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On some open problems in cone metric space over Banach algebra

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Abstract. In this paper we prove an analogue of Banach and Kannan fixed point theorems by generalizing the Lipschitz constat k, in generalized Lipschitz mapping on cone metric space over Banach algebra, which are answers for the open problems proposed by Sastry et al, [K. P. R. Sastry, G. A. Naidu, T. Bakeshie, Fixed point theorems in cone metric spaces with Banach algebra cones, Int. J. of Math. Sci. and Engg. Appl. (6) (2012), 129-136].

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1. Introduction

In 1922, Banach [1] proved the fixed point theorem for contraction mapping in a metric space, which becomes very famous due to its wide applications. Several authors generalized this results, one of the generalization is given by Liu and Xu [2] in 2013. They replaced the constant k with vector k, the element of cone P with the condition that $r(k) < 1$.

In 2012, Sastry et al [4] proved the following theorems.

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Theorem 1.1 Let (X, d) be a complete cone metric space and $P \subseteq E$ be a Banach algebra cone. Let T be a self mapping of X satisfying

$$d(Tx, Ty) \leqslant kd(x, y)$$

for some $k \in P$ with $0 < \|k\| < 1$ and for all $x, y \in X$. Then T has a unique fixed point.

Theorem 1.2 Let (X, d) be a complete cone metric space and $P \subseteq E$ be a Banach algebra cone. Let T be a self mapping of X. If there exists $k \in P$ such that k is invertible, $k^{-1} \in P$, $0 < \|k\| < \frac{1}{2}$ and

$$d(Tx, Ty) \leqslant k[d(Tx, x) + d(Ty, y)]$$

for all $x, y \in X$, then T has a unique fixed point.

In [4], the authors raised the following open problems.

Question 1.3 If the hypothesis $0 < \|k\| < 1$ in Theorem 1.1 is replaced by the vector $k \in P$, with $\theta \prec k \prec e$ or $\theta \prec k \ll e$, then do we still get the same conclusion?

Question 1.4 If the hypothesis $0 < \|k\| < \frac{1}{2}$ in Theorem 1.2, is replaced by the vector $k \in P$, with $\theta \prec k \prec \frac{1}{2}e$ or $\theta \prec k \ll \frac{1}{2}e$, then do we still get the same conclusion?

In this paper we give the positive answer to these problems and which are more general form of Banach and Kannan fixed point theorems, in which we replace the condition $r(k) < 1$ with $\theta \prec k \prec e$. The following definitions and results will be needed in the sequel.

Let A be a real Banach algebra that is a real Banach space in which an operation of multiplication is defined subject to the following properties:

(1) $(xy)z = x(yz)$,
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$,
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
(4) $||xy|| \leqslant \|x\|\|y\|$.

for all $x, y, z \in A$ and $\alpha \in \mathbb{R}$. An algebra A with unit element e, is called unital algebra A, i.e. multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

A subset P of a Banach algebra A is called a cone if

(1) P is non-empty, closed and $\{\theta, e\} \subset P$;
(2) $aP + bP \subset P$, for all non-negative real numbers a and b;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$;

where θ denotes the null of the Banach algebra A. For a given cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$ and we write $x \prec y$ if $x \preceq y$ and $x \neq y$ while $x \ll y$ will stands for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$ then P is called a solid cone. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

Definition 1.5 ([2]) Let X be a non-empty set. Let A be a Banach algebra and $P \subseteq A$ be a cone. Suppose the mapping $d\colon X \times X \to A$ satisfies

(1) $\theta \preceq d(x, y)$ for every $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for every $x, y \in X$,
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

Then (X, d) is called cone metric space over a Banach algebra.

***Example* 1.6** Let $A = \{a = (a_{ij})_{33} : a_{ij} \in \mathbb{R}, 1 \leqslant i, j \leqslant 3\}$ and

$$\|a\| = \frac{1}{3} \sum_{1 \leqslant i,j \leqslant 3} |a_{ij}|.$$

Take a cone $P = \{a \in A : a_{ij} \geqslant 0, 1 \leqslant i, j \leqslant 3\}$ in A. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \longrightarrow A$ by $d(1, 1) = d(2, 2) = d(3, 3) = (0)_{33}$ and

$$d(1,2) = d(2,1) = \begin{pmatrix} 2 & 2 & 4 \\ 3 & 4 & 3 \\ 2 & 5 & 3 \end{pmatrix},$$

$$d(3,1) = d(1,3) = \begin{pmatrix} 5 & 4 & 6 \\ 6 & 7 & 5 \\ 5 & 6 & 4 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 4 & 5 & 6 \\ 5 & 7 & 6 \\ 7 & 7 & 5 \end{pmatrix}.$$

Then (X, d) be a cone metric space over Banach algebra.

