Characterization of \((\delta, \varepsilon)\)-double derivations on rings and algebras

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Abstract. This paper is an attempt to prove the following result:
Let \(n > 1\) be an integer and let \(R\) be a \(n!\)-torsion-free ring with the identity element. Suppose that \(d, \delta, \varepsilon\) are additive mappings satisfying
\[
d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-1-j}\left(\delta(x)x^{j-i}\varepsilon(x) + \varepsilon(x)x^{j-i}\delta(x)\right)x^{i-1}
\]
for all \(x \in R\). If \(\delta(e) = \varepsilon(e) = 0\), then \(d\) is a Jordan \((\delta, \varepsilon)\)-double derivation. In particular, if \(R\) is a semiprime algebra and further, \(\delta(x)e + \varepsilon(x)d(x) = \frac{1}{2}\left[(\delta\varepsilon + \varepsilon\delta)(x^2) - (\delta\varepsilon(x) + \varepsilon\delta(x))x - x(\delta\varepsilon(x) + \varepsilon\delta(x))\right]\) holds for all \(x \in R\), then \(d - \frac{\delta(x) + \varepsilon(x)}{2}\) is a derivation on \(R\).

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1. Introduction and Preliminaries

In this paper, $\mathcal{R}$ represents an associative unital ring with center $Z(\mathcal{R})$ such that $e$ will show its unit element and $x^0 = e$ for all $x \in \mathcal{R}$. The center of $\mathcal{R}$ is

$$Z(\mathcal{R}) = \{ x \in \mathcal{R} | xy = yx, \forall y \in \mathcal{R} \}.$$

A ring $\mathcal{R}$ is $n$-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in \mathcal{R}$ implies $x = 0$. Like most authors, we denote the commutator $xy - yx$ by $[x, y]$ for all pair $x, y \in \mathcal{R}$. Recall that $\mathcal{R}$ is prime if $x\mathcal{R}y = \{0\}$ implies $x = 0$ or $y = 0$, and is semiprime if $x\mathcal{R}x = \{0\}$ implies $x = 0$.

As well, the above-mentioned statements are considered for algebras. An additive mapping $d : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is an arbitrary ring, is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in \mathcal{R}$, and is called a Jordan derivation when $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in \mathcal{R}$. A derivation $d$ is inner if there exists $a \in \mathcal{R}$ such that $d(x) = [a, x]$ holds for all $x \in \mathcal{R}$. Every derivation is a Jordan derivation. The converse is not true in general. A classical result of Herstein [7], asserts that any Jordan derivation on a 2-torsion free prime ring (prime ring with characteristic different from two) is a derivation. A brief proof of Herstein’s result can be found Brešar and Vukman [2]. Cusack [5] generalized Herstein’s result to 2-torsion free prime rings (see also [1] for an alternative proof). A series of results related to derivations on prime and semiprime rings, can be found in [3, 4, 10, 11, 13].

Mirzavaziri and Omidvar Tehrani [9] defined a $(\delta; \varepsilon)$-double derivation as follows. Suppose that $\delta, \varepsilon : \mathcal{R} \to \mathcal{R}$ are two additive mappings. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a $(\delta; \varepsilon)$-double derivation, when for all $x, y \in \mathcal{R}$,

$$d(xy) = d(x)y + xd(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y).$$

Similar to the Jordan derivations, an additive mapping $d$ is called a Jordan $(\delta; \varepsilon)$-double derivation if

$$d(x^2) = d(x)x + xd(x) + \delta(x)\varepsilon(x) + \varepsilon(x)\delta(x)$$

holds for all $x \in \mathcal{R}$. Clearly, this notion includes ordinary derivation (Jordan derivation), when $\delta(x)\varepsilon(y) + \varepsilon(x)\delta(y) = 0$ for all $x, y \in \mathcal{R}$.

