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On the energy of non-commuting graphs

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Abstract. For given non-abelian group G, the non-commuting (NC)-graph $\Gamma(G)$ is a graph with the vertex set $G \setminus Z(G)$ and two distinct vertices $x, y \in V(\Gamma)$ are adjacent whenever $xy \neq yx$. The aim of this paper is to compute the spectra of some well-known NC-graphs.

Keywords: Non-commuting graph, characteristic polynomial, linear group.

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1. Introduction

Paul Erdős was the first person who considered the non-commuting graph to answer a question on the size of the cliques of a graph in 1975, see [7]. In general, the non-commuting graph or briefly the $NC-$graph $\Gamma(G)$ associated to the non-abelian group G with center $Z(G)$, is a simple and undirected graph with the vertex set $G \setminus Z(G)$ in which two vertices join whenever they don't commute. For background materials about $NC-$graphs, we encourage the interested reader to see references [1, 2, 5, 6].

For given graph Γ, its characteristic polynomial is defined as $\chi(\Gamma, \lambda) = det(\lambda I - A)$, where A is the adjacency matrix of Γ. The eigenvalues of Γ are the roots of this polynomial and the multi-set $\{\lambda_1, \cdots, \lambda_n\}$ of eigenvalues of A forms the spectrum of Γ. By this notation, the energy of Γ is defined as [4]:

$$\mathcal{E}(\Gamma) = \sum_{i=1}^{n} |\lambda_i|.$$

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Here, in Section 2, we compute the spectrum of $NC-$graph of group G where $G \in \{QD_{2^n}, PSL(2,2^k), GL(2,q)\}$ and in Section 3, at first we compute the related Laplacian eigenvalues and then we compute the energy of these graphs.

2. Main Results

Here, we find the spectrum of NC-graph of three following groups: the projective special linear group $PSL(2,2^k)$, where $k \geqslant 2$, the general linear group $GL(2,q)$, where $q = p^n$ (p is a prime integer and $n \geqslant 4$) and the quasi-dihedral group QD_{2^n}, with the following presentations:

$$QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1,\, bab^{-1} = a^{2^{n-2}-1} \rangle.$$

Theorem 2.1 The spectrum of NC- graph of group QD_{2^n} is

$$\{[2^{n-2} - 1 - k]^1, [-2]^{2^{n-2}-1}, [0]^{3\times 2^{n-2}-3}, [2^{n-2} - 1 + k]^1\},$$

where $k = \sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}$.

Proof. The group QD_{2^n} is a non-abelian group of order 2^n and $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$. For non-central elements, the centralizers are as follows

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a \rangle \text{ for } 1 \leqslant i \leqslant 2^{n-1} - 1, i \neq 2^{n-2}$$

and

$$C_{QD_{2^n}}(a^j b) = \{1, a^{2^{n-2}}, a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leqslant j \leqslant 2^{n-2}.$$

Since the centralizers are abelian subgroups, QD_{2^n} is an AC-group and therefore $\Gamma(QD_{2^n})$ is a multipartite graph with parts V_i's ($0 \leqslant i \leqslant 2^{n-2}$) as follows:

$$\begin{aligned} V_0 &= \{a, a^2, \cdots, a^{2^{n-2}-1}, a^{2^{n-2}+1}, \cdots, a^{2^{n-1}-1}\}, \\ V_j &= \{a^j b, a^{j+2^{n-2}} b\},\ 1 \leqslant j \leqslant 2^{n-2}. \end{aligned}$$

By putting $p = 2^{n-1} - 2$ and $q = p/2$, the adjacency matrix of $\Gamma(QD_{2^n})$ is

$$M = \begin{pmatrix} 0_q & J_{q\times(q+1)} \\ J_{(q+1)\times q} & (J - I)_{q+1} \end{pmatrix} \otimes J_2 = A \otimes J_2.$$

The spectrum of J_2 is $\{[0]^1, [2]^1\}$ and by [3, Lemma 2], the spectrum of A is

$$\{[0]^{q-1}, [-1]^q, [\frac{q - \sqrt{q(5(q+1) - 1)}}{2}]^1, [\frac{q + \sqrt{q(5(q+1) - 1)}}{2}]^1\}.$$

