

## On the energy of non-commuting graphs

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**Abstract.** For given non-abelian group  $G$ , the non-commuting (NC)-graph  $\Gamma(G)$  is a graph with the vertex set  $G \setminus Z(G)$  and two distinct vertices  $x, y \in V(\Gamma)$  are adjacent whenever  $xy \neq yx$ . The aim of this paper is to compute the spectra of some well-known NC-graphs.

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### 1. Introduction

Paul Erdős was the first person who considered the non-commuting graph to answer a question on the size of the cliques of a graph in 1975, see [7]. In general, the non-commuting graph or briefly the  $NC$ -graph  $\Gamma(G)$  associated to the non-abelian group  $G$  with center  $Z(G)$ , is a simple and undirected graph with the vertex set  $G \setminus Z(G)$  in which two vertices join whenever they don't commute. For background materials about  $NC$ -graphs, we encourage the interested reader to see references [1, 2, 5, 6].

For given graph  $\Gamma$ , its characteristic polynomial is defined as  $\chi(\Gamma, \lambda) = \det(\lambda I - A)$ , where  $A$  is the adjacency matrix of  $\Gamma$ . The eigenvalues of  $\Gamma$  are the roots of this polynomial and the multi-set  $\{\lambda_1, \dots, \lambda_n\}$  of eigenvalues of  $A$  forms the spectrum of  $\Gamma$ . By this notation, the energy of  $\Gamma$  is defined as [4]:

$$\mathcal{E}(\Gamma) = \sum_{i=1}^n |\lambda_i|.$$

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Here, in Section 2, we compute the spectrum of NC-graph of group  $G$  where  $G \in \{QD_{2^n}, PSL(2, 2^k), GL(2, q)\}$  and in Section 3, at first we compute the related Laplacian eigenvalues and then we compute the energy of these graphs.

## 2. Main Results

Here, we find the spectrum of NC-graph of three following groups: the projective special linear group  $PSL(2, 2^k)$ , where  $k \geq 2$ , the general linear group  $GL(2, q)$ , where  $q = p^n$  ( $p$  is a prime integer and  $n \geq 4$ ) and the quasi-dihedral group  $QD_{2^n}$ , with the following presentations:

$$QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} \rangle.$$

**Theorem 2.1** The spectrum of NC- graph of group  $QD_{2^n}$  is

$$\{[2^{n-2} - 1 - k]^1, [-2]^{2^{n-2}-1}, [0]^{3 \times 2^{n-2}-3}, [2^{n-2} - 1 + k]^1\},$$

where  $k = \sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}$ .

**Proof.** The group  $QD_{2^n}$  is a non-abelian group of order  $2^n$  and  $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$ . For non-central elements, the centralizers are as follows

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a \rangle \text{ for } 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2}$$

and

$$C_{QD_{2^n}}(a^j b) = \{1, a^{2^{n-2}}, a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}.$$

Since the centralizers are abelian subgroups,  $QD_{2^n}$  is an AC-group and therefore  $\Gamma(QD_{2^n})$  is a multipartite graph with parts  $V_i$ 's ( $0 \leq i \leq 2^{n-2}$ ) as follows:

$$V_0 = \{a, a^2, \dots, a^{2^{n-2}-1}, a^{2^{n-2}+1}, \dots, a^{2^{n-1}-1}\},$$

$$V_j = \{a^j b, a^{j+2^{n-2}} b\}, 1 \leq j \leq 2^{n-2}.$$

By putting  $p = 2^{n-1} - 2$  and  $q = p/2$ , the adjacency matrix of  $\Gamma(QD_{2^n})$  is

$$M = \begin{pmatrix} 0_q & J_{q \times (q+1)} \\ J_{(q+1) \times q} & (J - I)_{q+1} \end{pmatrix} \otimes J_2 = A \otimes J_2.$$

