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# A solution of nonlinear fractional random differential equation via random fixed point technique

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**Abstract.** In this paper, we investigate a new type of random $F$-contraction and obtain a common random fixed point theorem for a pair of self stochastic mappings in a separable Banach space. The existence of a unique solution for nonlinear fractional random differential equation is proved under suitable conditions.



**Keywords:** Random fixed point, $F$-contraction, separable metric spaces, NFRDE.

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## 1. Introduction

Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and NFRDE will denote real numbers, positive real numbers, natural numbers and nonlinear fractional random differential equation respectively.

The contraction mapping principle is fundamental importance in the metrical fixed point theory, which states as:

**Theorem 1.1** [4] Let $(X,d)$ be a complete metric space and $T$ be a mapping of $X$ into itself satisfying:

$$d(Tx,Ty) \leqslant kd(x,y) \quad \forall x,y \in X, \tag{1}$$

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where $k$ is a constant in [0,1). Then $T$ has a unique fixed point $x \in X$.

There are various generalizations of the contraction principle, roughly obtained in two ways:

1) by weakening the contractive properties of the mapping and possibly, by simultaneously giving the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness,

2) by extending the structure of the ambient space.

Also, several fixed point theorems are obtained by combining the two ways previously described or by adding supplementary conditions (see [3, 8, 11, 16, 18, 22, 27]).

One of the most extension of it was introduced by Wardowski [9]. Abbas et al. [10] generalized the notion of $F$-contraction and proved certain fixed point results. Batra et al. [11, 12] extended the concept of $F$-contraction on graphs and altered distances. Recently, Cosentino and Vetro [13] followed the approach of $F$-contraction and obtained some fixed point theorems for Hardy-Rogers type self mappings in complete metric and ordered metric spaces. There are also many different types of fixed point theorems not mentioned above extending the Banach's result.

**Definition 1.2** [29] Let $\digamma$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

$(F_1)$ $F$ is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;

$(F_2)$ for every sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} F(a_n) = -\infty$;

$(F_3)$ there exist $\lambda \in (0,1)$ such that $\lim_{a\to 0^+} a^\lambda F(a) = 0$.

**Definition 1.3** [29] Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be an $F$-contraction on $(X, d)$ if there exists $F \in \digamma$ and $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)). \tag{2}$$

From $(F_1)$ and (2) it is easy to conclude that every $F$-contraction $T$ is a contractive mapping and hence necessarily continuous.

***Example* 1.4** The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ are the elements of $\digamma$

$$\begin{array}{ll} (i) F(t) = \ln(t) & (ii)\ F(t) = \ln(t) + t \\ (iii)\ F(t) = \frac{-1}{\sqrt{t}} & (iv)\ F(t) = \ln(t^2 + t), \quad t > 0. \end{array}$$

***Remark 1*** *Consider* $F(t) = \frac{-1}{t^{\frac{1}{p}}}$, $p > 1$, $t > 0$, *then* $F \in \digamma$.

**Proof.** Since $F^{'}(t) = \frac{1}{p.t^{1+\frac{1}{p}}} > 0$, $F$ satisfies $(F_1)$ and it is clear that the condition $(F_2)$ holds. Since $p > 1$, $\frac{1}{p} < 1$, if we take $\frac{1}{p} < \lambda < 1$, then $\lim_{t\to 0^+} t^\lambda F(t) = \lim_{t\to 0^+}(-t^{\lambda-\frac{1}{p}}) = 0$. So $F$ satisfies $(F_3)$. This gives $F \in \digamma$. ■

The Banach contraction (1) is a particular case of $F$-contraction. Meanwhile there exist $F$-contractions, which are not Banach contractions (Wardowski [29]).

**Theorem 1.5** [29] Let $T$ be a self mapping on a complete metric space $(X, d)$ satisfying the $F$-contraction condition (2), then $T$ has a unique fixed point $x^*$. Moreover, for any $x_\circ \in X$, the sequence $\{T^n x_\circ\}_{n\in\mathbb{N}}$ is convergent to $x^*$.

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems that appear in approximation theory, game and potential theory, theory of integral and differential equations and others. Random operator theory is needed for the study of various classes of random equations. These equations are much more difficult to handle mathematical than deterministic equations. Important contributions to the study of random equations have been presented in [7, 25] among others. The study of random fixed point problems was initiated by the Prague school of probability research. The first results were studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random kernel. In a separable metric space, random fixed point theorems for random contraction mappings were proved by Hanš [12] and Špaček [26].

