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Characterization of (δ, ε)-double derivations on rings and algebras

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Abstract. This paper is an attempt to prove the following result:
Let $n > 1$ be an integer and let $\mathcal{R}$ be a $n!$-torsion-free ring with the identity element. Suppose that d, δ, ε are additive mappings satisfying

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-1-j} \Big(\delta(x) x^{j-i} \varepsilon(x) + \varepsilon(x) x^{j-i} \delta(x) \Big) x^{i-1} \tag{1}$$

for all $x \in \mathcal{R}$. If $\delta(e) = \varepsilon(e) = 0$, then d is a Jordan (δ, ε)-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x) = \frac{1}{2}\Big[(\delta\varepsilon + \varepsilon\delta)(x^2) - (\delta\varepsilon(x) + \varepsilon\delta(x))x - x(\delta\varepsilon(x) + \varepsilon\delta(x))\Big]$ holds for all $x \in \mathcal{R}$, then $d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$ is a derivation on $\mathcal{R}$.

Keywords: Derivation, Jordan derivation, (δ, ε)-double derivation, n-torsion free semiprime ring.

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1. Introduction and Preliminaries

In this paper, $\mathcal{R}$ represents an associative unital ring with center $Z(\mathcal{R})$ such that e will show its unit element and $x^0 = e$ for all $x \in \mathcal{R}$. The center of $\mathcal{R}$ is

$$Z(\mathcal{R}) = \{x \in \mathcal{R} | xy = yx, \forall y \in \mathcal{R}\}.$$

A ring $\mathcal{R}$ is $n-torsion$ free, where $n > 1$ is an integer, in case $nx = 0$, $x \in \mathcal{R}$ implies $x = 0$. Like most authors, we denote the commutator $xy - yx$ by $[x, y]$ for all pair $x, y \in \mathcal{R}$. Recall that $\mathcal{R}$ is prime if $x\mathcal{R}y = \{0\}$ implies $x = 0$ or $y = 0$, and is semiprime if $x\mathcal{R}x = \{0\}$ implies $x = 0$.

As well, the above-mentioned statements are considered for algebras. An additive mapping $d : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is an arbitrary ring, is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in \mathcal{R}$, and is called a Jordan derivation when $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in \mathcal{R}$. A derivation d is inner if there exists $a \in \mathcal{R}$ such that $d(x) = [a, x]$ holds for all $x \in \mathcal{R}$. Every derivation is a Jordan derivation. The converse is not true in general. A classical result of Herstein [7], asserts that any Jordan derivation on a 2- torsion free prime ring (prime ring with characteristic different from two) is a derivation. A brief proof of Herstein's result can be found Brešar and Vukman [2]. Cusack [5] generalized Herstein's result to 2-torsion free prime rings (see also [1] for an alternative proof). A series of results related to derivations on prime and semiprime rings, can be found in [3, 4, 10, 11, 13].

Mirzavaziri and Omidvar Tehrani [9] defined a (δ, ε)-double derivation as follows. Suppose that $\delta, \varepsilon : \mathcal{R} \to \mathcal{R}$ are two additive mappings. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a (δ, ε)-double derivation, when for all $x, y \in \mathcal{R}$,

$$d(xy) = d(x)y + xd(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y).$$

Similar to the Jordan derivations, an additive mapping d is called a Jordan (δ, ε)-double derivation if

$$d(x^2) = d(x)x + xd(x) + \delta(x)\varepsilon(x) + \varepsilon(x)\delta(x)$$

holds for all $x \in \mathcal{R}$. Clearly, this notion includes ordinary derivation (Jordan derivation), when $\delta(x)\varepsilon(y) + \varepsilon(x)\delta(y) = 0$ for all $x, y \in \mathcal{R}$.

