Preclosure operator and its applications in general topology

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Abstract. In this paper, we show that a pointwise symmetric pre-isotonic preclosure function is uniquely determined the pairs of sets it separates. We then show that when the preclosure function of the domain is pre-isotonic and the preclosure function of the codomain is pre-isotonic and pointwise-pre-symmetric, functions which separate only those pairs of sets which are already separated are precontinuous.

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1. Introduction

Generalized open sets play a very important role in general topology and they are now the research topics of many topologist worldwide. Indeed a significant there in general topology and real analysis concerns the variously modified forms of continuity, separation axioms, compactness etc by utilizing generalized open sets. One of the most well-known
notations and also an inspiration source is the notion of preopen sets introduced by Moshour et al. [7]. Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by \( \text{Int}(A) \) and \( \text{Cl}(A) \), respectively. \( A \subset X \) is called a preopen [6,7] or nearly open [8] or locally dense [2] set of X if \( A \subset \text{Int}(\text{Cl}(A)) \). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing a set \( A \) is called the preclosure [3] of \( A \) and is denoted by \( p\text{Cl}(A) \). Notions and notations not described in this paper are standard and usual. This paper is closely related to [1].

**Definition 1.1** (1) A generalized preclosure space is a pair \((X, p\text{Cl})\) consisting of a set \( X \) and a preclosure function \( p\text{Cl} \), a function from the power set of \( X \) to itself.

(2) The preclosure of a subset \( A \) of \( X \), denoted \( p\text{Cl} \), is the image of \( A \) under \( p\text{Cl} \).

(3) The pre-exterior of \( A \) is \( p\text{Ext}(A) = X \setminus p\text{Cl}(A) \), and the pre-interior of \( A \) is \( p\text{Int}(A) = X \setminus \text{Cl}(X \setminus A) \).

(4) \( A \) is preclosed if \( A = p\text{Cl}(A) \), \( A \) is preopen if \( A = p\text{Int}(A) \) and \( N \) is a preneighborhood of a point \( x \in X \) [4], [5] if \( x \in p\text{Int}(N) \).

**Definition 1.2** A preclosure function \( p\text{Cl} \) defined on \( X \) is:

(1) pre-grounded if \( p\text{Cl}(\emptyset) = \emptyset \).

(2) pre-isotonic if \( p\text{Cl}(A) \subseteq p\text{Cl}(B) \) whenever \( A \subseteq B \).

(3) pre-enlarging if \( A \subseteq p\text{Cl}(A) \) for each subset \( A \) of \( X \).

(4) pre-idempotent if \( p\text{Cl}(A) = p\text{Cl} \), \( p\text{Cl}(p\text{Cl}(A)) \) for each subset \( A \) of \( X \).

(5) pre-sub-linear if \( p\text{Cl}(A \cup B) \subseteq p\text{Cl}(A) \cup p\text{Cl}(B) \) for all \( A, B \subseteq X \).

**Definition 1.3** (1) Subsets \( A \) and \( B \) of \( X \) are said to be pre-closure-separated in a generalized preclosure space \((X, p\text{Cl})\) (or simply \( p\text{Cl} \)-separated) if \( A \cap p\text{Cl}(B) = \emptyset \) and \( B \cap p\text{Cl}(A) = \emptyset \) or equivalently, if \( A \subseteq p\text{Ext}(B) \) and \( B \subseteq p\text{Ext}(A) \).

(2) Pre-Exterior points are said to be preclosure-separated in a generalized preclosure \((X, p\text{Cl})\) if for each \( A \subseteq X \) and for each \( x \in p\text{Ext}(A) \), \( \{x\} \) and \( A \) are \( p\text{Cl} \)-separated.

**Theorem 1.4** Let \((X, p\text{Cl})\) be a generalized preclosure space in which pre-Exterior points are \( p\text{Cl} \)-separated and let \( S \) be the pairs of \( p\text{Cl} \)-separated sets in \( X \). Then, for each subset \( A \) of \( X \), the preclosure of \( A \) is \( p\text{Cl}(A) = \{x \in X : \{x\}, A \notin S \} \).