Definition 1.7 ([5]) Let (X, d) be a cone metric space over a Banach algebra A, $x \in X$ and let $\{x_n\}$ be a sequence in X and $\{u_n\}$ a sequence in A. Then

(1) $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, c) \ll c$ for all $n \geqslant N$;
(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$;
(3) A cone metric space over Banach algebra X is said to be complete if every Cauchy sequence in X is convergent in X;
(4) $\{u_n\}$ is a c-sequence if for each $\theta \ll c$, there is a natural number N such that $u_n \ll c$ for all $n \geqslant N$.

Lemma 1.8 *([6])* Let E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$ then $u = \theta$.

Proposition 1.9 *([7])* Let P be a solid cone in a Banach space A and let $\{x_n\}$ and $\{y_n\}$ are sequences in P. If $\{x_n\}$ and $\{y_n\}$are c-sequence and $\alpha, \beta \in P$ then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.

2. Main Result

Now we define more generalized Lipschitz mapping on the cone metric space over Banach algebra.

Definition 2.1 Let (X, d) be a cone metric space over Banach algebra. A mapping $T: X \to X$ is called more generalized Lipschitz mapping if there exists a vector $k \in P$ with $k \prec e$ and for all $x, y \in X$, we have

$$d(Tx, Ty) \preceq kd(x, y).$$

Lemma 2.2 Let $P \subseteq A$ be a cone of Banach algebra A. If $a, k \in P$ are such that $k \prec e$, $a \preceq ka$ and $(e-k)^{-1}$ exists then $a = \theta$.

Proof. Since $a, k \in P$ and $a \preceq ka$, we have $ka - a \in P$. Consequently, $\theta \preceq (k-e)a \in P$. Since $(e-k)^{-1} \in P$ exists .We get it $-(e-k)^{-1}(e-k)a \in P$, and from here follows $-a \in P$. Since $P \cap (-P) = \{\theta\}$, we have $a = \theta$. ■

Lemma 2.3 Let (X, d) be a cone metric space over a Banach algebra A, $k \in X$ such that $\theta \preceq k \prec e$ and $(e-k)^{-1}$ exists. Then $\{k^n\}$ is a c-sequence.

Proof. How do $\theta \preceq k \prec e$ we get it

$$k^n \preceq k^i \tag{1}$$

for all $i \in \{1, \dots, n\}$. Therefore, we obtain

$$(n+1)k^n \preceq e + k + k^2 + \cdots + k^n. \tag{2}$$

From $\theta \preceq e - k$ and (2), we have

$$\begin{aligned}(e-k)(n+1)k^n &\preceq (e-k)(e + k + k^2 + \cdots + k^n) \\ &= e - k^{n+1} \\ &\preceq e.\end{aligned}$$

Thus, $(e-k)(n+1)k^n \preceq e$. Hence,

$$k^n \preceq \frac{1}{n+1}(e-k)^{-1}.$$

For any $\theta \ll c$ we get that there exists a positive integer n_0 such that $n \geqslant n_0$ implies $\frac{1}{n+1}(e-k)^{-1} \ll c$ (because $\left\{\frac{1}{n+1}(e-k)^{-1}\right\}$ is a convergent sequence). So $n \geqslant n_0$ implies $k^n \ll c$ and $\{k^n\}$ is c-sequence. ■

Now we shall prove Banach fixed point theorem for more generalized Lipschitz mapping in setting of a cone metric space over Banach algebra.

Theorem 2.4 Let (X, d) be a complete a cone metric space over Banach algebra A. Let $k \in P$ such that $k \prec e$ and $(e-k)^{-1}$ exists. Suppose that the mapping $T : X \to X$ satisfies more generalized Lipschitz condition

$$d(Tx, Ty) \preceq kd(x, y) \tag{3}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary, choose $x_n \in X$ such that $x_n = Tx_{n-1}$ for $n \geqslant 1$, we get the sequence $\{x_n\}$. From condition (3) we obtain,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq kd(x_n, x_{n-1}) \preceq ... \preceq k^n d(x_0, x_1). \tag{4}$$

If $x_{n+1} = x_n$, then x_n is a fixed point of T and hence the proof is complete. So, let $x_{n+1} \neq x_n$ for $n \geqslant 0$. Hence $x_{n+q} \neq x_n$ for $n \geqslant 0$ and $q \geqslant 1$. If possible suppose $x_{n+q} = x_n$ for some $n \geqslant 0$ and $q \geqslant 1$ then we have, $x_{n+q+1} = x_{n+1}$ then from condition (3) we obtain

$$d(x_{n+1}, x_n) = d(x_{n+q+1}, x_{n+q}) \preceq k^q d(x_{n+1}, x_n).$$

Hence, from Lemma 2.2, we have $d(x_{n+1}, x_n) = \theta$. So, $x_{n+1} = x_n$, which is contradiction. From triangular inequality and condition (3), we have

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\preceq k^m d(x_0, x_1) + kd(x_m, x_n) + k^n d(x_0, x_1), \end{aligned}$$

which implies that

$$(e - k)d(x_m, x_n) \preceq (k^m + k^n)d(x_0, x_1).$$

Thus, $d(x_m, x_n) \preceq (e-k)^{-1}(k^m + k^n)d(x_0, x_1)$, for all $m, n \in \mathbb{N}$ and $m < n$. Hence, from Lemma 2.3 and Proposition 1.9 we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). By completeness of (X, d), there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. We have