Bharucha-Reid [7] attracted the attention of several mathematicians and gave wings to this theory. Itoh [14] extended the results of Špaček and Hanš in multi-valued contractive mappings and obtained random fixed point theorems with an application to random differential equations in Banach spaces. Now, it has became a full fledged research area and a vast amount of mathematical activities have been carried out in this direction (see [23, 24]).

Recently, some authors ([19–21]) applied a random fixed point theorem to prove the existence of a solution in a sparable Banach space of a random nonlinear integral equations.

The aim of this paper is to establish a random version of fixed point theorem for a pair of self stochastic mappings satisfying $F$-contraction in a separable Banach space. Finally, we apply our result to obtain the existence unique solution of NFRDE.

## 2. Some basic concepts on a measurable space

Let $(X, \beta_X)$ be a separable complete metric space, where $\beta_X$ is a $\sigma-$algebra of Borel subsets of $X$ and let $(\Omega, \beta, \mu)$ denote a complete probability measure space with a nonempty set $\Omega$, a measurable $\mu$ and a $\sigma-$algebra $\beta$ of subsets of $\Omega$.

**Definition 2.1** [15] A measurable mapping $x : \Omega \to X$ is called:

(i) $X-$valued random variable if the inverse image under the mapping $x$ of every Borel set $B$ of $X$ belongs to $\beta$, that is, $x^{-1}(B) \in \beta$ for all $B \in \beta$.

(ii) finitely valued random variable if it is constant on each of a finite number of disjoint sets $A_i \in \beta$ and is equal to 0 on $\Omega - \left(\bigcup_{i=1}^{n} A_i\right)$.

(iii) simple random variable if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

(iv) strong random variable if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

(v) weak random variable if the function $x^*(x(\omega))$ is a real valued random variable for each $x^* \in X^*$, the space $X^*$ denoting the dual space of $X$.

**Definition 2.2** [15] Let $Y$ be a Banach space.

(i) A measurable mapping $f : \Omega \times X \to Y$ is said to be a random mapping if $f(\omega, x) = Y(\omega)$ is a $Y-$valued random variable for every $x \in X$.

(ii) A measurable mapping $f : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $f(\omega, x)$ is a continuous function of $x$ has measure one.

(iii) An equation of the type $f(\omega, x(\omega)) = x(\omega)$, where $f : \Omega \times X \to X$ is a random mapping is called a random fixed point equation.

(iv) Any measurable mapping $x : \Omega \to X$ which satisfies the random fixed point equation $f(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

(v) Any $X-$valued random variable $x(\omega)$ which satisfies $\mu\{\omega : f(\omega, x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of $f$.

(vi) A measurable mapping $x : \Omega \to X$ is called:

(a) a random fixed point of a random operator $f : \Omega \times X \to X$ if $f(\omega, x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

(b) a random coincidence of random operators $T, f : \Omega \times X \to X$ if $T(\omega, x(\omega)) = f(\omega, x(\omega))$ for every $\omega \in \Omega$.

(c) a common random fixed point of random mappings $T, f : \Omega \times X \to X$ if $T(\omega, x(\omega)) = f(\omega, x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

***Example* 2.3** [15] Let $X = \mathbb{R}$ and $C$ be a non-measurable subset of $X$. Consider a random mapping $f : \Omega \times X \to Y$ defined as $f(\omega, x(\omega)) = x^2(\omega) + x(\omega) - 1$ for all $\omega \in \Omega$. It's clearly that, the real-valued function $x(\omega) = 1$ is a random fixed point of $f$. However, the real-valued function $y(\omega) = \begin{cases} -1, & \omega \notin C \\ 1, & \omega \in C \end{cases}$ is a wide sense solution of the fixed point equation $f(\omega, x(\omega)) = x(\omega)$ without being a random fixed point of $f$. Therefore, a random solution is a wide sense solution of the fixed point equation but the converse is not necessarily true.