**Proof.** In any generalized preclosure space \( p\text{Cl}(A) \subseteq \{x \in X : \{x\}, A \notin S \} \). Suppose that \( y \notin \{x \in X : \{x\}, A \notin S \} \); that is, \( \{y\}, A \in S \). Then \( \{y\} \cap p\text{Cl}(A) = \emptyset \), and so \( y \notin p\text{Cl}(A) \). Now, let \( y \notin p\text{Cl}(A) \). By hypothesis, \( \{y\}, A \in S \). Therefore, \( y \notin \{x \in X : \{x\}, A \notin S \} \).

\[ \square \]

2. Some basic properties

**Definition 2.1** A preclosure function \( p\text{Cl} \) defined on a set \( X \) is said to be pointwise pre-symmetric when, for all \( x, y \in X \), if \( x \in p\text{Cl}(\{y\}) \), then \( y \in p\text{Cl}(\{x\}) \).

A generalized preclosure space \((X, p\text{Cl})\) is said to be pre-\( R_0 \) when, for all \( x, y \in X \), if \( x \) is in each preneighborhood of \( y \), then \( y \) is in each preneighborhood of \( x \).

**Corollary 2.2** Let \((X, p\text{Cl})\) be a generalized preclosure space in which \( p\text{Ex} \)-Exterior points are \( p\text{Cl} \)-separated. Then \( p\text{Cl} \) is pointwise pre-symmetric and \((X, p\text{Cl})\) is pre-\( R_0 \).

**Proof.** Let pre-Exterior points be \( p\text{Cl} \)-separated in \((X, p\text{Cl})\). If \( x \in p\text{Cl}(\{y\}) \), then \( \{x\} \) and \( \{y\} \) are not \( p\text{Cl} \)-separated. This means that \( y \in p\text{Cl}(\{x\}) \). Hence, \( p\text{Cl} \) is pointwise pre-symmetric. Suppose that \( x \) belongs to every preneighborhood of \( y \); that is, \( x \in M \).
whenever \( y \in pInt(M) \). Letting \( A = X \setminus M \) and rewriting contrapositively, \( y \in pCl(A) \) whenever \( x \in A \). Let \( x \in pInt(N) \) consequently \( x \notin pCl(X \setminus N) \), so \( x \) is \( pCl \)-separated from \( X \setminus N \). Hence \( pCl(\{x\}) \subseteq N, x \in \{x\}, \) so \( y \in pCl(\{x\}) \subseteq N \). Hence, \((X, pCl)\) is pre-\( R_0 \).

Observe that these three axioms are not equivalent in general, but they are equivalent when the preclosure function is pre-isotonic.

**Theorem 2.3** Let \((X, pCl)\) be a generalized preclosure space with \( pCl \) pre-isotonic. Then the following are equivalent:

1. \( pExterior \) points are \( pCl \)-separated.
2. \( pCl \) is pointwise pre-symmetric.
3. \((X, pCl)\) is pre-\( R_0 \).

**Proof.** Suppose that (2) is true. Let \( A \subseteq X \), and let \( x \in pExt(A) \). Then, as \( pCl \) is pre-isotonic, for each \( y \in A, x \notin pCl(\{y\}) \), and thus, \( y \notin pCl(\{x\}) \). Hence \( A \cap pCl(\{x\}) = \phi \). Therefore (2) implies (1). Moreover, by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let \( x, y \in X \) such that \( x \) is in every preneighbohood of \( y \), i.e. \( x \in N \) whenever \( y \in pInt(N) \). Then \( y \in pCl(A) \) whenever \( x \in A \), and in particular, since \( x \in \{x\}, y \in pCl(\{x\}) \). It follows that \( x \in pCl(\{y\}) \). Thus if \( y \in B \), then \( x \in pCl(\{y\}) \subseteq pCl(B) \), as \( pCl \) is pre-isotonic. Therefore, if \( x \in pInt(C) \), then \( y \in C \), that is, \( y \) is in every preneighbohood of \( x \). Hence, (2) implies (3).

