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On some Frobenius groups with the same prime graph as the almost simple group PGL(2, 49)

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Abstract. The prime graph of a finite group G is denoted by $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct primes p and q are adjacent in $\Gamma(G)$, whenever G contains an element with order pq. We say that G is unrecognizable by prime graph if there is a finite group H with $\Gamma(H) = \Gamma(G)$, in while $H \ncong G$. In this paper, we consider finite groups with the same prime graph as the almost simple group $\mathrm{PGL}(2, 49)$. Moreover, we construct some Frobenius groups whose prime graphs coincide with $\Gamma(\mathrm{PGL}(2, 49))$, in particular, we get that $\mathrm{PGL}(2, 49)$ is unrecognizable by its prime graph.

Keywords: Almost simple group, prime graph, Frobenius group, element order.

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1. Introduction

Let N denotes the set of natural numbers. If $n \in N$, then we denote by $\pi(n)$, the set of all prime divisors of n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of G is denoted by $\pi_e(G)$. We denote by $\mu(G)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (and we write $p \sim q$), whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. A finite group G is called *unrecognizable by prime graph* if there exists a finite group H such that $\Gamma(H) = \Gamma(G)$, however $H \ncong G$.

In [10], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11$, 19 and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition

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factor which is isomorphic to $\mathrm{PSL}(2,p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,13)$ or $\mathrm{PSL}(2,27)$. In [3], it is proved that if $q = p^\alpha$, where p is a prime and $\alpha > 1$ is an integer, then $\mathrm{PGL}(2,q)$ is uniquely determined by its element orders. Also, in [1], it is proved that if $q = p^\alpha$, where p is an odd prime and α is an odd natural number, then $\mathrm{PGL}(2,q)$ is uniquely determined by its prime graph. However, in this paper as the main result we prove that the almost simple group $\mathrm{PGL}(2,49)$ is unrecognizable by prime graph. Finally we put a question about the existence of Frobenius groups with the same prime graph as the almost simple groups $\mathrm{PGL}(2,q)$.

We note that in this paper, if G is a group, then $Fit(G)$ is the Fitting subgroup of G that is the product of all nilpotent normal subgroups of G.

2. Preliminary Results

Lemma 2.1 ([17, Lemma 1]) Let G be a finite group and $N \trianglelefteq G$ such that G/N is a Frobenius group with kernel F and cyclic complement C. If $(|F|,|N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.

Lemma 2.2 ([8, Main Theorem]) Let G be a finite group and $|\pi(G)| \geqslant 3$. If there exist prime numbers r, s, $t \in \pi(G)$, such that $\{tr, ts, rs\} \cap \pi_e(G) = \emptyset$, then G is non-solvable.

We note that a Z-group is a group in which every sylow subgroup is cyclic. Now we have the following lemma:

Lemma 2.3 ([19, Theorem 18.6]) Let G be a nonsolvable Frobenius complement. Then G has a normal subgroup G_0 with $|G : G_0| = 1$ or 2 such that $G_0 = \mathrm{SL}(2,5) \times M$ with M a Z-group of order prime to 2, 3 and 5.

A finite group G is called a 2-Frobenius group when there exists a normal series, $1 \leqslant H \leqslant K \leqslant G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Using [14, Theorem A], we have the following result:

Lemma 2.4 Let G be a finite group with $t(G) \geqslant 2$. Then one of the following holds:
(a) G is a Frobenius or 2-Frobenius group;
(b) there exists a nonabelian simple group S such that $S \leqslant \overline{G} := G/N \leqslant \mathrm{Aut}(S)$ for some nilpotent normal subgroup N of G.

Lemma 2.5 ([20]) Let $G = L_n^\varepsilon(q)$, $q = p^m$, be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Denote $H = W \leftthreetimes G$. If $n = 2$ and q is odd then $2p \in \pi_e(H)$.

3. Main Results

Lemma 3.1 There are infinitely many finite Frobenius group G such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,49))$.

Proof. Let F be a finite field of characteristic 7. Also there are some elements α and β included in F such that $\alpha^2 = -1$ and $\beta^2 = 5$. We know that such a finite field does exist and moreover there are infinitely many filed with these properties.