Vukman and Ulbl [12] considered the following result: Let $n > 1$ be an integer and let $\mathcal{R}$ be an $n!$-torsion free semiprime ring with identity element. Suppose that there exists an additive mapping $D : \mathcal{R} \to \mathcal{R}$ such that

$$D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x) x^{j-1}$$

is fulfilled for all $x \in \mathcal{R}$. In this case, D is a derivation. In [8], Hosseini presented some characterizations of δ-double derivations on rings and algebras. In this note, by methods mentioned above, we prove notions of a general characterization of (δ, ε)- double derivations on rings and algebra by some equations. Let $n > 1$ be an integer and d, δ, ε :

$\mathcal{R} \to \mathcal{R}$ be additive mappings such that

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-1-j} \Big(\delta(x) x^{j-i} \varepsilon(x) + \varepsilon(x) x^{j-i} \delta(x) \Big) x^{i-1}$$

is fulfilled for all $x \in \mathcal{R}$. If $\mathcal{R}$ is a unital $n!$-torsion free ring and $\delta(e) = \varepsilon(e) = 0$, then d is a Jordan (δ, ε)-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further,

$$\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x) = \frac{1}{2}\Big[(\delta\varepsilon + \varepsilon\delta)(x^2) - (\delta\varepsilon(x) + \varepsilon\delta(x))x - x(\delta\varepsilon(x) + \varepsilon\delta(x))\Big],$$

for all $x \in \mathcal{R}$, then $d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$ is an ordinary derivation on $\mathcal{R}$.

2. Main results

Let $\mathcal{A}$ be an algebra, and δ, ε be two additive mappings on $\mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called a (δ, ε)-double derivation, if

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$$

for every pair $a, b \in \mathcal{A}$. Similar to Jordan derivation, an additive mapping d is called Jordan (δ, ε)-double derivation if

$$d(a^2) = d(a)a + ad(a) + \delta(a)\varepsilon(a) + \varepsilon(a)\delta(a)$$

holds for all $a \in \mathcal{A}$. By a δ-double derivation we mean a (δ, δ)-double derivation.

Also, an additive mapping $D : \mathcal{A} \to \mathcal{A}$ is called a (δ, ε)-double left derivation, if $D(ab) = aD(b) + bD(a) + \delta(a)\varepsilon(b) + \delta(b)\varepsilon(a)$ for each pair $a, b \in \mathcal{A}$ and is called a Jordan (δ, ε)-double left derivation in case $D(a^2) = 2aD(a) + 2\delta(a)\varepsilon(a)$ is fulfilled for all $a \in \mathcal{A}$.

Lemma 2.1 If δ, ε are derivations on $\mathcal{A}$, then each (δ, ε)-double derivation $d : \mathcal{A} \to \mathcal{A}$ is of the form $d = \frac{\delta\varepsilon + \varepsilon\delta}{2} + \gamma$, where $\gamma : \mathcal{A} \to \mathcal{A}$ is a derivation.

Proof. Suppose that $\gamma = d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$. It is routine to show that γ is a derivation. ■

Lemma 2.2 Let $\mathcal{A}$ be a semiprime algebra and let δ, ε be Jordan derivations on $\mathcal{A}$. If $d : \mathcal{A} \to \mathcal{A}$ is a Jordan (δ, ε)-double derivation, then d is a (δ, ε)-double derivation.

Proof. By using Lemma 2.1 and Theorem 1 of [1], we deduce that $d = \frac{\delta\varepsilon + \varepsilon\delta}{2} + \gamma$, where $\gamma : \mathcal{A} \to \mathcal{A}$ is a derivation. Therefore,

$$\begin{aligned} d(xy) &= \big(\frac{\delta\varepsilon + \varepsilon\delta}{2}\big)(xy) + \gamma(xy) \\ &= \big(\frac{\delta\varepsilon + \varepsilon\delta}{2} + \gamma\big)(x)y + x\big(\frac{\delta\varepsilon + \varepsilon\delta}{2} + \gamma\big)(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y) \\ &= d(x)y + xd(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y), \quad x, y \in \mathcal{R}. \end{aligned}$$

Hence d is a (δ, ε)-double derivation. ■

Theorem 2.3 Let $n > 1$ be an integer and let $\mathcal{R}$ be a $n!$-torsion-free ring with the identity element. Suppose that d, δ, ε are additive mappings satisfying

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{j=1}^{n-1} \sum_{i=1}^{j} x^{n-1-j} \Big(\delta(x) x^{j-i} \varepsilon(x) + \varepsilon(x) x^{j-i} \delta(x) \Big) x^{i-1} \tag{2}$$

for all $x \in \mathcal{R}$. If $\delta(e) = \varepsilon(e) = 0$, then d is a Jordan (δ, ε)-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x) = \frac{1}{2}\Big[(\delta\varepsilon + \varepsilon\delta)(x^2) - (\delta\varepsilon(x) + \varepsilon\delta(x))x - x(\delta\varepsilon(x) + \varepsilon\delta(x))\Big]$ holds for all $x \in \mathcal{R}$, then $d - \frac{\delta\varepsilon + \varepsilon\delta}{2}$ is an ordinary derivation on $\mathcal{R}$.