Now, let \((X, pCl)\) be pre-\( R_0 \) and \( x \in pCl(\{y\}) \). Since \( pCl \) is pre-isotonic, \( x \in pCl(B) \) whenever \( y \in B \), or equivalently, \( y \) is in every preneighbohood of \( x \). Since \((X, pCl)\) is pre-\( R_0 \), \( x \in N \) whenever \( y \in pInt(N) \). Therefore, \( y \in pCl(\{A\}) \) whenever \( x \in A \), and in particular, since \( x \in \{x\}, y \in pCl(\{x\}) \). It follows that (3) implies (2).

**Theorem 2.4** Let \( S \) be a set of unordered pairs of subsets of a set \( X \) such that, for all \( A, B, C \subseteq X \),

1. if \( A \subseteq B \) and \( \{B, C\} \in S \), then \( \{A, C\} \in S \) and
2. if \( \{x\}, B \in S \) for each \( x \in A \) and \( \{y\}, A \in S \) for each \( y \in B \), then \( \{A, B\} \in S \).

Then there exists a unique pointwise pre-symmetric pre-isotonic preclosure function \( pCl \) on \( X \) which pre-closure-separates the elements of \( S \).

**Proof.** Define \( pCl \) by \( pCl(A) = \{x \in X : \{x\}, A \notin S \} \) for every \( A \subseteq X \). If \( A \subseteq \emptyset \subseteq X \) and \( x \in pCl(A) \), then \( \{x\}, A \notin S \). Thus \( \{x\}, B \notin S \), that is, \( x \in \emptyset \). Hence \( pCl \) is pre-isotonic. Moreover \( x \in pCl(\{y\}) \) if and only if \( \{x\}, \{y\} \notin S \) if and only if \( y \in pCl(\{x\}) \). Thus \( pCl \) is pointwise pre-symmetric. Suppose that \( \{A, B\} \in S \). Then \( A \cap pCl(B) = A \cap \{x \in X : \{x\}, B \notin S \} = \{x \in A : \{x\}, A \notin S \} = \phi \). Similarly, \( pCl(A) \cap B = \phi \). Therefore, if \( \{A, B\} \in S \), then \( A \) and \( B \) are \( pCl \)-separated.

Now suppose that \( A \) and \( B \) are \( pCl \)-separated. Then \( \{x \in A : \{x\}, B \notin S \} = A \cap pCl(B) = \emptyset \) and \( \{x \in B : \{x\}, A \notin S \} = pCl(A) \cap B = \emptyset \). Hence, \( \{x\}, B \in S \) for each \( x \in A \) and \( \{y\}, A \in S \) for each \( y \in B \). Therefore, \( \{A, B\} \in S \).

In the following we show that many properties of preclosure functions can be expressed in terms of the sets they separate.

**Theorem 2.5** Let \( S \) be the pairs of \( pCl \)-separated sets of a generalized preclosure space \((X, pCl)\) in which pre-exterior points are preclosure-separates. Then \( pCl \) is

1. pre-grounded if and only if for all \( x \in X, \{x\}, \phi \in S \).
2. pre-enlarging if and only if for all \( \{A, B\} \in S, A \) and \( B \) are disjoint.
3. pre-sub-linear if and only if \( \{A, B \cup C\} \in S \) whenever \( \{A, B\} \in S \) and \( \{A, C\} \in S \).

Furthermore, if \( pCl \) is pre-enlarging and for all \( A, B \subseteq X, \{x\}, A \notin S \) whenever
\[ \{\{x\}, B\} \notin S \text{ and } \{\{y\}, A\} \notin S \text{ for each } y \in B, \text{ then } pCl \text{ is pre-idempotent. Now, if} \\
pCl \text{ is pre-isotonic and pre-idempotent, then } \{\{x\}, A\} \notin S \text{ whenever } \{\{x\}, B\} \notin S \text{ and} \\
\{\{y\}, A\} \notin S \text{ for each } y \in B. \]

**Proof.** (1) By Theorem 1.4, \( pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\} \) for every \( A \subseteq X \). Suppose that for all \( x \in X, \{\{x\}, \phi\} \in S \). Then \( pCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\} = \phi \).

Hence \( pCl \) is pre-grounded. Conversely, if \( \phi = pCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\} \), then \( \{\{x\}, \phi\} \in S \), for all \( x \in X \).