## 3. New random fixed point for F-contractions

We begin with the following definition:

**Definition 3.1** Let $X$ be a separable Banach space and $(\Omega, \beta)$ be a measurable space. The random mappings $T, S : \Omega \times X \to X$ are called $F$-contractions if for all $x, y \in X$ and $\omega \in \Omega$, we have

$$\tau + F(\|T(\omega, x) - S(\omega, y)\|) \leqslant F(N(x(\omega), y(\omega))), \tag{3}$$

where $F \in \digamma$, $\tau > 0$ and $N(x(\omega), y(\omega))$ is one of the following inequalities:

$(i)$

$$\begin{aligned} N(x(\omega), y(\omega)) = a\,\|x(\omega) - y(\omega)\| &+ b[\|x(\omega) - T(\omega, x)\| + \|y(\omega) - S(\omega, y)\|] \\ &+ c[\|x(\omega) - S(\omega, y)\| + \|y(\omega) - T(\omega, x)\|], \end{aligned}$$

where $1 - b - c > 0$, $a > 0$, $b \geqslant 0$ and $c \geqslant 0$,

$(ii)$

$$N(x(\omega), y(\omega)) = q \max \left\{ \begin{array}{c} \|x(\omega) - y(\omega)\|, \|x(\omega) - T(\omega, x)\| + \|y(\omega) - S(\omega, y)\|, \\ \|x(\omega) - S(\omega, y)\| + \|y(\omega) - T(\omega, x)\| \end{array} \right\},$$

where $0 < q < 1$.

$(iii)$

$$N(x(\omega), y(\omega)) = \alpha \max \left\{ \begin{array}{c} \|x(\omega) - y(\omega)\|, \|x(\omega) - T(\omega, x)\|, \|y(\omega) - S(\omega, y)\|, \\ \|x(\omega) - S(\omega, y)\|, \|y(\omega) - T(\omega, x)\| \end{array} \right\},$$

where $0 < \alpha < 1$.

**Theorem 3.2** Let $X$ be a separable Banach space and $(\Omega, \beta, \mu)$ be a probability measure space. Suppose that $T, S : \Omega \times X \to X$ are random mappings satisfying the contractive condition (3). If the axioms (i)-(ii) are hold, then there exists a common random fixed point $p(\omega)$ of $S$ and $T$. Moreover, if (iii) holds, then this common fixed point is unique.

**Proof.** Consider a measure mapping $\xi_\circ(\omega) : \Omega \to X$. We can define a sequence of measurable mappings $\{\xi_n(\omega)\}$ from $\Omega$ to $X$ as follows:

$$\xi_{2n+1}(\omega) = T(\omega, \xi_{2n}(\omega)) \text{ and } \xi_{2n+2}(\omega) = S(\omega, \xi_{2n+1}(\omega)),\ \omega \in \Omega,\ n = 0, 1, 2, ..$$

Applying the contractive condition (3), we get

$$\begin{aligned} F(\|\xi_{2n+1}(\omega) - \xi_{2n+2}(\omega)\|) &= F(\|T(\omega, \xi_{2n}(\omega)) - S(\omega, \xi_{2n+1}(\omega))\|) \\ &\leqslant F(N(\xi_{2n}(\omega), \xi_{2n+1}(\omega))) - \tau. \end{aligned} \tag{4}$$

If $T, S$ satisfy (i) and (ii), then by (4), we have respectively

$$\begin{aligned} F(\|\xi_{2n+1}(\omega) - \xi_{2n+2}(\omega)\|) &\leqslant F\left(\tfrac{a+b+c}{1-b-c} \|\xi_{2n}(\omega) - \xi_{2n+1}(\omega)\|\right) - \tau, \\ F(\|\xi_{2n+1}(\omega) - \xi_{2n+2}(\omega)\|) &\leqslant F\left(q \|\xi_{2n}(\omega) - \xi_{2n+1}(\omega)\|\right) - \tau. \end{aligned} \tag{5}$$

From (5), we get

$$F(\|\xi_{2n+1}(\omega) - \xi_{2n+2}(\omega)\|) \leqslant F(\gamma \|\xi_{2n}(\omega) - \xi_{2n+1}(\omega)\|) - \tau,$$

where $\gamma = \max\{r, q\} < 1$ and $r = \frac{a+b+c}{1-b-c} < 1$. Hence,

$$F(\|\xi_{n+1}(\omega) - \xi_{n+2}(\omega)\|) \leqslant F(\gamma \|\xi_n(\omega) - \xi_{n+1}(\omega)\|) - \tau, \tag{6}$$

and by (6), we obtain

$$F(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|) \leqslant F(\gamma^2 \|\xi_{n-2}(\omega) - \xi_{n-1}(\omega)\|) - 2\tau.$$