Now we construct some linear groups as follow:

$$C := \left\langle \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & \frac{\beta+1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

$$K := \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

By the above definition, $C \cong \langle x, y, z | x^3 = y^5 = z^2 = 1, x^z = z, y^z = y, (xy)^2 = z\rangle$. This implies that $C \cong \mathrm{SL}(2,5)$. Also we have $K \cong F \oplus F$, is a direct sum of additive group F by itself. This means K is isomorphic to a vector space of dimension 2 over F and so $|K| = |F|^2$. It is obvious that C belongs to the normalizer of K in $\mathrm{GL}(3, F)$.

Now we define $G := K \rtimes C$. Since K is an elementary abelian 7-group, it is easy to prove that C acts fixed point freely on K by conjugation. Hence G is a Frobenius group with kernel K and complement C. This implies that in the prime graph of G, 7 is an isolated vertex. Also by $\Gamma(\mathrm{SL}(2,5))$, we get that 2 is adjacent to 3 and 5 and there is no edge between 3 and 5 in $\Gamma(G)$. Therefore, $\Gamma(G)$ coincides to $\Gamma(\mathrm{PGL}(2,49))$, which completes the proof. ■

Lemma 3.2 Each following group G is an almost simple group related to the simple group S. Moreover, G has a prime graph which coincide with the prime graph of the almost simple group $\mathrm{PGL}(2,49)$:

(1) $G = S_7$ and $S = A_7$.

(2) $G = U_4(3) \cdot 2$ and $S = U_4(3)$.

(3) $G = U_3(5)$ or $G = U_3(5) \cdot 2$ and $S = U_3(5)$.

Proof. Using [4], it is straightforward. ■

Theorem 3.3 Let G be a finite group with the prime graph as same as the prime graph of $PGL(2,49)$. Then G is isomorphic to one of the following groups:

(1) A Frobenius group $K \rtimes C$, such that K is a 7-group and C contains a subgroup C_0 whose index in C is at most 2 and C_0 is isomorphic to $\mathrm{SL}(2,5)$.

(2) One of the almost simple groups: S_7, $U_4(3) \cdot 2$, $U_3(5) \cdot 2$ or $\mathrm{PGL}(2,49)$.

(3) The simple group: $U_3(5)$.

In particular, $\mathrm{PGL}(2,49)$ is unrecognizable by prime graph.

Proof. By [18, Lemma 7], it follows that $\mu(\mathrm{PGL}(2,49)) = \{7, 48, 50\}$. Hence, the connected components of the prime graph of $\mathrm{PGL}(2,49)$ are exactly $\{7\}$ and $\{2,3,5\}$. Also by $\mu(\mathrm{PGL}(2,49))$, there is no edge between 3 and 5 in $\Gamma(\mathrm{PGL}(2,49))$. Now since $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 49))$, we deduce that these relations hold in the prime graph of G.

First we claim that G is not solvable. On the contrary, let G be a solvable group. So there is a Hall $\{3,5,7\}$-subgroup in G, say H. On the other hand $\{3,5,7\}$ is an independent subset of $\Gamma(G)$, which is a contradiction by Lemma 2.2. Therefore, G is not solvable and so by Lemma 2.4, either G is a Frobenius group or there is a nonabelian simple group S such that $S \leqslant \bar{G} := G/\mathrm{Fit}(G) \leqslant \mathrm{Aut}(S)$.

Let G be a Frobenius group with kernel K and complement C. By Lemma 2.3, we know that K is nilpotent and $\pi(C)$ is a connected component of the prime graph of G. Hence we conclude that $\pi(K) = \{7\}$ and $\pi(C) = \{2,3,5\}$, since 7 is an isolated vertex in

$\Gamma(G)$. Hence if C is solvable, then G is a solvable which is a contradiction by the above argument.

Thus we suppose that C is non-solvable. Then by Lemma 2.3, the complement C has a normal subgroup C_0 with index at most 2 which is isomorphic to $\mathrm{SL}(2,5) \times M$, where $\pi(M) \cap \{2,3,5\} = \emptyset$. On the other hand, by the previous argument, we know that $\pi(C) = \{2,3,5\}$. This implies that $M = 1$ and so $C_0 \cong \mathrm{SL}(2,5)$. Also by Lemma 3.1, we know that this such Frobenius complement exists. Hence G can be isomorphic to a Frobenius group $K : C$, where K is a 7-subgroup and C contains a subgroup isomorphic to $\mathrm{SL}(2,5)$ whose index is at most 2, Therefore if G is a Frobenius group, then we get Case (1).