Vukman and Ulbl [12] considered the following result: Let $n > 1$ be an integer and let $\mathcal{R}$ be an $n!$-torsion free semiprime ring with identity element. Suppose that there exists an additive mapping $D : \mathcal{R} \to \mathcal{R}$ such that

$$D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x)x^{j-1}$$

is fulfilled for all $x \in \mathcal{R}$. In this case, $D$ is a derivation. In [8], Hosseini presented some characterizations of $\delta$-double derivations on rings and algebras. In this note, by methods mentioned above, we prove notions of a general characterization of $(\delta; \varepsilon)$-double derivations on rings and algebra by some equations. Let $n > 1$ be an integer and $d, \delta, \varepsilon :$
$\mathcal{R} \to \mathcal{R}$ be additive mappings such that

$$d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-j-i} \left( \delta(x)x^{j-i} + \varepsilon(x)x^{j-i} \delta(x) \right)x^{i-1}$$

is fulfilled for all $x \in \mathcal{R}$. If $\mathcal{R}$ is a unital $n$-torsion free ring and $\delta(e) = \varepsilon(e) = 0$, then $d$ is a Jordan $(\delta, \varepsilon)$-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further,

$$\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x) = \frac{1}{2} \left[ (\delta \varepsilon + \varepsilon \delta)(x^2) - (\delta \varepsilon(x) + \varepsilon \delta(x))x - x(\delta \varepsilon(x) + \varepsilon \delta(x)) \right],$$

for all $x \in \mathcal{R}$, then $d - \frac{\delta \varepsilon + \varepsilon \delta}{2}$ is an ordinary derivation on $\mathcal{R}$.

2. Main results

Let $\mathcal{A}$ be an algebra, and $\delta, \varepsilon$ be two additive mappings on $\mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called a $(\delta, \varepsilon)$-double derivation, if

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$$

for every pair $a, b \in \mathcal{A}$. Similar to Jordan derivation, an additive mapping $d$ is called Jordan $(\delta, \varepsilon)$-double derivation if

$$d(a^2) = d(a)a + ad(a) + \delta(a)\varepsilon(a) + \varepsilon(a)\delta(a)$$

holds for all $a \in \mathcal{A}$. By a $\delta$-double derivation we mean a $(\delta, \delta)$-double derivation.

Also, an additive mapping $D : \mathcal{A} \to \mathcal{A}$ is called a $(\delta, \varepsilon)$-double left derivation, if

$$D(ab) = aD(b) + bD(a) + \delta(a)\varepsilon(b) + \delta(b)\varepsilon(a)$$

for each pair $a, b \in \mathcal{A}$ and is called a Jordan $(\delta, \varepsilon)$-double left derivation in case $D(a^2) = 2aD(a) + 2\delta(a)\varepsilon(a)$ is fulfilled for all $a \in \mathcal{A}$.

**Lemma 2.1** If $\delta, \varepsilon$ are derivations on $\mathcal{A}$, then each $(\delta, \varepsilon)$-double derivation $d : \mathcal{A} \to \mathcal{A}$ is of the form $d = \frac{\delta \varepsilon + \varepsilon \delta}{2} + \gamma$, where $\gamma : \mathcal{A} \to \mathcal{A}$ is a derivation.

**Proof.** Suppose that $\gamma = d - \frac{\delta \varepsilon + \varepsilon \delta}{2}$. It is routine to show that $\gamma$ is a derivation. ■

**Lemma 2.2** Let $\mathcal{A}$ be a semiprime algebra and let $\delta, \varepsilon$ be Jordan derivations on $\mathcal{A}$. If $d : \mathcal{A} \to \mathcal{A}$ is a Jordan $(\delta, \varepsilon)$-double derivation, then $d$ is a $(\delta, \varepsilon)$-double derivation.