Proof. Let c be an element of $Z(\mathcal{R})$ so that $d(c)$, $\delta(c)$, and $\varepsilon(c)$ are zero. By putting $x + c$ instead of x in (1), we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} d(x^{n-i} c^i) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i} c^i d(x) + \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i d(x)(x+c) + \cdots \\
&+ d(x) \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i} c^i + \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i \Big[\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x) \Big] \\
&+ \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i} c^i \Big[\delta(x) \Big((x+c)\varepsilon(x) + \varepsilon(x)(x+c) \Big) \\
&+ \varepsilon(x) \Big((x+c)\delta(x) + \delta(x)(x+c) \Big) \Big] + \cdots + \Big[\delta(x) \Big(\sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i \varepsilon(x) \\
&+ \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i} c^i \varepsilon(x)(x+c) + \cdots + \varepsilon(x) \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i \Big) \\
&+ \varepsilon(x) \Big(\sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i \delta(x) + \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i} c^i \delta(x)(x+c) + \cdots \\
&+ (x+c)\delta(x) \sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i} c^i + \delta(x) \sum_{i=0}^{n-2} \binom{n-2}{i} x^{n-2-i} c^i \Big) \Big],
\end{aligned}$$

for all $x \in \mathcal{R}$. Using (1) and collecting together terms of the above-mentioned relations involving the same number of factors of c, we achieve

$$\sum_{i=1}^{n-1} f_i(x, c) = 0 \qquad x \in \mathcal{R}, \tag{3}$$

where

$$
\begin{aligned}
f_i(x,c) = &\binom{n}{i} d(x^{n-i}c^i) - \binom{n-1}{i} x^{n-1-i}c^i d(x) - \Big(\binom{n-2}{i}\binom{1}{0} x^{n-2-i}c^i d(x)x \\
&+ \binom{n-2}{i-1}\binom{1}{1} x^{n-1-i}c^i d(x)\Big) - \cdots - \binom{n-1}{i} d(x)x^{n-1-i}c^i \\
&- \binom{n-2}{i} x^{n-2-i}c^i \Big(\delta(x)\varepsilon(x) + \varepsilon(x)\delta(x)\Big) \\
&- \Big[\binom{n-3}{i} x^{n-3-i}c^i \Big(\big(\delta(x)x\varepsilon(x) + \delta(x)\varepsilon(x)x\big) + \big(\varepsilon(x)x\delta(x) + \varepsilon(x)\delta(x)x\big)\Big) \\
&+ \binom{n-3}{i-1} x^{n-2-i}c^i \Big(2\delta(x)\varepsilon(x) + 2\varepsilon(x)\delta(x)\Big)\Big] - \cdots \\
&- \Big[\delta(x)\Big(\binom{n-2}{i} x^{n-2-i}c^i\varepsilon(x) + \binom{n-3}{i}\binom{1}{0} x^{n-3-i}c^i\varepsilon(x)x \\
&+ \binom{n-3}{i-1}\binom{1}{1} x^{n-2-i}c^i\varepsilon(x) + \cdots + \binom{n-2}{i}\varepsilon(x)x^{n-2-i}c^i\Big) \\
&+ \varepsilon(x)\Big(\binom{n-2}{i} x^{n-2-i}c^i\delta(x) + \binom{n-3}{i}\binom{1}{0} x^{n-3-i}c^i\delta(x)x \\
&+ \binom{n-3}{i-1}\binom{1}{1} x^{n-2-i}c^i\delta(x) + \cdots + \binom{n-2}{i}\delta(x)x^{n-2-i}c^i\Big)\Big]
\end{aligned}
$$

