}$ is an $(N+1)-$vector with zero entries and C is an $n \times [n\,(N+1)]$ diagonal matrix.

4.1 *Algorithm*

(i) Substituting Eq. (9) in Eq. (6) yields

$$C^T \tilde{\mathbf{\Phi}}(s) = G(s) + \lambda \int_0^1 U(s,t,C^T \tilde{\mathbf{\Phi}}(t))\,dt, \quad s \in [0,1], \tag{10}$$

where C^T and $\tilde{\Phi}(s)$ are defined in Eq. (9).

(ii) Now, we collocate Eq.(10) at points x_i for $i=1,\ldots,N$. Hence we have

$$C^T \tilde{\mathbf{\Phi}}(x_i) = G(x_i) + \lambda \int_0^1 U(x_i,t,C^T \tilde{\mathbf{\Phi}}(t))\,dt. \tag{11}$$

(iii) We first transform the integral over $[0,1]$ into integral over $[-1,1]$ using the following transformation:

$$y = 2t-1, \quad t \in [0,1].$$

(iv) Then Eq.(11) may be restated as

$$C^T\tilde{\mathbf{\Phi}}(x_i) = G(x_i) + \frac{\lambda}{2}\int_{-1}^{1} U(x_i, \frac{y+1}{2}, C^T\tilde{\mathbf{\Phi}}(\frac{y+1}{2}))\, dt. \tag{12}$$

(v) By using the Legendre-Gauss-Lobatto integration formula, we have

$$C^T\tilde{\mathbf{\Phi}}(x_i) = G(x_i) + \frac{\lambda}{2}\sum_{k=0}^{r_1} w_{1k} U(x_i, \frac{y_k+1}{2}, C^T\tilde{\mathbf{\Phi}}(\frac{y_k+1}{2}))\, dt. \tag{13}$$

For $i = 1, 2, \ldots, n$, Eq. (13) generates a nonlinear equations system that can be solved via Newton's iteration method to obtain the unknown vectors $C^j, j = 1, 2, \ldots, n$.

5. Convergence Analysis

Assume $(C[T], ||.||)$ be the Banach space of all continuous functions on $J = [0, 1]$ with norm $||f(s)|| = \max_{s\in J} |f(s)|$. Let there exist a positive constant M such that $|k_{i,j}(s,t)| \leqslant M, \forall s, t \in [0, 1], i, j = 1, 2, \ldots, n$. Suppose the nonlinear term $u_{i,j}(f_j(t))$ is satisfied in the Lipschitz condition, so there is a positive constant K such that

$$|u_{i,j}(u) - u_{i,j}(v)| \leqslant K|u - v|.$$

Theorem 5.1 The solution of nonlinear Fredholm integral equations system by using multiquadric radial basis functions methods converges if $0 < \alpha < 1$.

Proof. Suppose $f_i(s)$ and $\bar{f}_i(s)$ are exact and approximate solutions of the ith integral equation, respectively, so we have

$$\begin{aligned}
||f_i - \bar{f}_i|| &= \max_{0\leqslant s\leqslant 1} |f_i(s) - \bar{f}_i(s)| \\
&\leqslant |\lambda| \max_{0\leqslant s<1} \sum_{j=1}^{n} \int_0^1 |k_{ij}(s,t)|.|u_{ij}(f_j(t)) - u_{ij}(\bar{f}_j(t))| \\
&= |\lambda| \sum_{j=1}^{n} M.K.||f_i - \bar{f}_i|| \\
&= |\lambda|.n.M.K.||f_i - \bar{f}_i||
\end{aligned}$$

where $\alpha = |\lambda|.n.M.K$, it implies that $(1-\alpha)||f_i - \bar{f}_i|| \leqslant 0$ and choose $0 < \alpha < 1$, by increasing n, it implies $||f_i - \bar{f}_i|| \to 0$ as $n \to \infty$ and this completes the proof. ■

A similar proof can be performed for nonlinear Volterra integral equations system.

6. Numerical examples

In order to illustrate the performance of the MQs method for solving nonlinear Fredholm integral equations system and justify the accuracy and efficiency of the presented method, we compare our results with obtained solutions in mentioned references to prove that

MQs method is simpler and more effective than previous methods. So the absolute errors are defined as

$$e_i(s) = |f_i(s) - \bar{f}_i(s)|, \quad s \in [0,1], \quad i = 1, 2, \ldots, n,$$

where $f_i(s)$ and $\bar{f}_i(s)$ are exact and approximate solutions of the ith integral equation, respectively.

***Example* 6.1** Consider the system of linear Fredholm integral equations [15]-[11]

$$\begin{cases} f_1(s) = g_1(s) - \int_0^1 e^{s-t} f_1(t)dt - \int_0^1 e^{(s+2)t} f_2(t)dt, \\ f_2(s) = g_2(s) - \int_0^1 e^{st} f_1(t)dt - \int_0^1 e^{s+t} f_2(t)dt, \end{cases}$$

where $g_1(s) = 2e^s + \frac{e^{s+1}-1}{s+1}$ and $g_2(s) = e^s + e^{-s} + \frac{e^{s+1}-1}{s+1}$ with the exact solution $(f_1(s), f_2(s)) = (e^s, e^{-s})$. The results are listed in tables 1-6. In tables 3-6, we compare presented method with four other methods [4, 7, 11, 15]. Numerical results are shown in figures 1 -4 as well.