(2) Assume that for all \( \{A, B\} \subseteq S \), \( A \) and \( B \) are disjoint. Since \( \{\{a\}, A\} \notin S \) if \( a \in A, A \subseteq pCl(A) \) for each \( A \subseteq X \). Therefore, \( pCl \) is pre-enlarging. Conversely, let \( pCl \) be pre-enlarging and \( \{A, B\} \subseteq S \). Then \( A \cap B \subseteq pCl(A) \cap B = \phi \).

(3) Suppose that \( \{A, B \cup C\} \subseteq S \) whenever \( \{A, B\} \subseteq S \) and \( \{A, C\} \subseteq S \). Let \( x \in X \) and \( B, C \subseteq X \) such that \( \{\{x\}, B \cup C\} \notin S \). Then \( \{\{x\}, B\} \notin S \) or \( \{\{x\}, C\} \notin S \).

Hence \( pCl(B \cup C) \subseteq pCl(B) \cup pCl(C) \). Therefore, \( pCl \) is pre-sub-linear. Conversely, suppose that \( pCl \) is pre-sub-linear and let \( \{A, B\}, \{A, C\} \subseteq S \). Then \( pCl(B \cup C) \cap A \subseteq (pCl(B) \cup pCl(C)) \cap A \subseteq (pCl(B) \cap A) \cup (pCl(C) \cap A) = \phi \) and \( (B \cup C) \cap pCl(A) = (B \cap pCl(A)) \cup (C \cap pCl(A)) = \phi \).

Let \( pCl \) be pre-enlarging and suppose that \( \{\{x\}, A\} \notin S \) whenever \( \{\{x\}, B\} \notin S \) and \( \{\{y\}, A\} \notin S \) for each \( y \in B \). Then \( pCl(pCl(A)) \subseteq pCl(A) \).

If \( x \in pCl(pCl(A)) \), then \( \{\{x\}, pCl(A)\} \notin S \). \( \{\{y\}, A\} \notin S \), for each \( y \in pCl(A) \); hence \( \{\{x\}, A\} \notin S \). Since \( pCl \) is pre-enlarging, then \( pCl(A) \subseteq pCl(pCl(A)) \).

Therefore, \( pCl(pCl(A)) = pCl(A) \) for each \( A \subseteq X \). Suppose that \( pCl \) is pre-isotonic and pre-idempotent. Let \( x \in X \) and \( A, B \subseteq X \) such that \( \{\{x\}, B\} \notin S \) and for each \( y \in B \), \( \{\{y\}, A\} \notin S \). Then \( x \in pCl(B) \) and for each \( y \in B \), \( y \in pCl(A) \), i.e. \( B \subseteq pCl(A) \). Therefore, \( x \in pCl(B) \subseteq pCl(pCl(A)) = pCl(A) \).

**Definition 2.6** If \( (X, (pCl)_X) \) and \( (Y, (pCl)_Y) \) are generalized preclosure spaces, then a function \( f : X \to Y \) is said to be

1. preclosure preserving if \( f((pCl)_X(A)) \subseteq (pCl)_Y(f(A)) \) for each \( A \subseteq X \).
2. precontinuous if \( (pCl)_X(f^{-1}(B)) \subseteq f^{-1}((pCl)_Y(B)) \) for each \( B \subseteq Y \).

Observe that in general, neither condition implies the other. Now, we have the following result:

**Theorem 2.7** Let \( (X, (pCl)_X) \) and \( (Y, (pCl)_Y) \) be generalized preclosure spaces and let \( f : X \to Y \) be a function.

1. If \( f \) is preclosure preserving and \( (pCl)_Y \) is pre-isotonic, then \( f \) is precontinuous.
2. If \( f \) is precontinuous and \( (pCl)_X \) is pre-isotonic, then \( f \) is preclosure preserving.

**Proof.** Let \( f \) be preclosure preserving and \( (pCl)_Y \) is pre-isotonic. Let \( B \subseteq Y \). \( f((((pCl)_X(f^{-1}(B))) \subseteq (pCl)_Y(f(f^{-1}(B))) \subseteq (pCl)_Y(B) \) and hence, \( (pCl)_X(f^{-1}(B)) \subseteq f^{-1}(f((pCl)_X(f^{-1}(B)))) \subseteq f^{-1}((pCl)_Y(B)) \).