The spectrum of  $J_2$  is  $\{[0]^1, [2]^1\}$  and by [3, Lemma 2], the spectrum of  $A$  is

$$\{[0]^{q-1}, [-1]^q, \left[\frac{q - \sqrt{q(5(q+1) - 1)}}{2}\right]^1, \left[\frac{q + \sqrt{q(5(q+1) - 1)}}{2}\right]^1\}.$$

Now by using [3, Theorem 1], the proof is completed. ■

**Theorem 2.2** The spectrum of NC-graph of the projective special linear group  $PSL(2, 2^k)$ , where  $k \geq 2$  is

$$\{[-u]^{t-1}, [-u+1]^u, [-u+2]^{s-1}, [0]^{(u+1)(u-2)+s(u-3)+t(u-1)}, [x_1]^1, [x_2]^1, [x_3]^1\},$$

where  $u = 2^k$ ,  $s = 2^{k-1}(2^k + 1)$ ,  $t = 2^{k-1}(2^k - 1)$  and  $x_1, x_2, x_3$  are the roots of the equation  $x^3 + (2^{k+2} - 2^{3k} - 2)x^2 + (2^{3k+1} - 2^{4k+1} + 5 \times 2^k(2^k - 1))x + 2^{4k+1} + 2^{3k} - 2^{5k} - 2^{2k+1}$ .

**Proof.** The center of non-abelian group  $PSL(2, 2^k)$  of order  $2^k(2^{2k} - 1)$  is trivial. By [1, Proposition 3.21], for non-central elements of  $PSL(2, 2^k)$  the set of centralizers is  $\{gPg^{-1}, gAg^{-1}, gBg^{-1} : g \in PSL(2, 2^k)\}$ , where  $P$  is an elementary abelian 2-group of order  $2^k$  and  $A, B$  are cyclic subgroups of order  $2^k - 1$  and  $2^k + 1$ , respectively. Let  $u = 2^k$ ,  $s = 2^{k-1}(2^k + 1)$  and  $t = 2^{k-1}(2^k - 1)$ . The number of conjugates of  $P, A$  and  $B$  in  $PSL(2, 2^k)$  are  $u + 1, s$  and  $t$  respectively. All centralizers of  $PSL(2, 2^k)$  are abelian, since it is an AC-group. Also, it is not difficult to see that they are disjoint from each other. This yields  $\Gamma(PSL(2, 2^k))$  is a multipartite graph. Let  $p = u - 1$ ,  $q = u - 2$ , then the adjacency matrix of  $\Gamma(PSL(2, 2^k))$  is the following block matrix:

$$M = \begin{pmatrix} 0_p & J_p & \cdots & J_p & J_{p \times q} & \cdots & J_{p \times q} & J_{p \times u} & \cdots & J_{p \times u} \\ J_p & & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ J_p & \cdots & J_p & 0_p & J_{p \times q} & \cdots & J_{p \times q} & J_{p \times u} & \cdots & J_{p \times u} \\ J_{q \times p} & \cdots & & J_{q \times p} & 0_q & & J_q & J_{q \times u} & \cdots & J_{q \times u} \\ \vdots & & & & & \ddots & & & & \\ J_{q \times p} & \cdots & & J_{q \times p} & J_q & \cdots & 0_q & J_{q \times u} & \cdots & J_{q \times u} \\ J_{u \times p} & \cdots & & J_{u \times p} & J_{u \times q} & \cdots & J_{u \times q} & 0_u & & J_u \\ \vdots & & & & & & & & \ddots & \\ J_{u \times p} & \cdots & & J_{u \times p} & J_{u \times q} & \cdots & J_{u \times q} & J_u & & 0_u \end{pmatrix}_{(u+1)+s+t}$$

Let  $N^r = xI + rJ$ , then

$$\det(xI - M) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(A)\det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} N_p^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & N_p^1 & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & N_p^1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & N_q^1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & N_q^1 & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & N_q^1 \end{pmatrix}_{(u+1)+s}$$

$$B = \begin{pmatrix} -N_{p \times u}^1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \\ -N_{q \times u}^1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}_{((u+1)+s) \times t},$$

$$C = \begin{pmatrix} -(u+1)J_{u \times p} & -J_{u \times p} & \cdots & -J_{u \times p} & -sJ_{u \times q} & -J_{u \times q} & \cdots & -J_{u \times q} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{t \times (u+s+1)}$$

and

$$D = \begin{pmatrix} N_u^{-t+1} & -J_u & \cdots & -J_u \\ 0 & N_u^1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & N_u^1 \end{pmatrix}_t.$$