***Example* 6.2** Consider the system of nonlinear Fredholm integral equations [15]-[7]

$$\begin{cases} f_1(s) = s - \frac{5}{18} + \int_0^1 \frac{1}{3}[f_1(t) + f_2(t)]dt, \\ f_2(s) = s^2 - \frac{2}{9} + \int_0^1 \frac{1}{3}([f_1(t)]^2 + f_2(t))dt, \end{cases}$$

with the exact solution $(f_1(s), f_2(s)) = (s, s^2)$. The results are shown in tables 7-11. In tables 10 and 11, we compare presented method with three other methods in [15]-[7] as well. Figures 5-8 show the numerical results.

***Example* 6.3** Consider the system of nonlinear Fredholm integral equations [15]

$$\begin{cases} f_1(s) = g_1(s) - \int_0^1 (s+t)[f_1(t) + [f_1(t)]^2]dt - \int_0^1 (s+2t^2)[f_2(t) + [f_2(t)]^3]dt, \\ f_2(s) = g_2(s) - \int_0^1 (st^2)[f_1(t) + [f_1(t)]^2]dt - \int_0^1 (s^2 t)[f_2(t) + [f_2(t)]^3]dt, \end{cases}$$

where $g_1(s) = \frac{97}{42}s + \frac{217}{180}$ and $g_2(s) = \frac{11}{8}s^2 + \frac{9}{20}s$ with the exact solution $(f_1(s), f_2(s)) = (s, s^2)$. The results are shown in tables 12-15. In table 15, we compare presented method with delta functions method in [15]. Figures 9 and 10 show the numerical results.

***Example* 6.4** Consider the system of linear Fredholm integral equations [4, 11]

$$\begin{cases} f_1(s) = \frac{11}{6}s + \frac{11}{15} - \int_0^1 (s+t) f_1(t)dt - \int_0^1 (s+2t^2) f_2(t)dt, \\ f_2(s) = \frac{5}{4}s^2 + \frac{1}{4}s - \int_0^1 st^2 f_1(t)dt - \int_0^1 s^2 t f_2(t)dt, \end{cases}$$

with exact solution $(f_1(s), f_2(s)) = (s, s^2)$. The results are shown in tables 16-19. In tables 18 and 19, we compare presented method with two other methods in [4, 11] as well. Figures 11-14 show the numerical results.

***Example* 6.5** Consider the system of linear Fredholm integral equations [6, 12, 16, 17]

$$\begin{cases} f_1(s) = \frac{1}{18}s + \frac{17}{38} + \int_0^1 \frac{t+s}{3}(f_1(t) + f_2(t))\,dt, \\ f_2(s) = s^2 - \frac{19}{12}s + 1 + \int_0^1 ts(f_1(t) + f_2(t))\,dt, \end{cases}$$

The exact solutions to this problem are $f_1(s) = s + 1$ and $f_2(s) = s^2 + 1$. The results are shown in tables 20-23. We compare presented method with six other methods in [7, 14] and [6, 12, 16, 17]. Figures 15-18 show the numerical results.

We computed errors in tables 1, 7, 12 and 16 for examples 6.1, 6.2, 6.3 and 6.4 with $N = 2, 8$, $N = 4, 8$, $N = 2, 6$ and $N = 2, 10$, respectively. We denoted that by increasing N and $c \in [0, 2]$, max errors can be decreased effectively shown in tables 2, 8, 10 and 13 for examples 6.1, 6.2, 6.3 and 6.4, respectively. The results have been evaluated in Legendre-Gauss-Lobatto roots shown in tables 9 and 15 with $N = 4$ and $c = 2$ for all examples as well.

Table 1.
Errors for example 6.1 with $c = 2$.

s	$N = 2$		$N = 8$	
	e_1	e_2	e_1	e_2
0	$1.7433E-02$	$1.0364E-02$	$9.1945E-07$	$5.4424E-07$
0.1	$1.0632E-03$	$5.6589E-04$	$5.0440E-07$	$2.5819E-07$
0.2	$7.7204E-03$	$3.8127E-03$	$3.6652E-07$	$1.7068E-07$
0.3	$9.8885E-03$	$4.4275E-03$	$2.6190E-07$	$1.1507E-07$
0.4	$6.8609E-03$	$2.7901E-03$	$6.6741E-07$	$2.7915E-07$
0.5	$6.4089E-04$	$2.4053E-04$	$4.6665E-06$	$1.8399E-07$
0.6	$6.2100E-03$	$2.0731E-03$	$2.3726E-07$	$9.7018E-08$
0.7	$1.0572E-02$	$3.2084E-03$	$7.1050E-07$	$2.3177E-07$
0.8	$8.8024E-03$	$2.4315E-03$	$5.8856E-07$	$1.7833E-07$
0.9	$3.2059E-03$	$7.8972E-04$	$3.7816E-07$	$1.0905E-07$

Table 2.
Maximum errors for example 6.1 with $N = 6$ for various values of c.