Repeating these steps, we can get

$$F(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|) \leqslant F(\gamma^n \|\xi_\circ(\omega) - \xi_1(\omega)\|) - n\tau. \tag{7}$$

Taking the limit in (7), we obtain $\lim_{n\to\infty} F(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|) = -\infty$. Since $F \in \digamma$,

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi_{n+1}(\omega)\| = 0. \tag{8}$$

From the axiom $(F_3)$ of $F$-contraction, there exists $\lambda \in (0, 1)$ such that

$$\lim_{n\to\infty} (\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^\lambda F(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|) = 0. \tag{9}$$

Multiplying $(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^\lambda$ to (7), we get

$$\begin{aligned} &(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^\lambda . [F(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|) - F(\gamma^n \|\xi_\circ(\omega) - \xi_1(\omega)\|)] \\ &\qquad \leqslant -(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^\lambda . n\tau \leqslant 0. \end{aligned} \tag{10}$$

Considering (8) and taking the limit as $n \to \infty$ in (10), one can write

$$\lim_{n\to\infty}(n(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^{\lambda}) = 0. \tag{11}$$

By (11), there exist $n_1 \in \mathbb{N}$ such that $n(\|\xi_n(\omega) - \xi_{n+1}(\omega)\|)^{\lambda} \leqslant 1$ for all $n \geqslant n_1$ or

$$\|\xi_n(\omega) - \xi_{n+1}(\omega)\| \leqslant \frac{1}{n^{\frac{1}{\lambda}}} \text{ for all } n \geqslant n_1. \tag{12}$$

Using (12), for $m > n \geqslant n_1$, we have

$$\begin{aligned}\|\xi_n(\omega) - \xi_m(\omega)\| &\leqslant \|\xi_n(\omega) - \xi_{n+1}(\omega)\| + \|\xi_{n+1}(\omega) - \xi_{n+2}(\omega)\| + ... + \|\xi_{m-1}(\omega) - \xi_m(\omega)\| \\ &= \sum_{i=1}^{m-1} \|\xi_i(\omega), \xi_{i+1}(\omega)\| \leqslant \sum_{i=1}^{\infty} \|\xi_i(\omega), \xi_{i+1}(\omega)\| \leqslant \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}.\end{aligned}$$

Since the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ is convergent, taking limit as $n \to \infty$, we get $\lim_{n\to\infty} \|\xi_n(\omega) - \xi_m(\omega)\| = 0$. This shows that $\{\xi_n(\omega)\}$ is a Cauchy sequence. Since $X$ is complete, there exists a measurable function $p(\omega) : \Omega \to X$ such that $\lim_{n\to\infty} \xi_n(\omega) = p(\omega)$ and moreover,

$$\lim_{n\to\infty} \xi_{2n+1}(\omega) = \lim_{n\to\infty} \xi_{2n+2}(\omega) = p(\omega).$$

The continuity of $S$ yields

$$p(\omega) = \lim_{n\to\infty} \xi_{2n+2}(\omega) = \lim_{n\to\infty} S(\omega, \xi_{2n+1}(\omega)) = S(\omega, \lim_{n\to\infty} \xi_{2n+1}(\omega)) = S(\omega, p(\omega)).$$

Similarly, $p(\omega) = T(\omega, p(\omega))$. Thus we get $p(\omega) = T(\omega, p(\omega)) = S(\omega, p(\omega))$. Hence the pair $(S, T)$ has a common random fixed point.