**Proof.** By using Lemma 2.1 and Theorem 1 of [1], we deduce that $d = \frac{\delta \varepsilon + \varepsilon \delta}{2} + \gamma$, where $\gamma : \mathcal{A} \to \mathcal{A}$ is a derivation. Therefore,

$$d(xy) = \left( \frac{\delta \varepsilon + \varepsilon \delta}{2} \right)(xy) + \gamma(xy)$$

$$= \left( \frac{\delta \varepsilon + \varepsilon \delta}{2} + \gamma \right)(xy) + x\left( \frac{\delta \varepsilon + \varepsilon \delta}{2} + \gamma \right)(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y)$$

$$= d(x)y + xd(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y), \quad x, y \in \mathcal{R}.$$ 

Hence $d$ is a $(\delta, \varepsilon)$-double derivation. ■
Theorem 2.3 Let $n > 1$ be an integer and let $\mathcal{R}$ be a $n!$-torsion-free ring with the identity element. Suppose that $d, \delta, \varepsilon$ are additive mappings satisfying

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-j-i} \left( \delta(x) x^{j-i} \varepsilon(x) + \varepsilon(x) x^{j-i} \delta(x) \right) x^{i-1}$$

(2)

for all $x \in \mathcal{R}$. If $\delta(c) = \varepsilon(c) = 0$, then $d$ is a Jordan $(\delta, \varepsilon)$-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) = \frac{1}{2} \left( \delta \varepsilon + \varepsilon \delta \right) (x^2) - (\varepsilon \delta(x) + \delta \varepsilon(x)) x - x (\delta \varepsilon(x) + \varepsilon \delta(x)) \right)$ holds for all $x \in \mathcal{R}$, then $d - \frac{\delta \varepsilon + \varepsilon \delta}{2}$ is an ordinary derivation on $\mathcal{R}$.

Proof. Let $c$ be an element of $Z(\mathcal{R})$ so that $d(c), \delta(c),$ and $\varepsilon(c)$ are zero. By putting $x + c$ instead of $x$ in (1), we obtain

$$\sum_{i=0}^{n} \binom{n}{i} d(x^{n-i}c^i)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}c^i d(x) + \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i}c^i d(x)(x + c) + \cdots$$

$$+ d(x) \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}c^i + \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i}c^i \left[ \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right]$$

$$+ \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i}c^i \left[ \delta(x) \left( (x + c) \varepsilon(x) + \varepsilon(x)(x + c) \right) \right]$$

$$+ \varepsilon(x) \left( (x + c) \delta(x) + \delta(x)(x + c) \right) \right] + \cdots + \left[ \delta(x) \left( \sum_{i=0}^{n-3} \binom{n-2}{i} x^{n-2-i}c^i \varepsilon(x) \right)$$

$$+ \varepsilon(x) \left( \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i}c^i \delta(x) + \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i}c^i \delta(x)(x + c) + \cdots$$

$$+ (x + c) \delta(x) \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i}c^i + \delta(x) \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i}c^i \right],$$

for all $x \in \mathcal{R}$. Using (1) and collecting together terms of the above-mentioned relations involving the same number of factors of $c$, we achieve

$$\sum_{i=1}^{n-1} f_i(x, c) = 0 \quad x \in \mathcal{R},$$

(3)
where

\[ f_i(x, c) = \binom{n}{i} d(x^{n-i}c^i) - \binom{n-1}{i} x^{n-1-i}c^i d(x) - \binom{n-2}{i} x^{n-2-i}c^i d(x) + \sum_{i=1}^{n-1} \left( \binom{n-2}{i} \right) x^{n-2-i}c^i d(x) + \left( \binom{n-3}{i} \right) x^{n-3-i}c^i d(x) \]

Having replaced \( c, 2c, 3c, \ldots, (n-1)c \) instead of \( c \) in (2), we obtain a system of \( n-1 \) homogeneous equations as follows:

\[
\begin{align*}
\sum_{i=1}^{n-1} f_i(x, c) &= 0 \\
\sum_{i=1}^{n-1} f_i(x, 2c) &= 0 \\
\sum_{i=1}^{n-1} f_i(x, 3c) &= 0 \\
&\vdots \\
\sum_{i=1}^{n-1} f_i(x, (n-1)c) &= 0
\end{align*}
\]