Now by using [3, Theorem 1], the proof is completed. ■

Theorem 2.2 The spectrum of NC-graph of the projective special linear group $PSL(2,2^k)$, where $k \geqslant 2$ is

$$\{[-u]^{t-1}, [-u+1]^u, [-u+2]^{s-1}, [0]^{(u+1)(u-2)+s(u-3)+t(u-1)}, [x_1]^1,\ [x_2]^1,\ [x_3]^1\},$$

where $u = 2^k,\ s = 2^{k-1}(2^k+1),\ t = 2^{k-1}(2^k-1)$ and $x_1,\ x_2,\ x_3$ are the roots of the equation $x^3+(2^{k+2}-2^{3k}-2)x^2+(2^{3k+1}-2^{4k+1}+5\times 2^k(2^k-1))x+2^{4k+1}+2^{3k}-2^{5k}-2^{2k+1}$.

Proof. The center of non-abelian group $PSL(2,2^k)$ of order $2^k(2^{2k}-1)$ is trivial. By [1, Proposition 3.21], for non-central elements of $PSL(2,2^k)$ the set of centralizers is $\{gPg^{-1}, gAg^{-1}, gBg^{-1} : g \in PSL(2,2^k)\}$, where P is an elementary abelian 2-group of order 2^k and A, B are cyclic subgroups of order 2^k-1 and 2^k+1, respectively. Let $u = 2^k,\ s = 2^{k-1}(2^k+1)$ and $t = 2^{k-1}(2^k-1)$. The number of conjugates of P, A and B in $PSL(2,2^k)$ are $u+1, s$ and t respectively. All centralizers of $PSL(2,2^k)$ are abelian, since it is an AC-group. Also, it is not difficult to see that they are disjoint from each other. This yields $\Gamma(PSL(2,2^k))$ is a multipartite graph. Let $p = u-1,\ q = u-2$, then the adjacency matrix of $\Gamma(PSL(2,2^k))$ is the following block matrix:

$$M = \begin{pmatrix} 0_p & J_p & \cdots & J_p & J_{p\times q} & \cdots & J_{p\times q} & J_{p\times u} & \cdots & J_{p\times u} \\ J_p & & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ J_p & \cdots & J_p & 0_p & J_{p\times q} & \cdots & J_{p\times q} & J_{p\times u} & \cdots & J_{p\times u} \\ J_{q\times p} & \cdots & & J_{q\times p} & 0_q & & J_q & J_{q\times u} & \cdots & J_{q\times u} \\ \vdots & & & & & \ddots & & & & \\ J_{q\times p} & \cdots & & J_{q\times p} & J_q & \cdots & 0_q & J_{q\times u} & \cdots & J_{q\times u} \\ J_{u\times p} & \cdots & & J_{u\times p} & J_{u\times q} & \cdots & J_{u\times q} & 0_u & & J_u \\ \vdots & & & & & & & & \ddots & \\ J_{u\times p} & \cdots & & J_{u\times p} & J_{u\times q} & \cdots & J_{u\times q} & J_u & & 0_u \end{pmatrix}_{(u+1)+s+t}.$$

Let $N^r = xI + rJ$, then

$$det(xI - M) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = det(A)det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} N_p^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & N_p^1 & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & N_p^1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & N_q^1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & N_q^1 & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & N_q^1 \end{pmatrix}_{(u+1)+s},$$

$$B = \begin{pmatrix} -N^1_{p\times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ -N^1_{q\times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}_{((u+1)+s)\times t},$$

$$C = \begin{pmatrix} -(u+1)J_{u\times p} & -J_{u\times p} & \cdots & -J_{u\times p} & -sJ_{u\times q} & -J_{u\times q} & \cdots & -J_{u\times q} \\ 0 & & \cdots & 0 & 0 & & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & & \cdots & 0 & 0 & & \cdots & 0 \end{pmatrix}_{t\times(u+s+1)}$$

and

$$D = \begin{pmatrix} N_u^{-t+1} & -J_u & \cdots & -J_u \\ 0 & N^1_u & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & N^1_u \end{pmatrix}_t .$$

We have

$$det(A) = det(N^1_p)^{u+1} det(N^1_q)^s = x^{(u+1)(u-2)+s(u-3)}(x+(u-1))^{u+1}(x+(u-2))^s$$

and so

$$det(D - CA^{-1}B) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x+(u-1))(x+(u-2))} x^{t(u-1)}(x+u)^{t-1},$$

where, $x_1,\ x_2,\ x_3,$ are the roots of the following equation

$$x^3 + (2^{k+2} - 2^{3k} - 2)x^2 + (2^{3k+1} - 2^{4k+1} + 5\times 2^k(2^k-1))x + 2^{4k+1} + 2^{3k} - 2^{5k} - 2^{2k+1}.$$