c	e_1	e_2
0	$1.2622E-01$	$8.4790E-03$
0.4	$3.5203E-03$	$8.6888E-04$
0.8	$4.0016E-04$	$2.1030E-04$
1.2	$9.8645E-05$	$6.5280E-05$
1.6	$3.6619E-05$	$2.4216E-05$
2	$8.2349E-06$	$1.0789E-05$

Table 3.
Comparison between presented method with $c = 2$ and the method in [4] for example 6.1.

method in [4]			presented method		
m	e_1	e_2	N	e_1	e_2
4	$1.4223E-01$	$5.1993E-02$	4	$5.4614E-04$	$3.3612E-04$
8	$3.5697E-02$	$1.2066E-02$	6	$1.7113E-05$	$1.0789E-05$
16	$8.4343E-03$	$3.0757E-03$	8	$9.1945E-07$	$5.4424E-07$
32	$2.1993E-03$	$7.4296E-04$			
64	$5.2513E-04$	$1.9146E-04$			
128	$1.3722E-04$	$4.6412E-05$			
258	$3.2806E-05$	$1.1966E-05$			

Table 4.
Comparison between presented method with $c = 2$ and method in [14] for example 6.1.

method in [14]			presented method		
N	e_1	e_2	N	e_1	e_2
5	$1.4591E-03$	$1.5766E-03$	4	$5.4614E-04$	$3.3612E-04$
10	$8.6744E-05$	$9.7819E-05$	6	$1.7113E-05$	$1.0789E-05$
20	$1.6540E-06$	$1.8732E-06$	8	$9.1945E-07$	$5.4424E-07$

Table 5.
Comparison between presented method with $c = 2$ and method in [15] for example 6.1.

method in [15]			presented method		
m	e_1	e_2	N	e_1	e_2
4	$8.2831E-03$	$4.8707E-03$	4	$5.4614E-04$	$3.3612E-04$
8	$2.0719E-03$	$1.2588E-03$	6	$1.7113E-05$	$1.0789E-05$
16	$5.2905E-04$	$3.2008E-04$	8	$9.1945E-07$	$5.4424E-07$
32	$1.3319E-04$	$8.0690E-04$			
64	$3.3409E-05$	$2.0273E-05$			
128	$8.3591E-06$	$5.5880E-06$			

Table 6.
Comparison between presented method with $c = 2$ and $N = 8$ and methods in [4, 11] for example 6.1.

s	BPF method $(m = 32)$		TF method $(m = 32)$		presented method$(N = 8)$	
	e_1	e_2	e_1	e_2	e_1	e_2
0	$1.047E{-}02$	$1.530E{-}02$	$1.151E{-}03$	$6.699E{-}04$	$9.1945E{-}07$	$5.4424E{-}07$
0.1	$1.124E{-}02$	$8.260E{-}03$	$1.151E{-}03$	$7.307E{-}04$	$5.0440E{-}07$	$2.5819E{-}07$
0.2	$3.560E{-}03$	$2.370E{-}03$	$1.192E{-}03$	$7.430E{-}04$	$3.6652E{-}07$	$1.7068E{-}07$
0.3	$4.930E{-}03$	$2.700E{-}03$	$1.285E{-}03$	$7.131E{-}04$	$2.6190E{-}07$	$1.1507E{-}07$
0.4	$1.406E{-}02$	$6.500E{-}03$	$1.446E{-}03$	$6.484E{-}04$	$6.6741E{-}07$	$2.7915E{-}07$
0.5	$2.572E{-}02$	$9.680E{-}03$	$1.694E{-}03$	$5.549E{-}04$	$4.6665E{-}07$	$1.8399E{-}07$
0.6	$1.699E{-}02$	$4.950E{-}03$	$1.699E{-}03$	$5.440E{-}04$	$2.3726E{-}07$	$9.7018E{-}07$
0.7	$6.060E{-}03$	$1.390E{-}03$	$1.769E{-}03$	$4.917E{-}04$	$7.1050E{-}07$	$2.3177E{-}07$
0.8	$6.540E{-}03$	$7.800E{-}04$	$1.927E{-}03$	$4.0234E{-}04$	$5.8856E{-}07$	$1.7833E{-}07$
0.9	$2.309E{-}02$	$4.130E{-}03$	$2.199E{-}03$	$2.786E{-}04$	$3.7816E{-}07$	$1.0905E{-}07$

Table 7.
Errors for example 6.2 with $c = 2$.

s	$N = 4$		$N = 8$	
	e_1	e_2	e_1	e_2
0	$1.2569E-04$	$2.0816E-05$	$1.0990E-07$	$5.6461E-09$
0.1	$5.1743E-05$	$2.5439E-06$	$4.0016E-08$	$2.4806E-08$
0.2	$2.0199E-05$	$5.5379E-06$	$1.7321E-08$	$2.6347E-08$
0.3	$3.6275E-05$	$1.8446E-05$	$1.8583E-09$	$1.8245E-08$
0.4	$4.1378E-05$	$3.1073E-05$	$4.9005E-08$	$4.5260E-08$
0.5	$2.6006E-12$	$5.0022E-12$	$2.2053E-08$	$2.6717E-08$
0.6	$4.0588E-05$	$5.0893E-05$	$4.3889E-09$	$7.8720E-09$
0.7	$3.4909E-05$	$5.2739E-05$	$4.1492E-08$	$5.7907E-08$
0.8	$1.9078E-05$	$3.3739E-05$	$2.6509E-08$	$3.5230E-08$
0.9	$4.7989E-05$	$9.7188E-05$	$5.3202E-09$	$9.7812E-09$