For uniqueness, let $v(\omega) \in \Omega \times X$ be another common random fixed point of $S$ and $T$ such that $p(\omega) \neq v(\omega)$. Then using (iii) and the contractive condition (3), we obtain that

$$\tau + F(\|T(\omega, p) - S(\omega, v)\|) \leqslant F(N(p(\omega), v(\omega))), \tag{13}$$

where

$$\begin{aligned}N(p(\omega), v(\omega)) &= \alpha \max \left\{ \begin{array}{c} \|p(\omega) - v(\omega)\|, \|p(\omega) - T(\omega, p)\|, \|v(\omega) - S(\omega, v)\|, \\ \|p(\omega) - S(\omega, v)\|, \|v(\omega) - T(\omega, p)\| \end{array} \right\} \\ &= \alpha \max \{\|p(\omega) - v(\omega)\|, 0, 0, \|p(\omega) - v(\omega)\|, \|p(\omega) - v(\omega)\|\} \\ &= \alpha \|p(\omega) - v(\omega)\|.\end{aligned}$$

Applying this to (13), we get

$$\tau + F(\|p(\omega) - v(\omega)\|) \leqslant F(\alpha \|p(\omega) - v(\omega)\|),$$

which implies that $\|p(\omega) - v(\omega)\| < \alpha \|p(\omega) - v(\omega)\|$, which is a contradiction (since $0 < \alpha < 1$). Hence $p(\omega) = v(\omega)$. This complete the proof. ■

The following numerical example justifies our theorem:

***Example* 3.3** Let $(\Omega, \beta)$ be a measurable space and $\Omega = C = [0,1] \subset \mathbb{R}$ with the usual normed $d(x(\omega), y(\omega)) = \|x(\omega) - y(\omega)\|$. Consider a $\sigma-$algebra $\beta$ of Lebesgue's measurable subsets of $\Omega$. Define random mappings $T, S : \Omega \times C \to C$ for all $\omega \in \Omega$ by

$$T(\omega, x) = \begin{cases} \frac{2}{3}x(\omega) & \text{if } x(\omega) \in \Omega - \{0\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad S(\omega, x) = \begin{cases} \frac{1}{3}x(\omega) & \text{if } x(\omega) \in \Omega - \{0\} \\ 0 & \text{otherwise} \end{cases}.$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(t) = \ln(t)$ for all $t > 0$. Then the contractive condition (3) is satisfied. Indeed for all $x, y \in X$ and $\omega \in \Omega$, the following inequality

$$\tau + \ln(\|T(\omega, x) - S(\omega, y)\|) \leqslant \ln(N(x(\omega), y(\omega))), \tag{14}$$

holds for all $\tau > 0$. Particularly, for $x(\omega) \in \Omega - \{0\}$ and $y(\omega) = 0$, we obtain the two cases:

1. If the axiom (i) is given, then we get

$$N(x(\omega), 0) = (a + \frac{1}{3}b + \frac{5}{3}c)x(\omega)$$

and

$$\|T(\omega, x) - S(\omega, 0)\| = \left\| \frac{2}{3}x(\omega) - 0 \right\| = \frac{2}{3}x(\omega).$$

Thus

$$\tau + \ln(\|T(\omega, x) - S(\omega, 0)\|) = \tau + \ln(\frac{2}{3}x(\omega)) \leqslant \ln(N(x(\omega), 0)) = \ln((a + \frac{1}{3}b + \frac{5}{3}c)x(\omega)).$$

Since $a + \frac{1}{3}b + \frac{5}{3}c \geqslant \frac{2}{3}$ permanently, the inequality (14) is verified.

2. If the axiom (ii) holds, then we have

$$N(x(\omega), 0) = \frac{5q}{3}x(\omega),$$

and so

$$\tau + \ln(\|T(\omega, x) - S(\omega, 0)\|) = \tau + \ln(\frac{2}{3}x(\omega)) \leqslant \ln(N(x(\omega), 0)) = \ln(\frac{5q}{3}x(\omega)).$$

Since $\frac{5q}{3} \geqslant \frac{2}{3}$ at $q = \frac{2}{5} < 1$, this gives (14).

Therefore, $S$ and $T$ are $F$-contractions. Hence all the axioms of Theorem 3.2 are satisfied and $x(\omega) = 0$ is a unique common random fixed point of $T$ and $S$.

The proof of the following corollary is obtained by setting $S = T$ in Theorem 3.2.