It is observed that the coefficient matrix of the above system is equal to the following matrix:

\[
A = \begin{bmatrix}
\binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\
2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1}
\end{bmatrix}
\]
Since the determinant of $A$ is different from zero, it follows that the system has only a trivial solution. In particular, $f_{n-2}(x, e) = 0$, that is
\[
0 = \left( \frac{n}{n-2} \right) d(x^2) - \left( \frac{n-1}{n-2} \right) xd(x) - \left( \frac{n-2}{n-3} \right) d(x) x - \cdots - \left( \frac{n-1}{n-2} \right) d(x) x
- \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right) - 2 \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right) - \cdots - (n-1) \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right),
\]
for all $x \in \mathcal{R}$. The above equation reduces to
\[
\frac{n(n-1)}{2} d(x^2) = (n-1) xd(x) + d(x) x + (n-2) x d(x) + \cdots + (n-1) d(x) x + \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right)
+ 2 \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right) + \cdots + (n-1) \left( \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right),
\]
for all $x \in \mathcal{R}$. Thus, we have
\[
\frac{n(n-1)}{2} d(x^2) = \left( \sum_{i=1}^{n-1} i \right) \left( d(x) x + xd(x) + \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) \right),
\]
for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is $n!$-torsion free, it follows from (3) that
\[
d(x^2) = d(x) x + xd(x) + \delta(x) \varepsilon(x) + \varepsilon(x) \delta(x),
\]
for all $x \in \mathcal{R}$. In other words, $d$ is a Jordan $(\delta, \varepsilon)$-double derivation. Now suppose that $\mathcal{R}$ is a semiprime algebra and further $\delta(x) \varepsilon(x) + \varepsilon(x) \delta(x) = \frac{1}{2} \left( \delta \varepsilon + \varepsilon \delta \right)(x^2) - (\delta \varepsilon(x) + \varepsilon \delta(x)) x - x (\delta \varepsilon(x) + \varepsilon \delta(x))$ for all $x \in \mathcal{R}$. This equation with (4) imply that $d - \frac{\delta \varepsilon + \varepsilon \delta}{2}$ is a Jordan derivation. It follows from Theorem 1 of [1] that $d - \frac{\delta \varepsilon + \varepsilon \delta}{2}$ is an ordinary derivation on $\mathcal{R}$. 

Using the above theorem, we obtain the following result.

**Corollary 2.4** Let $n > 1$ be an integer, $\mathcal{A}$ be a unital semiprime algebra. Suppose that $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ are additive mappings satisfying
\[
d(a^n) = \sum_{j=1}^{n} a^{n-j} d(a) a^{j-1} + \sum_{j=1}^{n-1} a^{n-1-j} \left( \delta(a) a^{j-1} + \varepsilon(a) a^{j-2} \delta(a) \right) a^{j-1}
\]
for all $a \in \mathcal{A}$. If $\delta, \varepsilon$ are two derivations, then $d$ is a $(\delta, \varepsilon)$-double derivation.

**Proof.** According to the previous theorem and Lemma 2.2, $d$ is a $(\delta, \varepsilon)$-double derivation.

**Theorem 2.5** Let $\mathcal{A}$ be a unital Banach algebra and $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be additive mappings satisfying
\[
d(a) = -a d(a^{-1}) a - a \delta(a^{-1}) \varepsilon(a) - a \varepsilon(a^{-1}) \delta(a)
\]
for all invertible element $a \in \mathcal{A}$. If $\delta(a) = -a \delta(a^{-1}) a$ and $\varepsilon(a) = -a \varepsilon(a^{-1}) a$ for all invertible element $a$, then $d$ is a Jordan $(\delta, \varepsilon)$-double derivation.
In particular, if $\mathcal{A}$ is semiprime and $\delta(b)\varepsilon(b) + \varepsilon(b)\delta(b) = \frac{1}{2}\left[(\delta\varepsilon + \varepsilon\delta)(b^2) - (\delta\varepsilon(b) + \\
\varepsilon\delta(b))b - b(\delta\varepsilon(b) + \varepsilon\delta(b))\right]$ holds for all $b \in \mathcal{A}$, then $d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$ is an ordinary derivation.