Having replaced c, $2c$, $3c$,..., $(n-1)c$ instead of c in (2), we obtain a system of $n-1$ homogeneous equations as follows:

$$
\begin{cases}
\sum_{i=1}^{n-1} f_i(x,c) = 0 \\
\sum_{i=1}^{n-1} f_i(x,2c) = 0 \\
\sum_{i=1}^{n-1} f_i(x,3c) = 0 \\
\quad\vdots \\
\sum_{i=1}^{n-1} f_i(x,(n-1)c) = 0
\end{cases}
$$

It is observed that the coefficient matrix of the above system is equal to the following matrix:

$$
A = \begin{bmatrix}
\binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\
2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1}
\end{bmatrix}
$$

Since the determinant of A is different from zero, it follows that the system has only a trivial solution. In particular, $f_{n-2}(x,e)=0$, that is

$$0=\binom{n}{n-2}d(x^2)-\binom{n-1}{n-2}xd(x)-\binom{n-2}{n-2}d(x)x-\binom{n-2}{n-3}xd(x)-\cdots-\binom{n-1}{n-2}d(x)x$$
$$-\Big(\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big)-2\Big(\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big)-\cdots-(n-1)\Big(\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big),$$

for all $x\in\mathcal{R}$. The above equation reduces to

$$\frac{n(n-1)}{2}d(x^2)=(n-1)xd(x)+d(x)x+(n-2)xd(x)+\cdots+(n-1)d(x)x+\Big(\delta(x)\varepsilon(x)$$
$$+\varepsilon(x)\delta(x)\Big)+2\Big(\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big)+\cdots+(n-1)\Big(\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big),$$

for all $x\in\mathcal{R}$. Thus, we have

$$\frac{n(n-1)}{2}d(x^2)=\Big(\sum_{i=1}^{n-1}i\Big)\Big(d(x)x+xd(x)+\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)\Big), \tag{4}$$

for all $x\in\mathcal{R}$. Since $\mathcal{R}$ is $n!$-torsion free, it follows from (3) that

$$d(x^2)=d(x)x+xd(x)+\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x), \tag{5}$$

for all $x\in\mathcal{R}$. In other words, d is a Jordan (δ,ε)-double derivation. Now suppose that $\mathcal{R}$ is a semiprime algebra and further $\delta(x)\varepsilon(x)+\varepsilon(x)\delta(x)=\frac{1}{2}\Big[(\delta\varepsilon+\varepsilon\delta)(x^2)-(\delta\varepsilon(x)+\varepsilon\delta(x))x-x(\delta\varepsilon(x)+\varepsilon\delta(x))\Big]$ for all $x\in\mathcal{R}$. This equation with (4) imply that $d-\frac{\delta\varepsilon+\varepsilon\delta}{2}$ is a jordan derivation. It follows from Theorem 1 of [1] that $d-\frac{\delta\varepsilon+\varepsilon\delta}{2}$ is an ordinary derivation on $\mathcal{R}$. ■

Using the above theorem, we obtain the following result.

Corollary 2.4 Let $n>1$ be an integer, $\mathcal{A}$ be a unital semiprime algebra. Suppose that $d,\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ are additive mappings satisfying

$$d(a^n)=\sum_{j=1}^{n}a^{n-j}d(a)a^{j-1}+\sum_{j=1}^{n-1}\sum_{i=1}^{j}a^{n-1-j}\Big(\delta(a)a^{j-i}\varepsilon(a)+\varepsilon(a)a^{j-i}\delta(a)\Big)a^{i-1}$$

for all $a\in\mathcal{A}$. If δ, ε are two derivations, then d is a (δ,ε)-double derivation.

Proof. According to the previous theorem and Lemma 2.2, d is a (δ,ε)-double derivation. ■

Theorem 2.5 Let $\mathcal{A}$ be a unital Banach algebra and $d,\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ be additive mappings satisfying

$$d(a)=-ad(a^{-1})a-a\delta(a^{-1})\varepsilon(a)-a\varepsilon(a^{-1})\delta(a) \tag{6}$$

for all invertible element $a\in\mathcal{A}$. If $\delta(a)=-a\delta(a^{-1})a$ and $\varepsilon(a)=-a\varepsilon(a^{-1})a$ for all invertible element a, then d is a Jordan (δ,ε)-double derivation.