We have

$$\det(A) = \det(N_p^1)^{u+1} \det(N_q^1)^s = x^{(u+1)(u-2)+s(u-3)} (x + (u-1))^{u+1} (x + (u-2))^s$$

and so

$$\det(D - CA^{-1}B) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x + (u-1))(x + (u-2))} x^{t(u-1)} (x + u)^{t-1},$$

where,  $x_1, x_2, x_3$ , are the roots of the following equation

$$x^3 + (2^{k+2} - 2^{3k} - 2)x^2 + (2^{3k+1} - 2^{4k+1} + 5 \times 2^k(2^k - 1))x + 2^{4k+1} + 2^{3k} - 2^{5k} - 2^{2k+1}.$$

Thus

$$\begin{aligned} \det(xI - M) &= \det(A) \det(D - CA^{-1}B) \\ &= x^{(u+1)(u-2)+s(u-3)+t(u-1)} (x + (u-1))^u (x + (u-2))^{s-1} (x + u)^{t-1} \\ &\quad (x - x_1)(x - x_2)(x - x_3). \end{aligned}$$

Hence, the result follows. ■

**Theorem 2.3** The spectrum of NC- graph of the general linear group  $GL(2, q)$ , where  $q = p^n > 2$  and  $p$  is prime integer is given by

$$\{[-2s]^{s-1}, [-k]^{t-1}, [-l]^{u-1}, [0]^{l(u-1)+s(2s-1)+t(k-1)}, [x_1]^1, [x_2]^1, [x_3]^1\},$$

where  $u = \frac{q(q+1)}{2}$ ,  $s = \frac{q(q-1)}{2}$ ,  $t = q + 1$ ,  $l = (q - 1)(q - 2)$ ,  $k = (q - 1)^2$  and  $x_1, x_2, x_3$  are the roots of the equation

$$x^3 + (-q^4 + q^3 + 4q^2 + 2)x^2 + (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - q^8 + 5q^7 - 8q^6 + 2q^5 + 7q^4 - 7q^3 + 2q^2.$$

**Proof.** The order of non-abelian group  $GL(2, q)$  is  $(q^2 - 1)(q^2 - q)$  and it is well-known that  $|Z(GL(2, q))| = q - 1$ . By [1, Proposition 3.26], all non-central elements of  $GL(2, q)$  have the set of centralizers as  $\{gDg^{-1}, gIg^{-1}, gPZ(GL(2, q))g^{-1} : g \in GL(2, q)\}$ , where  $D$  is the subgroup of  $GL(2, q)$  consisting of all diagonal matrices,  $I$  is a cyclic subgroup  $GL(2, q)$  having order  $q^2 - 1$  and  $P$  is the Sylow  $p$ -subgroup of  $GL(2, q)$  consisting of all upper triangular matrices whose diagonal entries are 1. The orders of  $D$  and  $PZ(GL(2, q))$  are  $(q - 1)^2$  and  $q(q - 1)$  respectively. The number of conjugates of  $D$ ,  $I$  and  $PZ(GL(2, q))$  in  $GL(2, q)$  are  $\frac{q(q+1)}{2}$ ,  $\frac{q(q-1)}{2}$  and  $q + 1$  respectively. Let  $u = \frac{q(q+1)}{2}$ ,  $s = \frac{q(q-1)}{2}$ ,  $t = q+1$ ,  $l = (q-1)(q-2)$  and  $k = (q-1)^2$ . Similar to the last case  $\Gamma(GL(2, q))$  is a multipartite graph and the order of its parts are  $|D| - |Z(GL(2, q))| = l$ ,  $|I| - |Z(GL(2, q))| = 2s$  and  $|PZ(GL(2, q))| - |Z(GL(2, q))| = k$ . The adjacency matrix of  $\Gamma(GL(2, q))$  has the following form:

$$M = \begin{pmatrix} 0_l & J_l & \cdots & J_l & J_{l \times k} & \cdots & J_{l \times k} & J_{l \times 2s} & \cdots & J_{l \times 2s} \\ J_l & & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ J_l & \cdots & J_l & 0_l & J_{l \times k} & \cdots & J_{l \times k} & J_{l \times 2s} & \cdots & J_{l \times 2s} \\ J_{k \times l} & \cdots & & J_{k \times l} & 0_k & & J_k & J_{k \times 2s} & \cdots & J_{k \times 2s} \\ \vdots & & & & & \ddots & & & & \\ J_{k \times l} & \cdots & & J_{k \times l} & J_k & \cdots & 0_k & J_{k \times 2s} & \cdots & J_{k \times 2s} \\ J_{2s \times l} & \cdots & & J_{2s \times l} & J_{2s \times k} & \cdots & J_{2s \times k} & 0_{2s} & & J_{2s} \\ \vdots & & & & & & & & \ddots & \\ J_{2s \times l} & \cdots & & J_{2s \times l} & J_{2s \times k} & \cdots & J_{2s \times k} & J_{2s} & & 0_{2s} \end{pmatrix}_{u+t+s}$$

Let  $N^r = xI + rJ$ , similar to the proof of last theorem we have

$$\det(xI - M) = \det(A)\det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} N_l^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & N_l^1 & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & N_l^1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & N_k^1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & N_k^1 & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & N_k^1 \end{pmatrix}_{u+t}$$

$$B = \begin{pmatrix} -N_{l \times 2s}^1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \\ -N_{k \times 2s}^1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}_{(u+t) \times s},$$

$$C = \begin{pmatrix} -uJ_{2s \times l} - J_{2s \times l} & \cdots & -J_{2s \times l} & -tJ_{2s \times k} - J_{2s \times k} & \cdots & -J_{2s \times k} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{s \times (u+t)}$$

and

$$D = \begin{pmatrix} N_{2s}^{-s+1} & -J_{2s} & \cdots & -J_{2s} \\ 0 & N_{2s}^1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & N_{2s}^1 \end{pmatrix}_s.$$

We have

$$\det(A) = \det(N_l^1)^u \det(N_k^1)^t = x^{u(l-1)+t(k-1)}(x+l)^u(x+k)^t$$

and hence

$$\det(D - CA^{-1}B) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x+l)(x+k)} x^{s(2s-1)}(x+2s)^{s-1},$$

where,  $x_1, x_2, x_3$  are the roots of the equation

$$x^3 + (-q^4 + q^3 + 4q^2 + 2)x^2 + (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - q^8 + 5q^7 - 8q^6 + 2q^5 + 7q^4 - 7q^3 + 2q^2.$$

Thus

$$\det(xI - M) = x^{l(u-1)+s(2s-1)+t(k-1)}(x+l)^{u-1}(x+k)^{t-1}(x+2s)^{s-1}(x-x_1)(x-x_2)(x-x_3).$$

■

### 3. Laplacian Eigenvalues

The aim of this section is to find the Laplacian spectrum of NC-graph of three groups  $PSL(2, 2^k)$ ,  $GL(2, q)$  and  $QD_{2^n}$ .

**Theorem 3.1** The Laplacian spectrum of NC-graph of the projective special linear group  $PSL(2, 2^k)$ , where  $k \geq 2$ , is given by

$$\{[0]^1, [u^3 - 2u - 1]^{t(u-1)}, [u^3 - 2u]^{(u-2)(u+1)}, [u^3 - 2u + 1]^{s(u-3)}, [u^3 - u - 1]^{t+s+u}\},$$

where  $u = 2^k$ ,  $s = 2^{k-1}(2^k + 1)$ ,  $t = 2^{k-1}(2^k - 1)$ .