Table 8.
Maximum errors for example 6.2 with $N = 6$ for various values of c.

c	e_1	e_2
0	$5.6098E-02$	$2.9974E-02$
0.4	$1.9203E-03$	$1.3328E-03$
0.8	$2.2101E-04$	$1.7419E-04$
1.2	$4.0476E-05$	$2.9348E-05$
1.6	$1.0089E-05$	$4.9193E-06$
2	$3.1588E-06$	$2.0469E-06$

Table 9.
Errors for examples 6.1 and 6.2 with $N = 4$ and $c = 2$ in Legendre-Gauss-Lobatto roots.

s	example 6.1		example 6.2	
	e_1	e_2	e_1	e_2
0.9531	$2.5692E-10$	$7.0965E-10$	$2.6015E-12$	$5.0625E-12$
0.7692	$6.9429E-10$	$2.4412E-10$	$2.5779E-12$	$5.3654E-12$
0.5000	$7.7242E-10$	$1.1380E-10$	$2.6006E-12$	$5.0022E-12$
0.2308	$3.7809E-10$	$1.4676E-10$	$2.5999E-12$	$4.8162E-12$
0.0469	$6.0297E-10$	$1.7914E-10$	$2.5974E-12$	$4.6625E-12$

Table 10.
Comparison between presented method with $c = 2$ and method in [15] for example 6.2.

method in [15]			presented method		
m	e_1	e_2	N	e_1	e_2
10	$3.5776E-03$	$5.2253E-02$	4	$1.2569E-04$	$9.7188E-05$
			6	$3.1588E-06$	$2.0469E-06$
			8	$1.0990E-07$	$5.7907E-08$

Table 11.
Comparison between presented method with $c = 2$ and Adomian method [7] and TF method [4] for example 6.2.

s	Adomian method		TF method ($m = 32$)		presented method $N = 8$	
	e_1	e_2	e_1	e_2	e_1	e_2
0	$2.30E–02$	$4.43E–02$	$2.170E–04$	$2.713E–04$	$1.0990E–07$	$5.6461E–09$
0.1	$2.30E–02$	$4.43E–02$	$2.170E–04$	$4.276E–04$	$4.0016E–08$	$2.4806E–08$
0.2	$2.30E–02$	$4.43E–02$	$2.170E–04$	$5.057E–04$	$1.7321E–08$	$2.6347E–08$
0.3	$2.30E–02$	$4.43E–02$	$2.170E–04$	$5.057E–04$	$1.8583E–09$	$1.8245E–08$
0.4	$2.30E–02$	$4.43E–02$	$2.170E–04$	$4.276E–04$	$2.9005E–08$	$4.5260E–08$
0.5	$2.30E–02$	$4.43E–02$	$2.170E–04$	$2.713E–04$	$2.2053E–08$	$2.6717E–08$
0.6	$2.30E–02$	$4.43E–02$	$2.170E–04$	$4.276E–04$	$4.3889E–09$	$7.7820E–09$
0.7	$2.30E–02$	$4.43E–02$	$2.170E–04$	$5.057E–04$	$4.1792E–08$	$5.7909E–08$
0.8	$2.30E–02$	$4.43E–02$	$2.170E–04$	$2.276E–04$	$5.3202E–09$	$9.7812E–09$
0.9	$2.30E–02$	$4.43E–02$	$2.170E–04$	$2.786E–04$	$3.7816E–07$	$1.0905E–07$

Table 12.
Errors for example 6.3 with $c = 2$.

s	$N=2$		$N=6$	
	e_1	e_2	e_1	e_2
0	$6.7682E-03$	$1.6970E-04$	$2.7691E-06$	$9.0822E-07$
0.1	$1.3834E-03$	$7.2758E-05$	$1.2144E-06$	$5.2733E-08$
0.2	$1.2521E-03$	$8.7786E-04$	$7.7367E-07$	$2.4765E-07$
0.3	$1.7881E-03$	$1.5651E-03$	$3.8987E-07$	$6.2195E-10$
0.4	$9.2032E-04$	$1.3221E-03$	$1.3885E-06$	$7.6432E-07$
0.5	$6.3341E-04$	$6.8575E-06$	$2.7187E-07$	$1.3590E-08$
0.6	$2.1574E-03$	$1.9587E-03$	$8.2400E-07$	$1.3630E-06$
0.7	$2.9624E-03$	$3.5901E-03$	$1.5593E-07$	$6.2888E-08$
0.8	$2.4070E-03$	$3.3847E-03$	$1.2539E-06$	$2.0356E-06$
0.9	$8.4083E-05$	$5.9636E-04$	$6.1158E-07$	$1.5378E-06$

Table 13.
Maximum errors for example 6.3 with $N = 4$ for various values of c.