In particular, if $\mathcal{A}$ is semiprime and $\delta(b)\varepsilon(b) + \varepsilon(b)\delta(b) = \frac{1}{2}\Big[(\delta\varepsilon + \varepsilon\delta)(b^2) - (\delta\varepsilon(b) + \varepsilon\delta(b))b - b(\delta\varepsilon(b) + \varepsilon\delta(b))\Big]$ holds for all $b \in \mathcal{A}$, then $d - \frac{\delta\varepsilon+\varepsilon\delta}{2}$ is an ordinary derivation.

Proof. Let b be an arbitrary element from $\mathcal{A}$ and let n be a positive number so that $\|\frac{b}{n-1}\| < 1$. It is evident that $\|\frac{b}{n}\| < 1$, too. If we consider $a = ne + b$, then we have $\frac{a}{n} = e - \frac{-b}{n}$. Since $\|\frac{-b}{n}\| < 1$, it follows from Theorem 1.4.2 of [6] that $e - \frac{-b}{n}$ is invertible and consequently, a is invertible. Similarly, we can show that $e-a$ is also an invertible element of $\mathcal{A}$. In the following, we use the well-known Hua identity $a^2 = a - (a^{-1} + (e-a)^{-1})^{-1}$. Applying the relation (5) it is obtained that

$$\begin{aligned}
d(a^2) &= d(a) - d((a^{-1} + (e-a)^{-1})^{-1}) \\
&= d(a) + (a^{-1} + (e-a)^{-1})^{-1} d(a^{-1} + (e-a)^{-1})(a^{-1} + (e-a)^{-1})^{-1} \\
&+ (a^{-1} + (e-a)^{-1})^{-1}\delta(a^{-1} + (e-a)^{-1})\varepsilon((a^{-1} + (e-a)^{-1})^{-1}) \\
&+ (a^{-1} + (e-a)^{-1})^{-1}\varepsilon(a^{-1} + (e-a)^{-1})\delta((a^{-1} + (e-a)^{-1})^{-1}) \\
&= d(a) + a(e-a)d(a^{-1})a(e-a) + a(e-a)d((e-a)^{-1})a(e-a) \\
&+ a(e-a)\delta(a^{-1} + (e-a)^{-1})\varepsilon(a(e-a)) + a(e-a)\varepsilon(a^{-1} + (e-a)^{-1})\delta(a(e-a)) \\
&= d(a) - a(e-a)a^{-1}d(a)a^{-1}a(e-a) - a(e-a)a^{-1}\delta(a)\varepsilon(a^{-1})a(e-a) \\
&- a(e-a)a^{-1}\varepsilon(a)\delta(a^{-1})a(e-a) + a(e-a)(e-a)^{-1}d(a)(e-a)^{-1}a(e-a) \\
&+ a(e-a)(e-a)^{-1}\delta(a)\varepsilon((e-a)^{-1})a(e-a) \\
&+ a(e-a)(e-a)^{-1}\varepsilon(a)\delta((e-a)^{-1})a(e-a) \\
&+ a(e-a)a^{-1}\delta(a)a^{-1}(a-a^2)\varepsilon(a^{-1})(a-a^2) \\
&+ a(e-a)a^{-1}\delta(a)a^{-1}(a-a^2)\varepsilon((e-a)^{-1})(a-a^2) \\
&- a(e-a)(e-a)^{-1}\delta(a)(e-a)^{-1}(a-a^2)\varepsilon(a^{-1})(a-a^2) \\
&- a(e-a)(e-a)^{-1}\delta(a)(e-a)^{-1}(a-a^2)\varepsilon((e-a)^{-1})(a-a^2) \\
&+ a(e-a)a^{-1}\varepsilon(a)a^{-1}(a-a^2)\delta(a^{-1})(a-a^2) \\
&+ a(e-a)a^{-1}\varepsilon(a)a^{-1}(a-a^2)\delta((e-a)^{-1})(a-a^2) \\
&- a(e-a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}(a-a^2)\delta(a^{-1})(a-a^2) \\
&- a(e-a)(e-a)^{-1}\varepsilon(a)(e-a)^{-1}(a-a^2)\delta((e-a)^{-1})(a-a^2) \\
&= ad(a) + d(a)a - \delta(a)\varepsilon(a^{-1})a + \delta(a)\varepsilon(a^{-1})a^2 + a\delta(a)\varepsilon(a^{-1})a - a\delta(a)\varepsilon(a^{-1})a^2 \\
&- \varepsilon(a)\delta(a^{-1})a + \varepsilon(a)\delta(a^{-1})a^2 + a\varepsilon(a)\delta(a^{-1})a - a\varepsilon(a)\delta(a^{-1})a^2 \\
&+ a\delta(a)(e-a)^{-1}\varepsilon(e-a)(e-a)^{-1}a(e-a) + a\varepsilon(a)(e-a)^{-1}\delta(e-a)(e-a)^{-1}a(e-a) \\
&- \delta(a)a^{-1}\varepsilon(a) + \delta(a)a^{-1}\varepsilon(a)a + \delta(a)\varepsilon(a) + a\delta(a)a^{-1}\varepsilon(a) - a\delta(a)a^{-1}\varepsilon(a)a - a\delta(a)\varepsilon(a) \\
&+ a\delta(a)(e-a)^{-1}\varepsilon(a) - a\delta(a)(e-a)^{-1}\varepsilon(a)a - \varepsilon(a)a^{-1}\delta(a) \\
&- a\delta(a)(e-a)^{-1}a\varepsilon(a) + \varepsilon(a)a^{-1}\delta(a)a + \varepsilon(a)\delta(a) + a\varepsilon(a)a^{-1}\delta(a) - a\varepsilon(a)a^{-1}\delta(a)a \\
&- a\varepsilon(a)\delta(a) + a\varepsilon(a)(e-a)^{-1}\delta(a) - a\varepsilon(a)(e-a)^{-1}\delta(a)a - a\varepsilon(a)(e-a)^{-1}a\delta(a).
\end{aligned}$$