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(Tx^*, x_n) + d(x_n, x^*) \\ &= d(Tx^*, Tx_{n-1}) + d(x_n, x^*) \\ &\preceq kd(x^*, x_{n-1}) + d(x_n, x^*). \end{aligned}$$

Since $\{d(x_n, x^*)\}$ is a c-sequence, we obtain $d(Tx^*, x^*) = \theta$. So, we have $Tx^* = x^*$. Hence, x^* is a fixed point of T. Let y^* be the another fixed point of T. Then we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq kd(x^*, y^*).$$

Thus, by Lemma 2.2, we have $d(x^*, y^*) = \theta$. Hence x^* is an unique fixed point of T. ■

Now, we shall prove Kannan fixed point theorem for more generalized Lipschitz mapping in setting of a cone metric space over Banach algebra.

Theorem 2.5 Let (X, d) be a complete a cone metric space over Banach algebra A. Let $k \in P$ such that $\theta \preceq k \prec \frac{1}{2}e$ and $(e - k)^{-1}$ exists. Suppose that the mapping $T : X \to X$ satisfies more generalized Lipschitz condition

$$d(Tx, Ty) \preceq k[d(Tx, x) + d(Ty, y)], \tag{5}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary, choose $x_n \in X$ and set $x_n = Tx_{n-1}$ for $n \geqslant 1$, we get the sequence $\{x_n\}$. Now, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\preceq k[d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})], \end{aligned}$$

which implies that

$$d(x_{n+1}, x_n) \preceq k(e-k)^{-1} d(x_n, x_{n-1}).$$

Let $h = k(e-k)^{-1}$. Since $k \prec \frac{e}{2}$, we obtain $k \prec e - k$ and from here follows $h \prec e$. Therefore,

$$d(x_{n+1}, x_n) \preceq h^n d(x_1, x_0). \tag{6}$$

If $x_{n+1} = x_n$, then x_n is a fixed point of T. Hence the proof is complete. Thus, let $x_{n+1} \neq x_n$ for $n \geqslant 0$. Hence, $x_{n+q} \neq x_n$ for $n \geqslant 0$ and $q \geqslant 1$. If possible suppose $x_{n+q} = x_n$ for some $n \geqslant 0$ and $q \geqslant 1$, then we have $x_{n+q+1} = x_{n+1}$, and (6) gives

$$d(x_{n+1}, x_n) = d(x_{n+q+1}, x_{n+q}) \preceq h^q d(x_{n+1}, x_n).$$

Hence, from Lemma 2.2, we have $d(x_{n+1}, x_n) = \theta$. So, $x_{n+1} = x_n$, which is contradiction. Now, from conditions (5) and (6), we have

$$\begin{aligned} d(x_m, x_n) &\preceq k[d(x_{m+1}, x_m) + d(x_{n+1}, x_n) \\ &\preceq k[h^m d(x_1, x_0) + h^n d(x_1, x_0)] \\ &\preceq k(h^m + h^n) d(x_1, x_0) \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $m < n$. Consequently, from Lemma 2.3 and Proposition 1.9, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). By completeness of (X, d) there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Thus,

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(Tx^*, x_n) + d(x_n, x^*) \\ &= d(Tx^*, Tx_{n-1}) + d(x_n, x^*) \\ &\preceq k[d(Tx^*, x^*) + d(Tx_{n-1}, x_{n-1})] + d(x_n, x^*). \end{aligned}$$

Consequently,

$$d(Tx^*, x^*) \preceq (e-k)^{-1} k d(x_n, x_{n-1}) + (e-k)^{-1} d(x_n, x^*).$$

Since $\{d(x_n, x^*)\}$ is a c-sequence we obtain $d(Tx^*, x^*) = \theta$. So, $Tx^* = x^*$.

Now, we obtain that x^* is a unique fixed point of T. Let y^* be the another fixed point of T. Then we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq k[d(Tx^*, x^*) + d(Ty^*, y^*)] \preceq \theta,$$

which implies $x^* = y^*$. So, x^* is an unique fixed point of T. ∎

Remark 1 *Note that we have used some of the ideas from the paper [3].*

We present one example to illustrate the Theorem 2.4.

***Example* 2.6** Consider the Example 1.6. Let $T : X \to X$ be a mapping define by $T1 = T2 = 2$, $T3 = 1$ and $k = \begin{pmatrix} 0.7 \; 0.0 \; 0.0 \\ 0.0 \; 0.6 \; 0.0 \\ 0.0 \; 0.0 \; 0.9 \end{pmatrix} \in P$, then $k \prec e$ and $(e-k)^{-1}$ exists. By simple calculations we can see that all the conditions of Theorem 2.4 are satisfied. The point $x = 2$ is the unique fixed point of T.

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