c	e_1	e_2
0.1	$4.5557E-02$	$1.9347E-02$
0.5	$6.9149E-03$	$4.0102E-03$
0.9	$1.6885E-03$	$1.1954E-03$
1.3	$5.5422E-04$	$4.1596E-04$
1.7	$2.2406E-04$	$1.7186E-04$

Table 14.
Comparison between the presented method with $c = 2$ and method in [15] for example 6.3.

method in [15]			presented method		
m	e_1	e_2	N	e_1	e_2
4	$1.5081E-02$	$1.0252E-02$	2	$6.7682E-03$	$3.5901E-03$
8	$3.9386E-03$	$2.5931E-03$	4	$1.2569E-04$	$9.7188E-05$
16	$9.9562E-04$	$6.3034E-04$	6	$2.7691E-06$	$2.0356E-06$
32	$2.4960E-04$	$1.6272E-04$			
64	$6.2442E-05$	$4.0687E-05$			

Table 15.
Maximum Errors for examples 6.3 and 6.4 with $N = 4$ and $c = 2$ in Legendre-Gauss-Lobatto roots.

s	example 6.3		example 6.4	
	e_1	e_2	e_1	e_2
0.9531	$2.9754E-13$	$4.0146E-13$	$3.8280E-13$	$6.2172E-13$
0.7692	$9.1038E-14$	$9.1593E-14$	$5.1159E-13$	$1.1369E-13$
0.5000	$4.6896E-13$	$5.6843E-14$	$5.1159E-13$	$1.1369E-13$
0.2308	$7.0333E-14$	$3.8238E-13$	$3.2613E-13$	$6.4080E-13$
0.0469	$1.4796E-13$	$2.2868E-13$	$1.5047E-13$	$2.2868E-13$

Table 16.
Errors for example 6.4 with $c = 2$.

s	$N = 2$		$N = 10$	
	e_1	e_2	e_1	e_2
0	$5.8917E-03$	$1.7126E-04$	$5.7044E-09$	$2.2352E-08$
0.1	$5.6753E-04$	$9.5456E-05$	$6.3796E-09$	$1.4678E-08$
0.2	$2.0058E-03$	$8.4485E-04$	$8.6147E-10$	$2.7567E-09$
0.3	$2.4784E-03$	$1.5334E-03$	$9.7789E-10$	$1.5981E-08$
0.4	$1.5463E-03$	$1.3041E-03$	$6.0769E-09$	$1.4752E-07$
0.5	$7.2106E-05$	$1.5174E-05$	$4.3074E-09$	$1.7695E-08$
0.6	$1.6606E-03$	$1.9115E-03$	$1.0245E-09$	$8.9779E-09$
0.7	$2.5301E-03$	$3.4920E-03$	$6.5891E-09$	$2.0862E-09$
0.8	$2.0382E-03$	$3.2243E-03$	$3.6554E-09$	$2.0154E-08$
0.9	$3.9088E-04$	$8.2956E-04$	$1.1269E-08$	$7.8604E-09$

Table 17.
Maximum errors for example 6.4 with $N = 8$ for various values of c.

c	e_1	e_2
0	$3.1652E-02$	$1.6180E-02$
0.4	$3.1253E-04$	$3.3107E-04$
0.8	$1.8664E-05$	$1.7825E-05$
1.2	$1.9954E-06$	$1.4747E-06$
1.6	$3.1708E-07$	$2.2582E-07$
2	$6.7841E-08$	$4.7354E-08$

Table 18.
Comparison between the presented method with $c = 2$ and method in [4] for example 6.4.

method in [4]			presented method		
m	e_1	e_2	N	e_1	e_2
4	$7.7577E-03$	$1.4657E-02$	4	$1.5116E-04$	$1.1662E-04$
8	$1.9301E-03$	$3.5805E-03$	6	$3.1704E-06$	$1.9699E-06$
16	$4.8193E-04$	$9.1856E-04$	8	$3.7841E-08$	$4.7354E-08$
32	$1.2045E-04$	$2.2390E-04$	10	$1.1269E-08$	$2.2352E-08$
64	$3.0110E-05$	$5.7420E-05$			
128	$7.5272E-06$	$1.3994E-05$			
256	$1.8818E-06$	$3.5888E-06$			

Table 19.
Comparison between the presented method with $c = 2$ and $N = 10$ and methods in [4, 11] for example 6.4.

s	BPF method ($m = 32$)		TF method ($m = 32$)		presented method $N=10$	
	e_1	e_2	e_1	e_2	e_1	e_2
0	1.421E–02	4.331E–02	1.151E–04	0.000E–00	5.7044E–09	2.2352E–08
0.1	1.980E–03	2.900E–04	1.157E–04	1.156E–04	6.3796E–09	1.4678E–08
0.2	3.450E–03	1.720E–03	1.163E–04	2.239E–04	8.6147E–10	2.7567E–09
0.3	8.540E–03	1.340E–03	1.169E–04	2.170E–04	9.7789E–10	1.5981E–08
0.4	1.210E–02	8.120E–03	1.175E–04	2.310E–04	6.0769E–08	1.4752E–08
0.5	1.386E–02	1.411E–02	1.181E–04	3.426E–05	4.3074E–09	1.7695E–08
0.6	6.410E–03	2.405E–02	1.187E–04	1.119E–04	1.0245E–09	8.9779E–09
0.7	9.140E–03	9.000E–05	1.193E–04	1.178E–04	6.5891E–09	2.0862E–09
0.8	1.314E–02	1.180E–02	1.199E–04	1.666E–04	3.6554E–09	2.0154E–08
0.9	1.512E–02	4.490E–03	1.204E–04	7.512E–05	1.1269E–08	7.8604E–09