Suppose that \( f \) is precontinuous and \( (pCl)_X \) is pre-isotonic. Let \( A \subseteq X \). \( (pCl)_X(A) \subseteq (pCl)_X(f^{-1}(f(A))) \subseteq f^{-1}((pCl)_Y(f(f(A)))) \). Therefore, \( f( (pCl)_X(A)) \subseteq f(f^{-1}((pCl)_Y(f(A)))) \subseteq (pCl)_Y(f(A)) \).

**Definition 2.8** Let \( (X, (pCl)_X) \) and \( (Y, (pCl)_Y) \) be generalized preclosure spaces and let \( f : X \to Y \) be a function. If for all \( A, B \subseteq X, f(A) \) and \( f(B) \) are not \( (pCl)_Y \)-separated whenever \( A \) and \( B \) are not \( (pCl)_X \)-separated, then we say that \( f \) is non-pre-separating. Observe that \( f \) is non-pre-separating if and only if \( A \) and \( B \) are not \( (pCl)_X \)-separated whenever \( f(A) \) and \( f(B) \) are \( (pCl)_Y \)-separated.

**Theorem 2.9** Let \( (X, (pCl)_X) \) and \( (Y, (pCl)_Y) \) be generalized preclosure spaces and let \( f : X \to Y \) be a function.
(1) If \((pCl)_Y\) is pre-isotonic and \(f\) is non-pre-separating, then \(f^{-1}(C)\) and \(f^{-1}(D)\) are \((pCl)_Y\)-separated whenever \(C\) and \(D\) are \((pCl)_Y\)-separated.

(2) If \((pCl)_X\) is pre-isotonic and \(f^{-1}(C)\) and \(f^{-1}(D)\) are \((pCl)_X\)-separated whenever \(C\) and \(D\) are \((pCl)_Y\)-separated, then \(f\) is non-pre-separating.

**Proof.** Suppose that \(C\) and \(D\) are \((pCl)_Y\)-separated subsets, where \((pCl)_Y\) is pre-isotonic. Let \(A = f^{-1}(C)\) and \(B = f^{-1}(D)\). If \(f\) is pre-isotonic, \(f(A)\) and \(f(B)\) are also \((pCl)_Y\)-separated. It follows now that \(A\) and \(B\) are \((pCl)_X\)-separated in \(X\). Suppose that \((pCl)_X\) is pre-isotonic and let \(A, B \subseteq X\) such that \(C = f(A)\) and \(D = f(B)\) are \((pCl)_X\)-separated. Then \(f^{-1}(C)\) and \(f^{-1}(D)\) are \((pCl)_X\)-separated and since \((pCl)_X\) is pre-isotonic, \(A \subseteq f^{-1}(f(A)) = f^{-1}(C)\) and \(B \subseteq f^{-1}(f(B)) = f^{-1}(D)\) are \((pCl)_X\)-separated as well.

**Theorem 2.10** Let \((X, (pCl)_X)\) and \((Y, (pCl)_Y)\) be generalized preclosure spaces and let \(f : X \rightarrow Y\) be a function. If \(f\) is preclosure preserving, then \(f\) is non-pre-separating.

**Proof.** Suppose that \(f\) is preclosure preserving and \(A, B \subseteq X\) are not \((pCl)_X\)-separated. Suppose that \((pCl)_X(A) \cap B \neq \emptyset\). Then \(\phi \neq f((pCl)_X(A) \cap B) \subseteq f((pCl)_X(A)) \cap f(B) \subseteq (pCl)_Y(f(A)) \cap f(B)\). Similarly, if \(A \cap (pCl)_X(B) \neq \emptyset\), then \(f(A) \cap (pCl)_Y(f(B)) \neq \emptyset\). Hence \(f(A)\) and \(f(B)\) are not \((pCl)_Y\)-separated.