Thus

$$\begin{aligned} det(xI - M) &= det(A)det(D - CA^{-1}B) \\ &= x^{(u+1)(u-2)+s(u-3)+t(u-1)}(x+(u-1))^u(x+(u-2))^{s-1}(x+u)^{t-1} \\ &\quad (x-x_1)(x-x_2)(x-x_3). \end{aligned}$$

Hence, the result follows. ■

Theorem 2.3 The spectrum of NC- graph of the general linear group $GL(2,q)$, where $q = p^n > 2$ and p is prime integer is given by

$$\{[-2s]^{s-1}, [-k]^{t-1}, [-l]^{u-1}, [0]^{l(u-1)+s(2s-1)+t(k-1)}, [x_1]^1,\ [x_2]^1,\ [x_3]^1\},$$

where $u = \frac{q(q+1)}{2}$, $s = \frac{q(q-1)}{2}$, $t = q+1$, $l = (q-1)(q-2)$, $k = (q-1)^2$ and $x_1,\ x_2,\ x_3$ are the roots of the equation

$$x^3 + (-q^4 + q^3 + 4q^2 + 2)x^2 + (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - q^8 + 5q^7 - 8q^6 + 2q^5 + 7q^4 - 7q^3 + 2q^2.$$

Proof. The order of non-abelian group $GL(2,q)$ is $(q^2-1)(q^2-q)$ and it is well-known that $|Z(GL(2,q))| = q-1$. By [1, Proposition 3.26], all non-central elements of $GL(2,q)$ have the set of centralizers as $\{gDg^{-1}, gIg^{-1}, gPZ(GL(2,q))g^{-1} : g \in GL(2,q)\}$, where D is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, I is a cyclic subgroup $GL(2,q)$ having order q^2-1 and P is the Sylow p-subgroup of $GL(2,q)$ consisting of all upper triangular matrices whose diagonal entries are 1. The orders of D and $PZ(GL(2,q))$ are $(q-1)^2$ and $q(q-1)$ respectively. The number of conjugates of D, I and $PZ(GL(2,q))$ in $GL(2,q)$ are $\frac{q(q+1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively. Let $u = \frac{q(q+1)}{2}$, $s = \frac{q(q-1)}{2}$, $t = q+1$, $l = (q-1)(q-2)$ and $k = (q-1)^2$. Similar to the last case $\Gamma(GL(2,q))$ is a multipartite graph and the order of its parts are $|D| - |Z(GL(2,q))| = l$, $|I| - |Z(GL(2,q))| = 2s$ and $|PZ(GL(2,q))| - |Z(GL(2,q))| = k$. The adjacency matrix of $\Gamma(GL(2,q))$ has the following form:

$$M = \begin{pmatrix} 0_l & J_l & \cdots & J_l & J_{l\times k} & \cdots & J_{l\times k} & J_{l\times 2s} & \cdots & J_{l\times 2s} \\ J_l & & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ J_l & \cdots & J_l & 0_l & J_{l\times k} & \cdots & J_{l\times k} & J_{l\times 2s} & \cdots & J_{l\times 2s} \\ J_{k\times l} & \cdots & & J_{k\times l} & 0_k & & J_k & J_{k\times 2s} & \cdots & J_{k\times 2s} \\ \vdots & & & & & \ddots & & & & \\ J_{k\times l} & \cdots & & J_{k\times l} & J_k & \cdots & 0_k & J_{k\times 2s} & \cdots & J_{k\times 2s} \\ J_{2s\times l} & \cdots & & J_{2s\times l} & J_{2s\times k} & \cdots & J_{2s\times k} & 0_{2s} & & J_2s \\ \vdots & & & & & & & & \ddots & \\ J_{2s\times l} & \cdots & & J_{2s\times l} & J_{2s\times k} & \cdots & J_{2s\times k} & J_{2s} & & 0_2s \end{pmatrix}_{u+t+s}.$$