**Proof.** By Theorem 2, the degree of each element in the centralizer of the form  $xPx^{-1}$ ,  $xAx^{-1}$  and  $xBx^{-1}$  is  $u^3 - 2u$ ,  $u^3 - 2u + 1$  and  $u^3 - 2u - 1$ , respectively. Let  $z = u^3 - 2u$ ,  $p = u - 1$ ,  $q = u - 2$ ,  $X = x - z$ ,  $L = XI - J$  and  $L' = (X - 1)I - J$ , then

$$\det(xI - (\Delta - M)) = \det(A)\det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} L_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & L_p & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & L_p & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & L'_q & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & L'_q & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & L'_q \end{pmatrix}_{(u+1)+s},$$

$$B = \begin{pmatrix} (J - (X + 1)I)_{p \times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \\ (J - (X + 1)I)_{q \times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}_{((u+1)+s) \times t},$$

$$C = \begin{pmatrix} (u + 1)J_{u \times p} & J_{u \times p} & \cdots & J_{u \times p} & sJ_{u \times q} & J_{u \times q} & \cdots & J_{u \times q} \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \end{pmatrix}_{t \times (u+s+1)}$$

and

$$D = \begin{pmatrix} ((X + 1)I + (t - 1)J)_u & J_u & \cdots & J_u \\ 0 & ((X + 1) - J)_u & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & ((X + 1) - J)_u \end{pmatrix}_t.$$

We have

$$\begin{aligned} \det(A) &= \det((XI - J)_p)^{u+1} \det(((X - 1)I - J)_q)^s \\ &= X^{(u+1)(u-2)} (X - (u - 1))^{u+1} (x - 1)^{s(u-3)} (X - 1 - (u - 2))^s \\ &= (x - u^3 + 2u)^{(u-2)(u+1)} (x - u^3 + 2u - 1)^{s(u-3)} (x - u^3 + u + 1)^{u+1+s} \end{aligned}$$

and thus

$$\begin{aligned} \det(D - CA^{-1}B) &= x(x - u^3 + 2u + 1)^{t(u-1)} (x - u^3 + u + 1)^{t-1} \\ &= x(x - u^3 + 2u)^{(u+1)(u-2)} (x - u^3 + 2u - 1)^{s(u-3)} \\ &\quad (x - u^3 + u + 1)^{s+t+u} (x - u^3 + 2u + 1)^{t(u-1)}. \end{aligned}$$

■

**Theorem 3.2** The Laplacian spectrum of the NC-graph of general linear group  $GL(2, q)$ , where  $q = p^n > 2$  and  $p$  is prime integer is

$$\{[0]^1, [(q^3 - 2q - 1)(q - 1)]^{s(2s-1)}, [(q^3 - 2q)(q - 1)]^{t(k-1)}, [(q^3 - 2q + 1)(q - 1)]^{u(l-1)}, [(q^3 - q - 1)(q - 1)]^{u+t+s-1}\},$$

$$\text{where } u = \frac{q(q+1)}{2}, s = \frac{q(q-1)}{2}, t = q + 1, l = (q - 1)(q - 2), k = (q - 1)^2.$$

**Proof.** By Theorem 3, the degree of each element in centralizer of the form  $gDg^{-1}$ ,  $gPZ(GL(2, q))g^{-1}$  and  $gIg^{-1}$  is  $(q - 1)(q^3 - 2q + 1)$ ,  $(q - 1)(q^3 - 2q)$ ,  $(q - 1)(q^3 - 2q - 1)$ , respectively. Let  $R = (q - 1)(q^3 - 2q)$ ,  $N^r = (R + r)I$ ,  $X = x - R$ ,  $L = (X - (q - 1))I - J$  and  $L' = XI - J$ . In order to find the eigenvalues of  $\Delta - M$ , we compute  $\det(xI - (\Delta - M))$  which can easily comes to the following form

$$\det(xI - (\Delta - M)) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(A)\det(D - CA^{-1}B),$$



where

$$A = \begin{pmatrix} L_l & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & L_l & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & L_l & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & L'_k & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & L'_k & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & L'_k \end{pmatrix}_{u+t},$$