**Proof.** Let $b$ be an arbitrary element from $\mathcal{A}$ and let $n$ be a positive number so that $\|\frac{b}{n}\| < 1$. It is evident that $\|\frac{b}{n}\| < 1$, too. If we consider $a = ne + b$, then we have $\frac{a}{n} = e - \frac{b}{n}$. Since $\|\frac{b}{n}\| < 1$, it follows from Theorem 1.4.2 of [6] that $e - \frac{b}{n}$ is invertible and consequently, $a$ is invertible. Similarly, we can show that $e - a$ is also an invertible element of $\mathcal{A}$. In the following, we use the well-known Hua identity $a^2 = a - (a^{-1} + (e - a)^{-1})^{-1}$.

Applying the relation (5) it is obtained that

\[
d(a^2) = d(a) - d((a^{-1} + (e - a)^{-1})^{-1})
\]

\[
= d(a) + (a^{-1} + (e - a)^{-1})^{-1}d(a^{-1} + (e - a)^{-1})(a^{-1} + (e - a)^{-1})^{-1}
\]

\[
+ (a^{-1} + (e - a)^{-1})^{-1}\delta(a^{-1} + (e - a)^{-1})\varepsilon((a^{-1} + (e - a)^{-1})^{-1})
\]

\[
+ (a^{-1} + (e - a)^{-1})^{-1}\varepsilon(a^{-1} + (e - a)^{-1})\delta((a^{-1} + (e - a)^{-1})^{-1})
\]

\[
= d(a) + a(e-a)d(a^{-1})a(e-a) + a(e-a)d((e-a)^{-1})a(e-a)
\]

\[
+ a(e-a)\delta(a^{-1} + (e - a)^{-1})\varepsilon(a(e-a)) + a(e-a)\varepsilon(a^{-1} + (e - a)^{-1})\delta(a(e-a))
\]

\[
= d(a) - a(e-a)a^{-1}d(a)a^{-1}a(e-a) - a(e-a)a^{-1}\delta(a)(a^{-1})a(e-a)
\]

\[
- a(e-a)a^{-1}\varepsilon(a)\delta(a^{-1})a(e-a) + a(e-a)(e-a)^{-1}d(a)(e-a)^{-1}a(e-a)
\]

\[
+ a(e-a)(e-a)^{-1}\delta(a)\varepsilon((e-a)^{-1})a(e-a)
\]

\[
+ a(e-a)(e-a)^{-1}\varepsilon(a)\delta((e-a)^{-1})a(e-a)
\]

\[
+ a(e-a)a^{-1}\varepsilon(a)a^{-1}(a - a^2)\delta(a^{-1})(a - a^2)
\]

\[
+ a(e-a)a^{-1}\delta(a)a^{-1}(a - a^2)\varepsilon(a^{-1})(a - a^2)
\]

\[
- a(e-a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}(a - a^2)\delta(a^{-1})(a - a^2)
\]

\[
- a(e-a)(e-a)^{-1}\delta(a)(e-a)^{-1}(a - a^2)\varepsilon(a^{-1})(a - a^2)
\]

\[
+ a(e-a)a^{-1}\varepsilon(a)a^{-1}(a - a^2)\delta(a^{-1})(a - a^2)
\]

\[
+ a(e-a)a^{-1}\delta(a)\varepsilon((e-a)^{-1})(a - a^2)
\]

\[
- a(e-a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}(a - a^2)\delta(a^{-1})(a - a^2)
\]

\[
- a(e-a)(e-a)^{-1}\delta(a)(e-a)^{-1}(a - a^2)\varepsilon(a^{-1})(a - a^2)
\]

\[
= ad(a) + d(a)a - \delta(a)a^{-1}a + \delta(a)a^{-1}a^2 + a\delta(a)\varepsilon(a^{-1})a - a\delta(a)\varepsilon(a^{-1})a^2
\]