**Corollary 2.11** Let \((X, (pCl)_X)\) and \((Y, (pCl)_Y)\) be generalized preclosure spaces with \((pCl)_Y\) pre-isotonic and let \(f : X \rightarrow Y\) be a function. If \(f\) is precontinuous, then \(f\) is non-pre-separating.

**Proof.** If \(f\) is precontinuous and \((pCl)_X\) pre-isotonic, then by Theorem 2.9 (2) \(f\) is preclosure-preserving. Now, by Theorem 2.10, \(f\) is non-pre-separating.

**Theorem 2.12** Let \((X, (pCl)_X)\) and \((Y, (pCl)_Y)\) be generalized preclosure spaces which pre-Exterior points \((pCl)_Y\)-separated in \(Y\) and let \(f : X \rightarrow Y\) be a function. Then \(f\) is preclosure-preserving if and only if \(Y\) is non-pre-separating.

**Proof.** By Theorem 2.10, if \(f\) is preclosure-preserving, then \(f\) is non-pre-separating. Suppose that \(f\) is non-pre-separating and let \(A \subseteq X\). If \((pCl)_X = \emptyset\), then \(f((pCl)_X(A)) = \emptyset \subseteq (pCl)_Y(f(A))\). Suppose \((pCl)_X(A) \neq \emptyset\). Let \(S_X\) and \(S_Y\) denote the pairs of \((pCl)_X\)-separated subsets of \(X\) and the pairs of \((pCl)_Y\)-separated subsets of \(Y\), respectively. Let \(y \in f((pCl)_X(A))\) and let \(x \in (pCl)_X(A) \cap f^{-1}(\{y\})\). Since \(x \in (pCl)_X(A)\), \(\{x\} \notin S_X\) and since \(f\) non-pre-separating, \(\{y\}, f(A)\) \(\notin S_Y\). Since pre-Exterior points are \((pCl)_Y\)-separated, \(y \in (pCl)_Y(f(A))\). Thus \(f((pCl)_X(A)) \subseteq (pCl)_Y(f(A))\) for each \(A \subseteq X\).

**Corollary 2.13** Let \((X, (pCl)_X)\) and \((Y, (pCl)_Y)\) be generalized preclosure spaces which pre-isotonic closure functions and with \((pCl)_Y\)-pointwise-pre-symmetric and let \(f : X \rightarrow Y\) be a function. Then \(f\) is precontinuous if and only if \(f\) is non-pre-separating.

**Proof.** Since \((pCl)_Y\) is pre-isotonic and pointwise-pre-symmetric, pre-Exterior points are preclosure separated in \((Y, (pCl)_Y)\) (Theorem 2.3 (1)). Since both pre-closure functions are pre-isotonic, \(f\) is preclosure-preserving if and only if \(f\) is precontinuous. Hence, we can apply the Theorem 2.12.

### 3. Preconnected generalized preclosure spaces

**Definition 3.1** Let \((X, pCl)\) be generalized preclosure space. \(X\) is said to be preconnected if \(X\) is not a union of disjoint nontrivial preclosure-separated pair of sets.
Theorem 3.2 Let \((X, p\text{Cl})\) be generalized preclosure space with pre-grounded pre-isotonic pre-enlarging \(p\text{Cl}\). Then, the following are equivalent:
(1) \((X, p\text{Cl})\) is preconnected,
(2) \(X\) can not be a union of nonempty disjoint preopen sets.

Proof. (1) \(\Rightarrow\) (2): Let \(X\) be a union of nonempty disjoint preopen sets \(A\) and \(B\). Then, \(X = A \cup B\) and this implies that \(B = X \setminus A\) and \(A\) is a preopen set. Thus, \(B\) is preclosed and hence \(A \cap p\text{Cl}(B) = A \cap B = \emptyset\). By using similar way, we obtain \(B \cap p\text{Cl}(A) = \emptyset\). Hence, \(A\) and \(B\) are preclosure-separated and hence \(X\) is not preconnected. This is a contradiction.

