

Initial value problems for second order hybrid fuzzy differential equations

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Received 16 May 2017; Revised 19 August 2017; Accepted 13 September 2017.

Abstract. Usage of fuzzy differential equations (FDEs) is a natural way to model dynamical systems under possibilistic uncertainty. We consider second order hybrid fuzzy differential equations with initial value condition under generalized H-differentiability. We prove the existence and uniqueness of solution for nonlinearities satisfying a Lipschitz condition.

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Keywords: Initial value problem, hybrid fuzzy differential equations.

1. Introduction

Usage of fuzzy differential equations is a natural way to model dynamical systems under possibilistic uncertainty. Fuzzy differential equations were first formulated by Kaleva [21] and Seikkala [32] in time dependent form. Kaleva had formulated fuzzy differential equations, in terms of Hukuhara derivative [21]. Buckley and Feuring [11] have given a very general formulation of a fuzzy first order initial value problem. They first find the crisp solution, make it fuzzy and then check if it satisfies the FDE. Also, the fuzzy initial value problem have been studied by several authors [1, 2, 6–8, 10, 14, 26, 27]. Bede et al. [9, 10] introduced a more general definition of the derivative for fuzzy mappings, enlarging the class of differentiable fuzzy mappings, and Chalco-Cano et al. [12] solved these FDEs.

Hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behavior. The hybrid systems are devoted to modeling, design, and validation of

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interactive systems of computer programs and continuous systems. The differential equations containing fuzzy value functions and interaction with a discrete time controller are named as hybrid fuzzy differential equations (HFDEs) [28]. Also, the first order hybrid fuzzy initial value problem have been studied by [20, 28, 29]. In this paper, using properties of the fuzzy integral, the existence and uniqueness of solutions to the second order hybrid fuzzy differential equations will be examined.

2. Preliminaries

We denote by \mathbb{k}^n the family of all nonempty compact subsets of \mathbb{R}^n , the n-dimensional Euclidean space. If $A, B \in \mathbb{k}^n$ and $\lambda \in \mathbb{R}$, then the operations of additions and scalar multiplication are defined as

$$A + B = \{a + b | a \in A, b \in B\}, \quad \lambda A = \{\lambda a | a \in A\}.$$

If $A \in \mathbb{k}^n$ we define the ϵ -neighbourhood of A as the set

$$N(A, \epsilon) = \{t \in \mathbb{R} | d(t, A) < \epsilon\},$$

where $d(t, A) = \inf_{a \in A} \|t - a\|$ and $\|\cdot\|$ is the usual Euclidean norm on \mathbb{R}^n . The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathbb{k}(T)$ is defined by

$$\gamma(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

A fuzzy set u in an universe set T is a mapping $u : T \rightarrow [0, 1]$. We think u as assigning to each element $t \in T$ a degree of membership, $0 \leq u(t) \leq 1$. If u is a fuzzy set in \mathbb{R}^n , we define $[u]_\alpha = \{t \in \mathbb{R}^n | u(t) \geq \alpha\}$ the α -level of u , with $0 < \alpha \leq 1$. For $\alpha = 0$ the support of u is defined as $[u]_0 = \text{supp}(u)$.

We will denote by F^n the space of all compact and convex fuzzy sets on \mathbb{R}^n . If $u \in F$, then u is called a fuzzy number if the α -level set $[u]_\alpha$ is a nonempty compact interval for all $\alpha \in [0, 1]$. We can extend the Hausdorff metric γ to F^n by means

$$D(u, v) = \sup_{\alpha \in [0, 1]} \gamma([u]_\alpha, [v]_\alpha), \quad \forall u, v \in F^n,$$

and it is well known that (F^n, D) is a complete metric space.

Definition 2.1 Let $u, v \in F^n$. If there exists $w \in F^n$, such that $u = v + w$, then w is called the H-difference of u, v and it is denoted $u \ominus v$.

In this paper the sign \ominus always stands for the H-difference, and let us remark that $u \ominus v \neq u + (-1)v$. Usually we denote $u + (-1)v$ by $u - v$, while $u \ominus v$ stands for the H-difference. In what follows, we fix $T = (a, b)$, for $a, b \in \mathbb{R}$.

Definition 2.2 [31] Let $f : T \rightarrow F^n$ be a fuzzy function. We say f is differentiable at $t_0 \in T$ if there exists an element $f'(t_0) \in F^n$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h},$$

exist and are equal to $f'(t_0)$.