Table 20.
Comparison between the presented method with $c = 2$ and $N = 8$ and methods in [16] and [12] for example 6.5.

s	method in [16]		method in [12] ($n=20$)		presented method $N=8$	
	e_1	e_2	e_1	e_2	e_1	e_2
0	1.15E–02	0	6.53E–06	0	1.15E–07	2.52E–08
0.1	1.33E–02	3.45E–03	7.66E–06	1.96E–06	4.13E–06	4.96E–09
0.2	1.52E–02	6.90E–03	8.80E–06	3.92E–06	1.94E–8	1.10E–08
0.3	1.71E–02	1.03E–02	9.93E–06	5.88E–06	6.06E–09	8.09E–09
0.4	1.89E–02	1.38E–02	1.11E–05	7.84E–06	5.71E–08	4.26E–08
0.5	2.08E–02	1.72E–02	1.22E–05	9.79E–06	3.22E–08	2.91E–08
0.6	2.26E–02	2.07E–02	1.33E–05	1.18E–05	8.87E–09	8.59E–10
0.7	2.45E–02	2.41E–02	1.45E–05	1.37E–05	5.60E–08	7.12E–08
0.8	2.64E–02	2.76E–02	1.56E–05	1.57E–05	4.40E–08	5.49E–08
0.9	2.82E–02	3.10E–02	1.67E–05	1.76E–05	2.70E–08	1.71E–08

Table 21.
Comparison between the presented method with $c = 2$ and method in [14] for example 6.5.

method in [14]			presented method		
N	e_1	e_2	N	e_1	e_2
10	3.06161$E-04$	3.56890$E-04$	4	1.3683$E-04$	9.3435$E-05$
20	5.80340$E-06$	6.74044$E-06$	6	3.4495$E-06$	2.0219$E-06$
30	2.64859$E-07$	3.07582$E-07$	8	1.1500$E-07$	7.1237$E-08$
40	1.92790$E-08$	2.23885$E-08$	10	5.9837$E-09$	1.1113$E-08$

Table 22.
Comparison between the presented method with $c = 2$ and method in [17] for example 6.5.

method in [17]			presented method		
$2M$	e_1	e_2	N	e_1	e_2
8	$6.15E-03$	$8.22E-03$	4	$1.37E-04$	$9.34E-05$
16	$1.57E-03$	$2.13E-03$	6	$3.45E-06$	$2.02E-06$
32	$3.96E-04$	$5.42E-04$	8	$1.15E-07$	$7.12E-08$
64	$9.95E-05$	$1.36E-04$	10	$5.98E-09$	$1.12E-08$

Table 23.
Comparison between the presented method with $c = 2$ and $N = 6$ and methods in [7] and [6] for example 6.5.

s	method in [6] (m=32)		method in [7] (k=11)		presented method N=6	
	e_1	e_2	e_1	e_2	e_1	e_2
0	$8.8E$–05	0	$1.15E$–02	0	$3.45E$–06	$6.24E$–07
0.1	$1.04E$–04	$1.83E$–04	$1.34E$–02	$3.45E$–03	$9.15E$–07	$3.07E$–08
0.2	$1.19E$–04	$2.87E$–04	$1.52E$–02	$6.90E$–03	$1.16E$–06	$2.68E$–07
0.3	$1.34E$–04	$3.14E$–04	$1.17E$–02	$1.04E$–02	$1.08E$–07	$6.01E$–08
0.4	$1.50E$–04	$6.2E$–04	$1.90E$–02	$4.62E$–02	$1.16E$–06	$8.51E$–07
0.5	$1.65E$–04	$1.33E$–04	$2.08E$–02	$1.73E$–02	$6.26E$–08	$5.64E$–08
0.6	$1.80E$–04	$3.15E$–04	$2.27E$–02	$2.07E$–02	$9.75E$–07	$1.27E$–06
0.7	$1.96E$–04	$4.20E$–04	$2.46E$–02	$2.42E$–02	$2.34E$–08	$8.82E$–10
0.8	$2.11E$–04	$4.46E$–04	$2.64E$–02	$2.73E$–02	$1.09E$–06	$2.02E$–06
0.9	$2.26E$–04	$3.95E$–04	$2.83E$–02	$3.31E$–02	$6.12E$–07	$1.45E$–06