Now we may assume that G is neither a Frobenius nor a 2-Frobenius group. Hence by Lemma 2.4, there exists a nonabelian simple group S such that:

$$S \leqslant \overline{G} := G/K \leqslant \mathrm{Aut}(S)$$

in which K is the Fitting subgroup of G. Since $\{2,7\}$ is an independent subset of $\Gamma(G)$, by Lemma 2.4, we conclude that $7 \in \pi(S)$ and $7 \notin \pi(K) \cup \pi(\bar{G}/S)$. Also we know that $\pi(S) \subseteq \pi(G)$. Since $\pi(G) = \{2,3,5,7\}$, so by [13, Table 8], we get that S is isomorphic to A_7, A_8, A_9, A_{10}, $S_6(2)$, $O_8^+(2)$, $L_3(2^2)$, $L_2(2^3)$, $U_3(3)$, $U_4(3)$, $U_3(5)$, $L_2(7)$, $S_4(7)$, $L_2(7^2)$ or J_2. Now we consider each possibility for the simple group S.

Let $S \cong L_2(7)$. Then $5 \in \pi(K)$, since $5 \notin (\pi(S) \cup \pi(\bar{G}/S))$. On the other hand S contains a $\{3,7\}$-subgroup H. Hence G has a subgroup isomorphic to $K_5 : H$ where K_5 is 5-group. On the other hand $K_5 : H$ is solvable and so there is an edge between to prime numbers in $\Gamma(K_5 : H)$, which is impossible since $\Gamma(K_5 : H)$ is a subgraph of $\Gamma(G)$. Thus $S \ncong L_2(7)$.

Let $S \cong L_2(2^3)$. In this case, $5 \in \pi(K)$. Also we know that S contains a Frobenius group isomorphic to $8 : 7$. Hence by Lemma 2.1, we get that G has an element order $5 \cdot 7$, which is a contradiction.

Let $S \cong A_8$, A_9 or A_{10}. Thus S consists an element of order $3 \cdot 5$, which contradicts to the prime graph of G.

Let $S \cong J_2$, $O_8^+(2)$ or $S_6(2)$. In this case S contains an element of order 15, which is a contradiction.

By Lemma 3.2, the finite group S can be isomorphic to each simple group A_7, $U_3(3)$, $U_4(3)$ and $U_3(5)$.

Let S be isomorphic to $\mathrm{PSL}_2(49)$. Hence $\mathrm{PSL}_2(49) \leqslant \bar{G} \leqslant \mathrm{Aut}(\mathrm{PSL}_2(49))$.

Let $\pi(K)$ contains a prime r such that $r \neq 7$. Since K is nilpotent, we may assume that K is a vector space over a field with r elements (analogous to the proof of Lemma 3.1). Hence the prime graph of the semidirect product $K \rtimes \mathrm{PSL}_2(49)$ is a subgraph of $\Gamma(G)$. Let B be a Sylow 7-subgroup of $\mathrm{PSL}_2(49)$. We know that B is not cyclic. On the other hand $K \rtimes B$ is a Frobenius group such that $\pi(K \rtimes B) = \{r, 7\}$. Hence B should be cyclic which is a contradiction. This implies that $K = 1$, since $7 \notin \pi(K)$.

We know that $\mathrm{Out}(\mathrm{PSL}_2(49)) \cong Z_2 \times Z_2$. Since in the prime graph of $\mathrm{PSL}_2(49)$ there is not any edge between 2 and 5, we get that $G \ncong \mathrm{PSL}_2(49)$. If $G = \mathrm{PSL}_2(49) : \langle\theta\rangle$, where θ is a field automorphism, then we get that 2 and 7 are adjacent in G (for example see Page 26 of [1]), which is a contradiction. Also if $G = \mathrm{PSL}_2(49) : \langle\gamma\rangle$, where γ is a diagonal automorphism, then we get that G is isomorphic to $\mathrm{PGL}_2(49)$, which completes the proof. ■

Problem. Let $G = \mathrm{PGL}(2,q)$ be an almost simple group related to the simple group $\mathrm{PSL}(2,q)$. Find all Frobenius group H such that $\Gamma(H) = \Gamma(G)$.

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