The above definition is a straightforward generalization of the Hukuhara differentiability of a set-value function. From proposition 4.2.8 in [15], it follows that a Hukuhara differentiable function has increasing length of support. Note that this definition of a derivative is very restrictive [9]. The authors of [9] introduced a more general definition of a derivative for a fuzzy-number-valued function. In this paper we consider the following definition [12]:

Definition 2.3 Let $f : T \rightarrow F^n$. Fix $t_0 \in T$. We say f is differentiable at t_0 , if there exists an element $f'(t_0) \in F^n$ such that

(1) for all $h > 0$ sufficiently close to 0, there exist $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \rightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

or

(2) for all $h < 0$ sufficiently close to 0, there exist $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \rightarrow 0^-} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Remark 1 [9] This definition agrees with the one introduced in [9]. Indeed, if f is differentiable in the senses (1) and (2) simultaneously, then for $h > 0$ sufficiently small, we have $f(t_0 + h) = f(t_0) + u_1, f(t_0) = f(t_0 - h) + u_2, f(t_0) = f(t_0 + h) + v_1$ and $f(t_0) = f(t_0 + h) + v_2$, with $u_1, u_2, v_1, v_2 \in F^n$. Thus, $f(t_0) = f(t_0) + (u_2 + v_1)$, i.e., $u_2 + v_1 = \chi_{\{0\}}$, which implies two possibilities: $u_2 = v_1 = \chi_{\{0\}}$ if $f'(t_0) = \chi_{\{0\}}$; or $u_2 = \chi_{\{a\}} = -v_1$, with $a \in \mathbb{R}$, if $f'(t_0) \in \mathbb{R}$. Therefore, if there exists $f'(t_0)$ in the first form (second form) with $f'(t_0)$ is not in \mathbb{R} , then $f'(t_0)$ does not exist in the second form (first form, respectively).

Remark 2 In the previous definition, case (1) corresponds to the H-derivative introduced in [31], so this differentiability concept is a generalization of the H-derivative.

Remark 3 In [9], the authors consider four cases for derivatives. Here we only consider the two first cases of Definition 5 in [9]. In the other cases, the derivative is trivial because it is reduced to a crisp element (more precisely, $f' \in \mathbb{R}$; for details see Theorem 7 in [9]).

Definition 2.4 [7] Let $f : T \rightarrow F^n$. Fix $t_0 \in X$. We say f' is differentiable, if there exists an element $f''(t_0) \in F^n$ such that

(1) for all $h > 0$ sufficiently close to 0, there exist $f'(t_0 + h) \ominus f'(t_0), f'(t_0) \ominus f'(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \rightarrow 0^+} \frac{f'(t_0 + h) \ominus f'(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f'(t_0) \ominus f'(t_0 - h)}{h} = f''(t_0),$$

(2) for all $h < 0$ sufficiently close to 0, there exist $f'(t_0 + h) \ominus f'(t_0), f'(t_0) \ominus f'(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \rightarrow 0^-} \frac{f'(t_0 + h) \ominus f'(t_0)}{h} = \lim_{h \rightarrow 0^-} \frac{f'(t_0) \ominus f'(t_0 - h)}{h} = f''(t_0).$$

Theorem 2.5 Let $f : T \rightarrow F^n$. We say f' is differentiable and denote $[f'(t)]_\alpha = [[f'(t)]_\alpha^L, [f'(t)]_\alpha^U]$, for each $\alpha \in [0, 1]$. Then

(1) If f' is differentiable in the first form (1) of definition 2.4, then $[f']_\alpha^L$ and $[f']_\alpha^U$ are differentiable functions and $[f''(t)]_\alpha = [[f''(t)]_\alpha^L, [f''(t)]_\alpha^U]$.

(2) If f' is differentiable in the first form (2) of definition 2.4, then $[f']_\alpha^L$ and $[f']_\alpha^U$ are differentiable functions and $[f''(t)]_\alpha = [[f''(t)]_\alpha^U, [f''(t)]_\alpha^L]$.

Proof. (2) If $h > 0$ and $\alpha \in [0, 1]$, then we have

$$[f'(t+h) \ominus f'(t)]_\alpha = [[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L, [f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U]$$

and, multiplying by $\frac{1}{h}$ we have

$$\begin{aligned} \frac{[f'(t+h) \ominus f'(t)]_\alpha}{h} &= \frac{1}{h} [[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L, [f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U] \\ &= \left[\frac{[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L}{h}, \frac{[f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U}{h} \right]. \end{aligned}$$

Similarly we obtain

$$\frac{[f'(t) \ominus f'(t-h)]_\alpha}{h} = \left[\frac{[f'(t)]_\alpha^L - [f'(t-h)]_\alpha^L}{h}, \frac{[f'(t)]_\alpha^U - [f'(t-h)]_\alpha^U}{h} \right].$$