\[
- \varepsilon(a)\delta(a^{-1})a + \varepsilon(a)\delta(a^{-1})a^2 + a\varepsilon(a)\delta(a^{-1})a - a\varepsilon(a)\delta(a^{-1})a^2
\]

\[
+ a\delta(a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}a(e-a) + a\delta(a)(e-a)^{-1}\delta(a)(e-a)^{-1}(e-a)(e-a)^{-1}a(e-a)
\]

\[
- \delta(a)a^{-1}\varepsilon(a) + \delta(a)a^{-1}\varepsilon(a)a + \delta(a)\varepsilon(a) + a\delta(a)a^{-1}\varepsilon(a) - a\delta(a)a^{-1}\varepsilon(a)a - a\delta(a)\varepsilon(a)
\]

\[
+ a\delta(a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}a - a\delta(a)(e-a)^{-1}\delta(a)(e-a)^{-1}a
\]

\[
- a\varepsilon(a)(e-a)^{-1}a\delta(a) + \varepsilon(a)a^{-1}\delta(a)a + \varepsilon(a)\delta(a) + a\varepsilon(a)a^{-1}\delta(a) - a\varepsilon(a)a^{-1}\delta(a)a
\]

\[
- a\varepsilon(a)\delta(a) + a\varepsilon(a)(e-a)^{-1}\delta(a) - a\varepsilon(a)(e-a)^{-1}\delta(a)a - a\varepsilon(a)(e-a)^{-1}a\delta(a).
\]
Putting $\delta(x) = -x\delta(x^{-1})x$ and $\varepsilon(x) = -x\varepsilon(x^{-1})x$ Thus

$$d(a^2) = ad(a) + d(a)a + \delta(a)\varepsilon(a) - a\delta(a)\varepsilon(a) + a\delta(a)(e - a)^{-1}\varepsilon(a) + \varepsilon(a)\delta(a)$$

$$- a\delta(a)(e - a)^{-1}a\varepsilon(a) - a\varepsilon(a)\delta(a) + a\varepsilon(a)(e - a)^{-1}\delta(a) - a\varepsilon(a)(e - a)^{-1}a\delta(a)$$

$$= ad(a) + d(a)a + \delta(a)\varepsilon(a) + \varepsilon(a)\delta(a).$$

Note that $\delta(e) = -e\delta(e^{-1})e$ and $\varepsilon(e) = -e\varepsilon(e^{-1})e$. Hence $\delta(e) = \varepsilon(e) = 0$ and it implies that $d(e) = 0$. Having put $a = ne + b$ in the previous equation, we obtain

$$d(n^2 + 2nb + b^2) = d(b)(ne + b) + (ne + b)d(b) + \delta(b)\varepsilon(b) + \varepsilon(b)\delta(b).$$

We, therefore, have $d(b^2) = bd(b) + d(b)b + \delta(b)\varepsilon(b) + \varepsilon(b)\delta(b)$ for all $b \in A$, i.e. $d$ is a Jordan $(\delta, \varepsilon)$-derivation. Now, assume that $\delta(b)\varepsilon(b) + \varepsilon(b)\delta(b) = \frac{1}{2}[(\delta\varepsilon + \varepsilon\delta)(b^2) - (\delta\varepsilon)(b) + \varepsilon\delta(b))b - \delta\varepsilon(b) + \varepsilon\delta(b)]$ for all $b \in A$. Hence,

$$d(b^2) = bd(b) + d(b)b + \frac{1}{2}[(\delta\varepsilon + \varepsilon\delta)(b^2) - (\delta\varepsilon)(b) + \varepsilon\delta(b))b - \delta\varepsilon(b) + \varepsilon\delta(b)]$$

equivalently we have,

$$\left(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\right)(b^2) = b\left(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\right)(b) + \left(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\right)(b)b.$$  

It means that $d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$ is a Jordan derivation. At this point, Theorem 1 of [1] completes the argument.

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