Let $N^r = xI + rJ$, similar to the proof of last theorem we have

$$det(xI - M) = det(A)det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} N_l^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & N_l^1 & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & N_l^1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & N_k^1 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & N_k^1 & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & N_k^1 \end{pmatrix}_{u+t},$$

$$B = \begin{pmatrix} -N^1_{l\times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ -N^1_{k\times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}_{(u+t)\times s},$$

$$C = \begin{pmatrix} -uJ_{2s\times l} & -J_{2s\times l} & \cdots & -J_{2s\times l} & -tJ_{2s\times k} & -J_{2s\times k} & \cdots & -J_{2s\times k} \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \end{pmatrix}_{s\times(u+t)}$$

and

$$D = \begin{pmatrix} N^{-s+1}_{2s} & -J_{2s} & \cdots & -J_{2s} \\ 0 & N^1_{2s} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & N^1_{2s} \end{pmatrix}_s.$$

We have

$$det(A) = det(N^1_l)^u det(N^1_k)^t = x^{u(l-1)+t(k-1)}(x+l)^u(x+k)^t$$

and hence

$$det(D - CA^{-1}B) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x+l)(x+k)} x^{s(2s-1)}(x+2s)^{s-1},$$

where, $x_1,\ x_2,\ x_3$ are the roots of the equation

$$x^3 + (-q^4+q^3+4q^2+2)x^2 + (-2q^6+6q^5-q^4-13q^3+15q^2-5q)x - q^8 + 5q^7 \\ - 8q^6 + 2q^5 + 7q^4 - 7q^3 + 2q^2.$$

Thus

$$det(xI - M) = x^{l(u-1)+s(2s-1)+t(k-1)}(x+l)^{u-1}(x+k)^{t-1}(x+2s)^{s-1}(x-x_1) \\ (x-x_2)(x-x_3).$$

■

3. Laplacian Eigenvalues

The aim of this section is to find the Laplacian spectrum of NC-graph of three groups $PSL(2,2^k)$, $GL(2,q)$ and QD_{2^n}.

Theorem 3.1 The Laplacian spectrum of NC-graph of the projective special linear group $PSL(2,2^k)$, where $k \geqslant 2$, is given by

$$\{[0]^1, [u^3-2u-1]^{t(u-1)}, [u^3-2u]^{(u-2)(u+1)}, [u^3-2u+1]^{s(u-3)}, [u^3-u-1]^{t+s+u}\},$$

where $u = 2^k,\ s = 2^{k-1}(2^k+1),\ t = 2^{k-1}(2^k-1)$.

Proof. By Theorem 2, the degree of each element in the centralizer of the form xPx^{-1}, xAx^{-1} and xBx^{-1} is u^3-2u, u^3-2u+1 and u^3-2u-1, respectively. Let $z = u^3-2u,\ p = u-1,\ q = u-2,\ X = x-z,\ L = XI-J$ and $L' = (X-1)I-J$, then

$$det(xI-(\Delta-M)) = det(A)det(D-CA^{-1}B),$$

where

$$A = \begin{pmatrix} L_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & L_p & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & L_p & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & L'_q & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & L'_q & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & L'_q \end{pmatrix}_{(u+1)+s},$$

$$B = \begin{pmatrix} (J-(X+1)I)_{p\times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ (J-(X+1)I)_{q\times u} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}_{((u+1)+s)\times t},$$

$$C = \begin{pmatrix} (u+1)J_{u\times p} & J_{u\times p} & \cdots & J_{u\times p} & sJ_{u\times q} & J_{u\times q} & \cdots & J_{u\times q} \\ 0 & & \cdots & 0 & 0 & & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & & \cdots & 0 & 0 & & \cdots & 0 \end{pmatrix}_{t\times(u+s+1)}$$

and

$$D = \begin{pmatrix} ((X+1)I+(t-1)J)_u & J_u & \cdots & J_u \\ 0 & ((X+1)-J)_u & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & ((X+1)-J)_u \end{pmatrix}_t.$$

We have

$$\begin{aligned} det(A) &= det((XI-J)_p)^{u+1} det(((X-1)I-J)_q)^s \\ &= X^{(u+1)(u-2)}(X-(u-1))^{u+1}(x-1)^{s(u-3)}(X-1-(u-2))^s \\ &= (x-u^3+2u)^{(u-2)(u+1)}(x-u^3+2u-1)^{s(u-3)}(x-u^3+u+1)^{u+1+s} \end{aligned}$$

and thus

$$\begin{aligned} det(D-CA^{-1}B) &= x(x-u^3+2u+1)^{t(u-1)}(x-u^3+u+1)^{t-1} \\ &= x(x-u^3+2u)^{(u+1)(u-2)}(x-u^3+2u-1)^{s(u-3)} \\ &\quad (x-u^3+u+1)^{s+t+u}(x-u^3+2u+1)^{t(u-1)}. \end{aligned}$$