**Corollary 3.4** Let $X$ be a separable Banach space and $(\Omega, \beta, \mu)$ be a probability measure space. Suppose that $T : \Omega \times X \to X$ be a random mapping satisfying

$$\tau + F(\|T(\omega, x) - T(\omega, y)\|) \leqslant F(N(x(\omega), y(\omega))), \tag{15}$$

where $F \in \digamma$, $\tau > 0$ and $N(x(\omega), y(\omega))$ is one of the following inequalities:

(i)
$$N(x(\omega), y(\omega)) = a\|x(\omega) - y(\omega)\| + b[\|x(\omega) - T(\omega, x)\| + \|y(\omega) - T(\omega, y)\|] \\ +c[\|x(\omega) - T(\omega, y)\| + \|y(\omega) - T(\omega, x)\|],$$

where $1 - b - c > 0$, $a > 0$, $b \geqslant 0$ and $c \geqslant 0$,

(ii)
$$N(x(\omega), y(\omega)) = q \max\left\{ \begin{array}{c} \|x(\omega) - y(\omega)\|, \|x(\omega) - T(\omega, x)\| + \|y(\omega) - T(\omega, y)\|, \\ \|x(\omega) - T(\omega, y)\| + \|y(\omega) - T(\omega, x)\| \end{array} \right\},$$

where $0 < q < 1$.

(iii)
$$N(x(\omega), y(\omega)) = \alpha \max\left\{ \begin{array}{c} \|x(\omega) - y(\omega)\|, \|x(\omega) - T(\omega, x)\|, \|y(\omega) - T(\omega, y)\|, \\ \|x(\omega) - T(\omega, y)\|, \|y(\omega) - T(\omega, x)\| \end{array} \right\},$$

where $0 < \alpha < 1$.

Then $T$ has a unique random fixed point $p(\omega)$.

## 4. Application

In this section, we present an application of Corollary 3.4 to discuss the existence of a random solution for NFRDE as the form:

$$^{c}D^{r}(x(\omega, t)) = f(\omega, t, x(\omega, t)),\ t \in [0, 1],\ 0 < r \leqslant 1, \tag{16}$$

with the integral boundary conditions $x(\omega, 0) = 0$, $x(\omega, 1) = \int_0^{\epsilon} x(\omega, s)ds, 0 < \epsilon < 1$. Here $\omega \in \Omega$ is a supporting set of the measure space $(\Omega, \beta)$, $x(t, \omega)$ is unknown vector-valued random variables for each $t \in [0, 1]$ and $^{c}D^{r}$ denotes the Caputo fractional derivative of order $r$, for more extensively details (see [1, 10, 13, 17, 28]). Also, $f : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a measurable continuous function. Here $(X, \|.\|_{\infty})$, where $X = C([0, 1], \mathbb{R})$ is the Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}$ endowed with the supremum norm $\|x(\omega)\|_{\infty} = \sup_{t\in[0,1]} |x(\omega, t)|$. It is well known that $x(t, \omega) \in X$ is a solution of system (16) if and only if it is a solution of the following random integral equation for $t \in [0, 1]$

$$x(\omega, t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} f(\omega, s, x(\omega, s))ds - \frac{2t}{(2 - \epsilon^2)\Gamma(r)} \int_0^1 (1 - s)^{r-1} f(\omega, s, x(\omega, s))ds \\ + \frac{2t}{(2 - \epsilon^2)\Gamma(r)} \int_0^{\epsilon} \left( \int_0^s (s - m)^{r-1} f(\omega, m, x(\omega, m))dm \right) ds.$$

Next, we consider a system (16) under the following axioms. Suppose that

($H_1$) there exist a measurable continuous function $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\tau > 0$ such that

$$|f(\omega,t,a) - f(\omega,t,b)| \leqslant \frac{\Gamma(r+1)}{5} e^{-\tau} |a-b|,$$

for all $t \in [0,1]$ and $a, b \in \mathbb{R}$,

($H_2$)

$$\frac{\Gamma(r+1)}{5} \sup_{t\in(0,1)} \Big( \frac{1}{\Gamma(r)} \int_0^1 |t-s|^{r-1} ds + \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^1 (1-s)^{r-1} ds$$
$$+ \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^\epsilon \int_0^s |s-m|^{r-1} dm ds \Big) \leqslant 1.$$

Now, we prove the following existence theorem.

**Theorem 4.1** Let $(\Omega, \beta)$ be a measure space and $\mathbb{R}$ a Banach space. Then the system (16) under assumptions ($H_1$) and ($H_2$) has at least one random solution.