Putting $\delta(x) = -x\delta(x^{-1})x$ and $\varepsilon(x) = -x\varepsilon(x^{-1})x$ Thus

$$\begin{aligned} d(a^2) &= ad(a) + d(a)a + \delta(a)\varepsilon(a) - a\delta(a)\varepsilon(a) + a\delta(a)(e-a)^{-1}\varepsilon(a) + \varepsilon(a)\delta(a) \\ &\quad - a\delta(a)(e-a)^{-1}a\varepsilon(a) - a\varepsilon(a)\delta(a) + a\varepsilon(a)(e-a)^{-1}\delta(a) - a\varepsilon(a)(e-a)^{-1}a\delta(a) \\ &= ad(a) + d(a)a + \delta(a)\varepsilon(a) + \varepsilon(a)\delta(a). \end{aligned}$$

Note that $\delta(e) = -e\delta(e^{-1})e$ and $\varepsilon(e) = -e\varepsilon(e^{-1})e$. Hence $\delta(e) = \varepsilon(e) = 0$ and it implies that $d(e) = 0$. Having put $a = ne + b$ in the previous equation, we obtain

$$d(n^2 + 2nb + b^2) = d(b)(ne + b) + (ne + b)d(b) + \delta(b)\varepsilon(b) + \varepsilon(b)\delta(b).$$

We, therefore, have $d(b^2) = bd(b) + d(b)b + \delta(b)\varepsilon(b) + \varepsilon(b)\delta(b)$ for all $b \in \mathcal{A}$, i.e. d is a Jordan (δ, ε)-derivation. Now, assume that $\delta(b)\varepsilon(b)+\varepsilon(b)\delta(b) = \frac{1}{2}\Big[(\delta\varepsilon+\varepsilon\delta)(b^2) - (\delta\varepsilon(b) + \varepsilon\delta(b))b - b(\delta\varepsilon(b) + \varepsilon\delta(b))\Big]$ for all $b \in \mathcal{A}$. Hence,

$$d(b^2) = bd(b) + d(b)b + \frac{1}{2}\Big[(\delta\varepsilon + \varepsilon\delta)(b^2) - (\delta\varepsilon(b) + \varepsilon\delta(b))b - b(\delta\varepsilon(b) + \varepsilon\delta(b))\Big]$$

equivalently we have,

$$\Big(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\Big)(b^2) = b\Big(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\Big)(b) + \Big(d - \frac{1}{2}(\delta\varepsilon + \varepsilon\delta)\Big)(b)b.$$

It means that $d - \frac{\delta\varepsilon+\varepsilon\delta}{2}$ is a Jordan derivation. At this point, Theorem 1 of [1] completes the argument. ■

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