■

Theorem 3.2 The Laplacian spectrum of the NC-graph of general linear group $GL(2,q)$, where $q = p^n > 2$ and p is prime integer is

$$\{[0]^1, [(q^3-2q-1)(q-1)]^{s(2s-1)}, [(q^3-2q)(q-1)]^{t(k-1)}, [(q^3-2q+1)(q-1)]^{u(l-1)}, [(q^3-q-1)(q-1)]^{u+t+s-1}\},$$

where $u = \frac{q(q+1)}{2}$, $s = \frac{q(q-1)}{2}$, $t = q+1$, $l = (q-1)(q-2)$, $k = (q-1)^2$.

Proof. By Theorem 3, the degree of each element in centralizer of the form gDg^{-1}, $gPZ(GL(2,q))g^{-1}$ and gIg^{-1} is $(q-1)(q^3-2q+1)$, $(q-1)(q^3-2q)$, $(q-1)(q^3-2q-1)$, respectively. Let $R = (q-1)(q^3-2q)$, $N^r = (R+r)I$, $X = x-R$, $L = (X-(q-1))I-J$ and $L' = XI-J$. In order to find the eigenvalues of $\Delta - M$, we compute $det(xI-(\Delta-M))$ which can easily comes to the following form

$$det(xI-(\Delta-M)) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = det(A)det(D-CA^{-1}B),$$

where

$$A = \begin{pmatrix} L_l & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & L_l & & 0 & 0 & & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & L_l & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 & L'_k & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & L'_k & & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & \cdots & L'_k \end{pmatrix}_{u+t},$$

$$B = \begin{pmatrix} (J-(X+(q-1))I)_{l\times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ (J-(X+(q-1))I)_{k\times 2s} & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}_{(u+t)\times s},$$

$$C = \begin{pmatrix} uJ_{2s\times l} & J_{2s\times l} & \cdots & J_{2s\times l} & tJ_{2s\times k} & J_{2s\times k} & \cdots & J_{2s\times k} \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & & 0 \end{pmatrix}_{s\times(u+t)}$$

and

$$D = \begin{pmatrix} ((X+(q-1))I+(s-1)J)_{2s} & J_{2s} & \cdots & J_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & ((X+(q-1))I-J)_{2s} \end{pmatrix}_{s}.$$

We have

$$\begin{aligned} det(A) &= (det(L_l))^u (det(L'_k))^t \\ &= ((X-(q-1))^{l-1}(X-(q-1)-l))^u (X^{k-1}(X-k))^t \\ &= (x-(q-1)(q^3-2q+1))^{u(l-1)}(x-(q-1)(q^3-q-1))^{u+t} \\ &\quad (x-(q-1)(q^3-2q))^{(k-1)t}. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} det(D-CA^{-1}B) &= (X+R)(X+(q-1))^{s(2s-1)}(X+(q-1)-2s)^{s-1} \\ &= x(x-(q-1)(q^3-2q-1))^{s(2s-1)}(x-(q-1)(q^3-q-1))^{s-1} \end{aligned}$$

and the result follows. ■

Theorem 3.3 The Laplacian spectrum of the NC-graph of quasi-dihedral group QD_{2^n} is

$$\{[0]^1, [2^{n-1}]^{2^{n-1}-3}, [2^n-4]^{2^{n-2}}, [2^n-2]^{2^{n-2}}\}.$$

Proof. By Theorem 1, the NC-graph of group QD_{2^n} has the following parts

$$V_0 = \{a,\, a^2,\, \cdots,\, a^{2^{n-2}-1},\, a^{2^{n-2}+1},\, \cdots,\, a^{2^{n-1}-1}\}$$

and

$$V_j = \{a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leqslant j \leqslant 2^{n-2}.$$