$$B = \begin{pmatrix} (J - (X + (q - 1))I)_{l \times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ (J - (X + (q - 1))I)_{k \times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}_{(u+t) \times s},$$

$$C = \begin{pmatrix} uJ_{2s \times l} & J_{2s \times l} & \cdots & J_{2s \times l} & tJ_{2s \times k} & J_{2s \times k} & \cdots & J_{2s \times k} \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \end{pmatrix}_{s \times (u+t)}$$

and

$$D = \begin{pmatrix} ((X + (q - 1))I + (s - 1)J)_{2s} & J_{2s} & \cdots & & J_{2s} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & \cdots & ((X + (q - 1))I - J)_{2s} \end{pmatrix}_s.$$

We have

$$\begin{aligned}
 \det(A) &= (\det(L_l))^u (\det(L'_k))^t \\
 &= ((X - (q - 1))^{l-1} (X - (q - 1) - l))^u (X^{k-1} (X - k))^t \\
 &= (x - (q - 1)(q^3 - 2q + 1))^{u(l-1)} (x - (q - 1)(q^3 - q - 1))^{u+t} \\
 &\quad (x - (q - 1)(q^3 - 2q))^{(k-1)t}.
 \end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
 \det(D - CA^{-1}B) &= (X + R)(X + (q - 1))^{s(2s-1)} (X + (q - 1) - 2s)^{s-1} \\
 &= x(x - (q - 1)(q^3 - 2q - 1))^{s(2s-1)} (x - (q - 1)(q^3 - q - 1))^{s-1}
 \end{aligned}$$

and the result follows. ■

**Theorem 3.3** The Laplacian spectrum of the NC-graph of quasi-dihedral group  $QD_{2^n}$  is

$$\{[0]^1, [2^{n-1}]^{2^{n-1}-3}, [2^n - 4]^{2^{n-2}}, [2^n - 2]^{2^{n-2}}\}.$$

**Proof.** By Theorem 1, the NC-graph of group  $QD_{2^n}$  has the following parts

$$V_0 = \{a, a^2, \dots, a^{2^{n-2}-1}, a^{2^{n-2}+1}, \dots, a^{2^{n-1}-1}\}$$

and

$$V_j = \{a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}.$$

Each element in part  $V_0$  has degree  $2^{n-1}$  and every element in each  $V_j$  has degree  $2^n - 4$ . Let  $X = x - 2^{n-1}$  and  $X' = x - 2^n + 4$ . In order to find the Laplacian spectrum of  $\Gamma(QD_{2^n})$ , we compute  $\det(xI - (\Delta - M))$ , as follows:

$$\det(xI - (\Delta - M)) = \begin{vmatrix} XI_2 & 0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\ 0_2 & XI_2 & & 0_2 & J_2 & \cdots & J_2 \\ \vdots & & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_2 & \cdots & & XI_2 & J_2 & \cdots & J_2 \\ J_2 & \cdots & & J_2 & X'I_2 & & J_2 \\ \vdots & & & & & & \\ J_2 & \cdots & & J_2 & J_2 & \cdots & X'I_2 \end{vmatrix}.$$

It can be easily seen that:

$$\det(xI - (\Delta - M)) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(A)\det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} XI_2 & 0_2 & \cdots & 0_2 \\ 0_2 & XI_2 & & 0_2 \\ \vdots & & \ddots & \vdots \\ 0_2 & \cdots & & XI_2 \end{pmatrix}_{2p},$$

$$B = \begin{pmatrix} 2^{n-1}J_2 & J_2 & \cdots & J_2 \\ 0_2 & \cdots & & 0_2 \\ \vdots & & & \vdots \\ 0_2 & \cdots & & 0_2 \end{pmatrix}_{2p \times \frac{p+2}{2}},$$

$$C = \begin{pmatrix} (2^{n-2} - 1)J_2 & J_2 & \cdots & J_2 \\ 0_2 & \cdots & & 0_2 \\ \vdots & & & \vdots \\ 0_2 & \cdots & & 0_2 \end{pmatrix}_{\frac{p+2}{2} \times 2p}$$

and

$$D = \begin{pmatrix} X'I_2 + (2^{n-2} - 1)J_2 & J_2 & \cdots & J_2 \\ 0_2 & X'I_2 - J_2 & & 0_2 \\ \vdots & & \ddots & \vdots \\ 0_2 & \cdots & & X'I_2 - J_2 \end{pmatrix}_{\frac{p+2}{2}}.$$