Passing to the limit we have

$$[f''(t)]_\alpha = [[f''(t)]_\alpha^L, [f''(t)]_\alpha^U].$$

(2) If $h < 0$ and $\alpha \in [0, 1]$, then we have

$$[f'(t+h) \ominus f'(t)]_\alpha = [[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L, [f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U]$$

and, multiplying by $\frac{1}{h}$ we have

$$\begin{aligned} \frac{[f'(t+h) \ominus f'(t)]_\alpha}{h} &= \frac{1}{h} [[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L, [f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U] \\ &= \left[\frac{[f'(t+h)]_\alpha^U - [f'(t)]_\alpha^U}{h}, \frac{[f'(t+h)]_\alpha^L - [f'(t)]_\alpha^L}{h} \right]. \end{aligned}$$

Similarly we obtain

$$\frac{[f'(t) \ominus f'(t-h)]_\alpha}{h} = \left[\frac{[f'(t)]_\alpha^U - [f'(t-h)]_\alpha^U}{h}, \frac{[f'(t)]_\alpha^L - [f'(t-h)]_\alpha^L}{h} \right].$$

Passing to the limit we have

$$[f''(t)]_\alpha = [[f''(t)]_\alpha^U, [f''(t)]_\alpha^L],$$

and the proof is now complete. ■

3. The hybrid fuzzy differential equation

In this paper, we will study second order hybrid fuzzy differential equations IVPs

$$x''(t) = \begin{cases} f(t, x(t), x'(t), \lambda_k(x_k)) & t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \dots, \\ x(t_0) = p_{1,0}, \quad x'(t_0) = p_{2,0}, \end{cases} \quad (1)$$

where $\{t_k\}_{k=0}^\infty$ is strictly increasing and unbounded, x_k denotes $x(t_k)$, $f : [t_0, \infty) \times F \times F \times F \rightarrow F$ is continuous, and each $\lambda_k : F \rightarrow F$ is continuous. A solution to (1) will be a function $x : [t_0, \infty) \rightarrow F$ satisfying (1). For $k = 0, 1, 2, \dots$, let $f_k : [t_k, t_{k+1}] \times F \times F \rightarrow F$ where $f_k(t, x_k(t), x'_k(t)) = f(t, x_k(t), x'_k(t), \lambda_k(x_k))$. The hybrid fuzzy differential equation in (1) can be written in expanded form as

$$x''(t) = \begin{cases} x''_0(t) = f(t, x_0(t), x'_0(t), \lambda_0(x_0)) = f_0(t, x_0(t), x'_0(t)), & t_0 \leq t \leq t_1, \\ x''_1(t) = f(t, x_1(t), x'_1(t), \lambda_1(x_1)) = f_1(t, x_1(t), x'_1(t)), & t_1 \leq t \leq t_2, \\ \vdots \\ x''_k(t) = f(t, x_k(t), x'_k(t), \lambda_k(x_k)) = f_k(t, x_k(t), x'_k(t)), & t_k \leq t \leq t_{k+1}, \\ \vdots \end{cases} \tag{2}$$

and a solution of (1) can be expressed as

$$x(t) = \begin{cases} x_0(t), & t_0 \leq t \leq t_1, \\ x_1(t), & t_1 \leq t \leq t_2, \\ \vdots \\ x_k(t), & t_k \leq t \leq t_{k+1}, \\ \vdots \end{cases} \tag{3}$$

A solution $x(t)$ of (1) will be continuous and piecewise differentiable over $t_0, \infty)$ and differentiable in each interval (t_k, t_{k+1}) for $k = 0, 1, 2, \dots$

Based on definitions 2.3 and 2.4, there are two possible cases for each derivation, so there are four possible cases for the second-order derivation. The condition for the existence of solutions in each case can now be stated.

Theorem 3.1 Consider the hybrid fuzzy differential equation IVP (1) expanded as (2) where for $k = 0, 1, 2, \dots$, each $f_k : [t_k, t_{k+1}] \times F \times F \rightarrow F$ is continuous. A mapping $x : [t_0, \infty) \rightarrow F$ is a solution to the hybrid fuzzy differential equation IVP (1) if and only if $x_k(t)$ and $x'_k(t)$ are continuous and satisfy one of the following conditions for $t \in [t_k, t_{k+1}]$, $k = 0, 1, 2, \dots$:

(i) $x_k(t) = p_{2,k}(t - t_k) + \int_{t_k}^t (\int_{t_k}^z f_k(s, x_k(s), x'_k(s)) ds) dz + p_{1,k}$,

where $x'_k(t)$ and $x''_k(t)$ are (1)-differentials, or

(ii) $x_k(t) = \ominus(-1)(p_{2,k}(t - t_k) \ominus (-1) \int_{t_k}^t (\int_{t_k}^z f_k(s, x_k(s), x'_k(s)) ds) dz) + p_{1,k}$,

where $x'_k(t)$ and $x''_k(t)$ are (2)-differentials, or

(iii) $x_k(t) = (p_{2,k}(t - t_k) \ominus (-1) \int_{t_k}^t (\int_{t_k}^z f_k(s, x_k(s), x'_k(s)) ds) dz) + p_{1,k}$,

where $x'_k(t)$ is the (2)-differential and $x''_k(t)$ is the (1)-differential, or

(iv) $x_k(t) = \ominus(-1)(p_{2,k}(t - t_k) + \int_{t_k}^t (\int_{t_k}^z f_k(s, x_k(s), x'_k(s)) ds) dz) + p_{1,k}$,

where $x'_k(t)$ is the (2)-differential and $x''_k(t)$ is the (1)-differential.