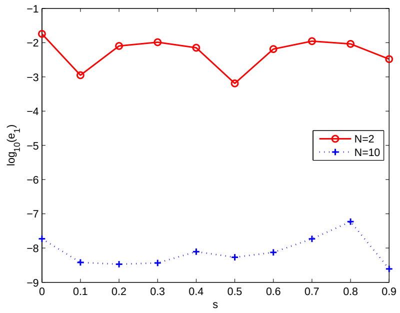

Figure 1. Error e_1 for example 6.1 with $N = 2$, $N = 10$ and $c = 1.9$

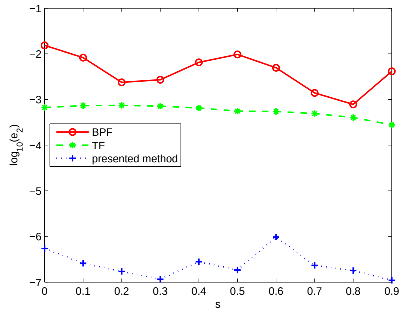

Figure 4. Comparison e_2 of the BPF method in [11] and TF method in [4] with presented method for example 6.1

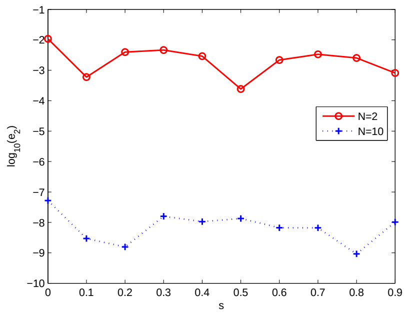

Figure 2. Error e_2 for example 6.1 with $N = 2$, $N = 10$ and $c = 1.9$

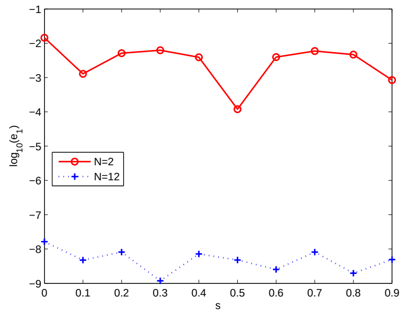

Figure 5. Error e_1 for example 6.2 with $N = 2$, $N = 12$ and $c = 1.2$

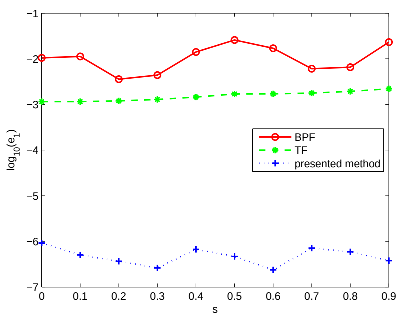

Figure 3. Comparison e_1 of the BPF method in [11] and TF method in [4] with presented method for example 6.1

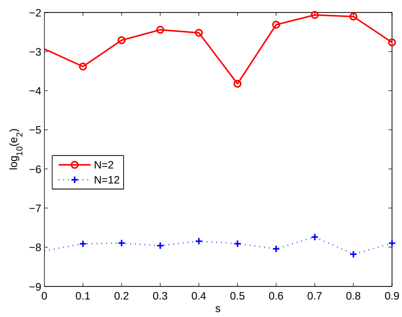

Figure 6. Error e_2 for example 6.2 with $N = 2$, $N = 12$ and $c = 1.2$

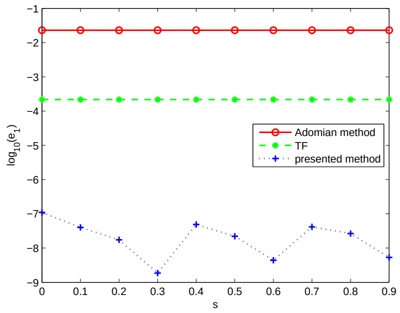

Figure 7. Comparison e_1 of the Adomian method in [7] and TF method in [4] with presented method for example 6.2

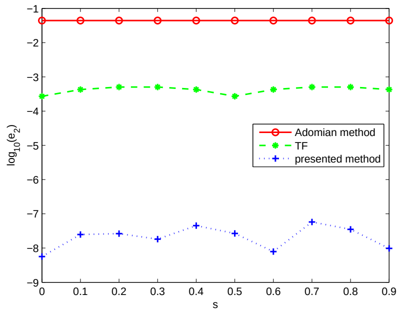

Figure 8. Comparison e_2 of the Adomian method in [7] and TF method in [4] with presented method for example 6.2