We have

$$\det(A) = (\det(XI_2))^{2^{n-2}-1} = ((x - 2^{n-1})^2)^{2^{n-2}-1} = (x - 2^{n-1})^{2^{n-1}-2}$$

and

$$\det(D - CA^{-1}B) = \frac{x(x - 2^n + 2)^{2^{n-2}}(x - 2^n + 4)^{2^{n-2}-1}}{x - 2^{n-1}}.$$

Therefore

$$\begin{aligned} \det(xI - (\Delta - M)) &= \det(A)\det(D - CA^{-1}B) \\ &= x(x - 2^{n-1})^{2^{n-1}-3}(x - 2^n + 4)^{2^{n-2}}(x - 2^n + 2)^{2^{n-2}}. \end{aligned}$$

Hence, the result follows. ■

Here, we find the energy of NC-graph of three groups  $PSL(2, 2^k)$ ,  $GL(2, q)$  and  $QD_{2^n}$ .

**Theorem 3.4** The energy of NC-graph of group  $QD_{2^n}$  is

$$\mathcal{E}(\Gamma(QD_{2^n})) = 2^{n-1} - 2 + 2\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}.$$

**Proof.** By using Theorem 1, we have

$$\begin{aligned} \mathcal{E}(\Gamma(QD_{2^n})) &= (2^{n-2} - 1)|-2| + 1|2^{n-2} - 1 - k| + 1|2^{n-2} - 1 + k| \\ &= 2^{n-1} - 2 - 2^{n-2} + 1 + k + 2^{n-2} - 1 + k \\ &= 2^{n-1} - 2 + 2k = 2^{n-1} - 2 + 2\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}. \end{aligned}$$

■

**Theorem 3.5**

$$\mathcal{E}(\Gamma(PSL(2, 2^k))) = 2^{3k} - 2^{k+2} + 2 + \alpha,$$

where  $\alpha = |x_1| + |x_2| + |x_3|$  and  $x_1, x_2, x_3$  are given in Theorem 2.

**Theorem 3.6** The energy of the NC-graph of general linear group  $GL(2, q)$ , where  $q = p^n > 2$  and  $p$  is prime integer is

$$\mathcal{E}(\Gamma(PSL(2, 2^k))) = (q - 1)(q^3 - 4q + 2) + \beta,$$

where  $\beta = |x_1| + |x_2| + |x_3|$ .

**Proof.** The proof follows from Theorem 3. ■

## References

- [1] A. Abdollahi, S. Akbari, H. R. Maimani, Non-commuting graph of a group, *J. Algebra* 298 (2006) 468-492.
- [2] M. R. Darafsheh, Groups with the same non-commuting graph, *Discrete Appl. Math.* 157 (2009) 833-837.
- [3] M. Ghorbani, Z. Gharavi-AlKhansari, An algebraic study of non-commuting graphs, *Filomat* 31 (2017) 663-669.
- [4] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sect. Forschungsz Graz* 103 (1978) 1-22.
- [5] A. R. Moghaddamfar, W. J. Shi, W. Zhou, A. R. Zokayi, On the non-commuting graph associated with a finite group, *Siberian Math. J.* 46 (2005) 325-332.
- [6] G. L. Morgan, C. W. Parker, The diameter of the commuting graph of a finite group with trivial centre, *J. Algebra* 393 (2013) 41-59.
- [7] B. H. Neumann, A problem of Paul Erdős on groups, *J. Austral. Math. Soc. Ser. A*, 21 (1976) 467-472.