**Proof.** For $x(\omega), y(\omega) \in C([0,1], \mathbb{R})$, $\omega \in \Omega$ and $t \in [0,1]$, we define a random operator $T : \Omega \times [0,1] \to \mathbb{R}$ by

$$T(x)(\omega,t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(\omega,s,x(\omega,s)) ds$$
$$- \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^1 (1-s)^{r-1} f(\omega,s,x(\omega,s)) ds$$
$$+ \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^\epsilon \left( \int_0^s (s-m)^{r-1} f(\omega,m,x(\omega,m)) dm \right) ds.$$

Then, problem (16) is equivalent to find $p(\omega) \in \Omega \times X$ which is a random fixed point of $T$. Now, let $x(\omega), y(\omega) \in C([0,1], \mathbb{R})$, $\omega \in \Omega$ for all $t \in [0,1]$. By the axioms ($H_1$) and ($H_2$), we have

$$|T(x)(\omega,t) - T(y)(\omega,t)|$$
$$= \Bigg| \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(\omega,s,x(\omega,s)) ds - \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^1 (1-s)^{r-1} f(\omega,s,x(\omega,s)) ds$$
$$+ \frac{2t}{(2-\epsilon^2)\Gamma(r)} \int_0^\epsilon \left( \int_0^s (s-m)^{r-1} f(\omega,m,x(\omega,m)) dm \right) ds$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(\omega,s,y(\omega,s))ds + \frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^1 (1-s)^{r-1} f(\omega,s,y(\omega,s))ds \\
&-\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^\epsilon \left(\int_0^s (s-m)^{r-1}(f(\omega,m,y(\omega,m))dm\right) ds\Bigg| \\
\leqslant\ & \frac{1}{\Gamma(r)}\int_0^t |t-s|^{r-1} \left|f(\omega,s,x(\omega,s)) - f(\omega,s,y(\omega,s))\right| ds \\
&+\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^1 (1-s)^{r-1} \left|f(\omega,s,x(\omega,s)) - f(\omega,s,y(\omega,s))\right| ds \\
&+\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^\epsilon \left|\int_0^s (s-m)^{r-1}(f(\omega,m,y(\omega,m)) - f(\omega,m,x(\omega,m)))dm\right| ds \\
\leqslant\ & \frac{1}{\Gamma(r)}\int_0^t |t-s|^{r-1} \frac{\Gamma(r+1)}{5} e^{-\tau} \left|x(\omega,s) - y(\omega,s)\right| ds \\
&+\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^1 (1-s)^{r-1} \frac{\Gamma(r+1)}{5} e^{-\tau} \left|x(\omega,s) - y(\omega,s)\right| ds \\
&+\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^\epsilon \left(\int_0^s |s-m|^{r-1} \frac{\Gamma(r+1)}{5} e^{-\tau} \left|x(\omega,m) - y(\omega,m)\right| dm\right) ds \\
\leqslant\ & e^{-\tau}\left\|x(\omega,s) - y(\omega,s)\right\|_\infty . \frac{\Gamma(r+1)}{5} \sup_{t\in(0,1)} \left(\frac{1}{\Gamma(r)}\int_0^1 |t-s|^{r-1} ds\right. \\
&\left.+\frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^1 (1-s)^{r-1} ds + \frac{2t}{(2-\epsilon^2)\Gamma(r)}\int_0^\epsilon\int_0^s |s-m|^{r-1} dmds\right) \\
\leqslant\ & e^{-\tau}\left\|x(\omega,s) - y(\omega,s)\right\|_\infty .
\end{aligned}
$$

This gives

$$\|T(x)(\omega,t) - T(y)(\omega,t)\|_\infty \leqslant e^{-\tau}\|x(\omega,s) - y(\omega,s)\|_\infty ,$$

or, equivalently

$$\tau + \ln(\|T(x)(\omega,t) - T(y)(\omega,t)\|) \leqslant \ln(N(x(\omega),y(\omega))).$$

Hence, the $F$-contraction (15) is satisfied by taking $F(t) = ln(t)$ and $N(x(\omega),y(\omega)) = \|x(\omega,s) - y(\omega,s)\|_\infty$ for every $\tau > 0$ and $t \in [0,1]$. So all the conditions of Corollary 3.4 are satisfied. Therefore, the system NFRDE (16) has a unique random solution $p(\omega)$. ■

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