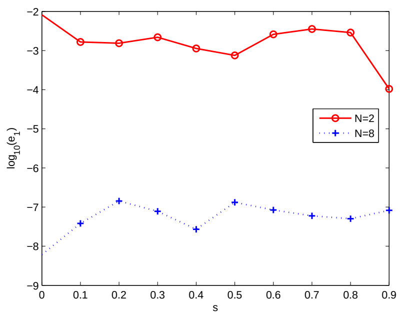

Figure 9. Error for e_1 example 6.3 with $N = 2$, $N = 8$ and $c = 1.8$

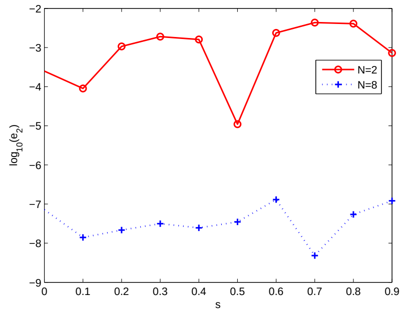

Figure 10. Error e_2 for example 6.3 with $N = 2$, $N = 8$ and $c = 1.8$

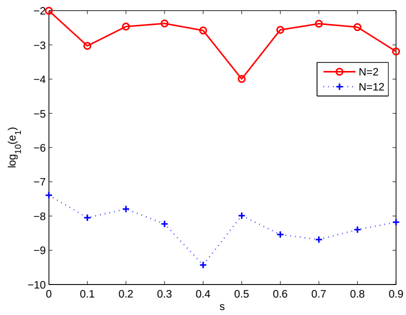

Figure 11. Error e_1 for example 6.4 with $N = 2$, $N = 12$ and $c = 1.5$

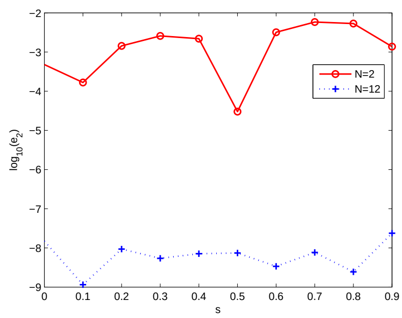

Figure 12. Error e_2 for example 6.4 with $N = 2$, $N = 12$ and $c = 1.5$

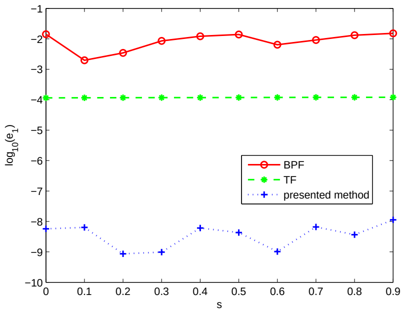

Figure 13. Comparison e_1 of the BPF method in [11] and TF method in [4] with presented method for example 6.4

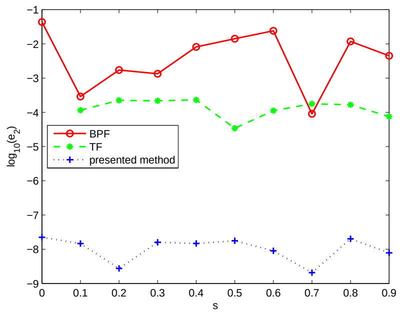

Figure 14. Comparison e_2 of the BPF method in [11] and TF method in [4] with presented method for example 6.4

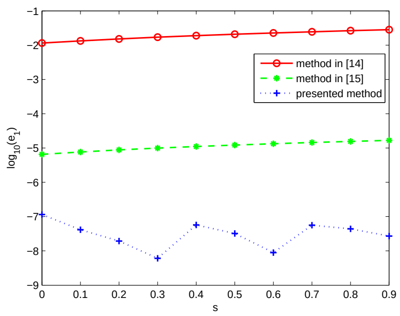

Figure 15. Comparison e_1 of the methods in [16] and [12] with presented method for example 6.5

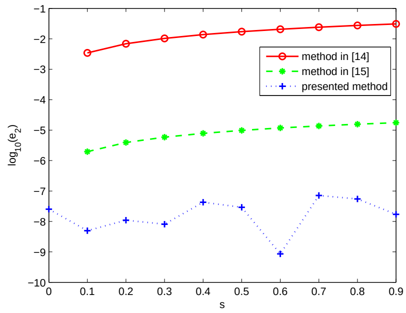

Figure 16. Comparison e_2 of the methods in [16] and [12] with presented method for example 6.5

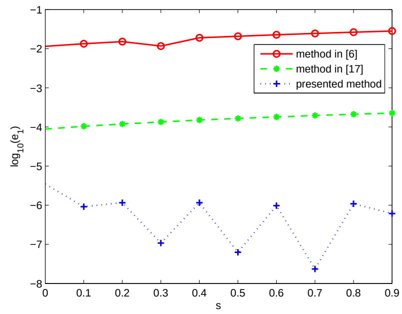

Figure 17. Comparison e_1 of the methods in [7] and [6] with presented method for example 6.5

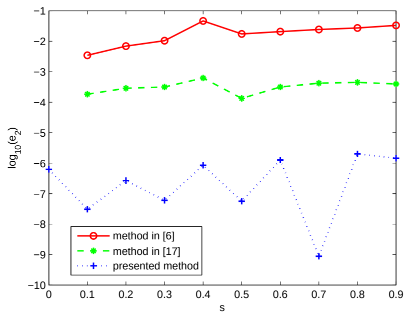

Figure 18. Comparison e_2 of the methods in [7] and [6] with presented method for example 6.5

7. Conclusion

In this paper, we presented a powerful numerical method based on multiquadric radial basis functions and zeros of Legendre-Gauss-Lobatto as collocation points for solving nonlinear Fredholm integral equations system. The stability interval for c is $[0, 2]$ and the optimal value of c in $[0, 2]$ is the same for all examples. Based on the obtained results shown in Section 6, we denoted that by increasing N and c, maximum errors are decreased. There are some methods for solving integral equations systems such as triangular functions method [4], delta functions method [15], block pulse functions method [11], Since collocation method [14] and Adomian method [7] but MQs method yields a better accuracy than all other methods, see tables 3, 4, 5, 6, 10, 11, 15, 18 and 19.

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