Each element in part V_0 has degree 2^{n-1} and every element in each V_j has degree $2^n - 4$. Let $X = x - 2^{n-1}$ and $X' = x - 2^n + 4$. In order to find the Laplacian spectrum of $\Gamma(QD_{2^n})$, we compute $det(xI - (\Delta - M))$, as follows:

$$det(xI-(\Delta-M)) = \begin{vmatrix} XI_2 & 0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\ 0_2 & XI_2 & & 0_2 & J_2 & \cdots & J_2 \\ \vdots & & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_2 & \cdots & & XI_2 & J_2 & \cdots & J_2 \\ J_2 & \cdots & & J_2 & X'I_2 & & J_2 \\ \vdots & & & & & & \\ J_2 & \cdots & & J_2 & J_2 & \cdots & X'I_2 \end{vmatrix}.$$

It can be easily seen that:

$$det(xI-(\Delta-M)) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = det(A)det(D - CA^{-1}B),$$

where

$$A = \begin{pmatrix} XI_2 & 0_2 & \cdots & 0_2 \\ 0_2 & XI_2 & & 0_2 \\ \vdots & & \ddots & \vdots \\ 0_2 & \cdots & & XI_2 \end{pmatrix}_{2p},$$

$$B = \begin{pmatrix} 2^{n-1}J_2 & J_2 & \cdots & J_2 \\ 0_2 & & \cdots & 0_2 \\ \vdots & & & \vdots \\ 0_2 & & \cdots & 0_2 \end{pmatrix}_{2p\times\frac{p+2}{2}},$$

$$C = \begin{pmatrix} (2^{n-2}-1)J_2 & J_2 & \cdots & J_2 \\ 0_2 & & \cdots & 0_2 \\ \vdots & & & \vdots \\ 0_2 & & \cdots & 0_2 \end{pmatrix}_{\frac{p+2}{2}\times 2p}$$

and

$$D = \begin{pmatrix} X'I_2 + (2^{n-2}-1)J_2 & J_2 & \cdots & J_2 \\ 0_2 & X'I_2 - J_2 & & 0_2 \\ \vdots & & \ddots & \vdots \\ 0_2 & \cdots & & X'I_2 - J_2 \end{pmatrix}_{\frac{p+2}{2}}.$$

We have

$$det(A) = (det(XI_2))^{2^{n-2}-1} = ((x-2^{n-1})^2)^{2^{n-2}-1} = (x-2^{n-1})^{2^{n-1}-2}$$

and

$$det(D - CA^{-1}B) = \frac{x(x-2^n+2)^{2^{n-2}}(x-2^n+4)^{2^{n-2}-1}}{x-2^{n-1}}.$$

Therefore

$$\begin{aligned} det(xI - (\Delta - M)) &= det(A)det(D - CA^{-1}B) \\ &= x(x-2^{n-1})^{2^{n-1}-3}(x-2^n+4)^{2^{n-2}}(x-2^n+2)^{2^{n-2}}. \end{aligned}$$

Hence, the result follows. ■

Here, we find the energy of NC-graph of three groups $PSL(2,2^k)$, $GL(2,q)$ and QD_{2^n}.

Theorem 3.4 The energy of NC-graph of group QD_{2^n} is

$$\mathcal{E}(\Gamma(QD_{2^n})) = 2^{n-1} - 2 + 2\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}.$$

Proof. By using Theorem 1, we have

$$\begin{aligned} \mathcal{E}(\Gamma(QD_{2^n})) &= (2^{n-2}-1)|-2| + 1|2^{n-2} - 1 - k| + 1|2^{n-2} - 1 + k| \\ &= 2^{n-1} - 2 - 2^{n-2} + 1 + k + 2^{n-2} - 1 + k \\ &= 2^{n-1} - 2 + 2k = 2^{n-1} - 2 + 2\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}. \end{aligned}$$

■

Theorem 3.5

$$\mathcal{E}(\Gamma(PSL(2,2^k))) = 2^{3k} - 2^{k+2} + 2 + \alpha,$$

where $\alpha = |x_1| + |x_2| + |x_3|$ and x_1, x_2, x_3 are given in Theorem 2.

Theorem 3.6 The energy of the NC-graph of general linear group $GL(2,q)$, where $q = p^n > 2$ and p is prime integer is

$$\mathcal{E}(\Gamma(PSL(2,2^k))) = (q-1)(q^3 - 4q + 2) + \beta,$$

where $\beta = |x_1| + |x_2| + |x_3|$.

Proof. The proof follows from Theorem 3. ■

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