Proof. Fix $k \in \{0, 1, 2, \dots\}$. Since the proof procedure is similar for all cases, we consider

case (ii) without loss of generality.

Assume the hypothesis of Theorem 3.1. Since f_k is continuous for each $k = 0, 1, 2, \dots$, it must be integrable [9]. Let $x'_k(t)$ and $x''_k(t)$ is the (2)-differential. By Theorem 3.1 [7], for

$$\begin{cases} x''_k(t) = f_k(t, x_k(t), x'_k(t)), & t \in [t_k, t_{k+1}], \\ x(t_k) = p_{1,k}, & x'(t_k) = p_{2,k}, \end{cases} \quad (4)$$

we have equivalently

$$x_k(t) = \ominus(-1)(p_{2,k}(t - t_k) \ominus (-1) \int_{t_k}^t (\int_{t_k}^z f_k(t, x_k(t), x'_k(t)) ds) dz) + p_{1,k}.$$

For each $k \in \{0, 1, 2, \dots\}$ define $x(t)$ as in (3) and the proof is now complete. \blacksquare

We denote, the space of continuous functions $x : I \rightarrow F$ by $C(I, F)$. $C(I, F)$ is a complete metric space with the distance $H(x, y) = \sup_{t \in I} \{D(x(t), y(t))e^{-\rho t}\}$, where $\rho \in \mathbb{R}$ is fixed [17]. Also, by $C(I, F)$, we denote the set of continuous functions $x : I \rightarrow F$ whose derivative $x' : I \rightarrow F$ exists as a continuous function. For $x, y \in C(I, F)$, consider the following distance:

$$H_1(x, y) = H(x, y) + H(x', y').$$

Lemma 3.2 [17] $(C(I, F), H_1)$ is a complete metric space.

Lemma 3.3 [7] For arbitrary $u, v, w \in F$ we have $D(u \ominus w, u \ominus v) = D(w, v)$.

Theorem 3.4 Let for $k = 0, 1, 2, \dots$, each $f_k : [t_k, t_{k+1}] \times F \times F \rightarrow F$ be continuous, and suppose that there exist $M_{1,k}, M_{2,k} > 0$ such that

$$D(f_k(t, x_{1,k}, x_{2,k}), f_k(t, y_{1,k}, y_{2,k})) \leq M_{1,k}D(x_{1,k}, y_{1,k}) + M_{2,k}D(x_{2,k}, y_{2,k})$$

for all $t \in [t_k, t_{k+1}]$, $x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k} \in F$. Then the hybrid fuzzy differential equation IVP (1) has a unique solution on $[t_0, \infty)$ for each case.

Proof. Fix $k \in \{0, 1, 2, \dots\}$. Since the proof procedure is similar for all cases, we consider case (ii) without loss of generality.

Let $I_k = [t_k, t_{k+1}]$, and consider the complete metric space $(C(I_k, F), H_1)$. Define the operator

$$\begin{aligned} G_k : C(I_k, F) &\rightarrow C(I_k, F) \\ x &\rightarrow G_k x, \end{aligned}$$

given by

$$\begin{aligned} (G_k x)(t) &= \ominus(-1)(p_{2,k}(t - t_k) \ominus (-1) \int_{t_k}^t (\int_{t_k}^z f_k(s, x_k(s), x'_k(s)) ds) dz) + p_{1,k}, \\ (G_k x)'(t) &= p_{2,k} \ominus (-1) \int_{t_k}^t f_k(s, x_k(s), x'_k(s)) ds, \quad t \in [t_k, t_{k+1}]. \end{aligned}$$

We can choose a value for ρ_k big enough such that G_k is a contraction, so that by Theorem 2.1 [17] and Theorem 3.2 [7] Banach fixed point theorem provides the existence of a unique fixed point for G_k , that is a unique solution for (4) for all $k \in \{0, 1, 2, \dots\}$.

For each $k \in \{0, 1, 2, \dots\}$ define $x(t)$ as in (3) Banach fixed point theorem provides the existence of a unique fixed point for (1). ■

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