(2) \(\Rightarrow\) (1): Suppose that \(X\) is not preconnected. Then \(X = A \cup B\), where \(A, B\) are disjoint preclosure-separated sets, i.e., \(A \cup p\text{Cl}(B) = p\text{Cl}(A) \cap B = \emptyset\). We have \(p\text{Cl}(B) \subseteq X \setminus A \subseteq B\). Since \(p\text{Cl}\) is pre-enlarging, we obtain \(p\text{Cl}(B) = B\) and hence, \(B\) is preclosed. By using \(p\text{Cl}(A) \cap B = \emptyset\) and similar way, it is obvious that \(A\) is preclosed. But this is a contradiction. \(\blacksquare\)

Definition 3.3 Let \((X, p\text{Cl})\) be a generalized preclosure space with pre-grounded pre-isotonic \(p\text{Cl}\). Then, \((X, p\text{Cl})\) is called a \(T_1\)-pre-grounded pre-isotonic \(p\text{Cl}\).

Theorem 3.4 Let \((X, p\text{Cl})\) be a generalized preclosure space with \(\lambda\)-grounded pre-isotonic \(p\text{Cl}\). Then, the following are equivalent:
(1) \((X, p\text{Cl})\) is preconnected,
(2) Any precontinuous function \(f : X \to Y\) is constant for all \(T_1\)-pre-grounded pre-isotonic spaces \(Y = \{0, 1\}\).

Proof. (1) \(\Rightarrow\) (2): Let \(X\) be preconnected. Suppose that \(f : X \to Y\) is pre-continuous and it is not constant. Then there exists a set \(U \subseteq X\) such that \(U = f^{-1}(\{0\})\) and \(X \setminus U = f^{-1}(\{1\})\). Since \(f\) is precontinuous and \(Y\) is \(T_1\)-\(\lambda\)-grounded pre-isotonic space, then we have \(p\text{Cl}(U) = p\text{Cl}(f^{-1}(\{0\})) \subset f^{-1}(p\text{Cl}(\{0\})) \subset f^{-1}(\{0\}) = U\) and hence \(p\text{Cl}(U) \cap (X \setminus U) = \emptyset\). By using similar way we have \(U \cap p\text{Cl}(X \setminus U) = \emptyset\). This is a contradiction. Thus, \(f\) is constant.

(2) \(\Rightarrow\) (1): Suppose that \(X\) is not preconnected. Then there exist preclosure-separated sets \(U\) and \(V\) such that \(U \cup V = X\). We have \(p\text{Cl}(U) \subseteq U\) and \(p\text{Cl}(V) \subseteq V\) and \(X \setminus U \subseteq V\). Since \(p\text{Cl}\) is pre-isotonic and \(U\) and \(V\) are preclosure-separated, then \(p\text{Cl}(X \setminus U) \subset p\text{Cl}(V) \subset X \setminus U\). If we consider the space \((Y, p\text{Cl})\) by \(Y = \{0, 1\}\), \(p\text{Cl}(\emptyset) = \emptyset\), \(p\text{Cl}(\{0\}) = \{0\}\), \(p\text{Cl}(\{1\}) = \{1\}\) and \(p\text{Cl}(Y) = Y\), then the space \((Y, p\text{Cl})\) is a \(T_1\)-pre-grounded pre-isotonic space. We define the function \(f : X \to Y\) as \(f(U) = \{0\}\) and \(f(X \setminus U) = \{1\}\). Let \(A \neq \emptyset\) and \(A \subseteq Y\). If \(A = Y\), then \(f^{-1}(A) = X\) and hence \(p\text{Cl}(X) = p\text{Cl}(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(p\text{Cl}(A))\). If \(A = \emptyset\), then \(f^{-1}(A) = U\) and hence \(p\text{Cl}(U) = p\text{Cl}(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(p\text{Cl}(A))\). If \(A = \{1\}\), then \(f^{-1}(A) = X \setminus U\) and so \(p\text{Cl}(X \setminus U) = p\text{Cl}(f^{-1}(A)) \subset X \setminus U = f^{-1}(A) = f^{-1}(p\text{Cl}(A))\). Hence, \(f\) is precontinuous. Since \(f\) is not constant, this is a contradiction. \(\blacksquare\)

Theorem 3.5 Let \(f : (X, p\text{Cl}) \to (Y, p\text{Cl})\) and \(g : (Y, p\text{Cl}) \to (Z, p\text{Cl})\) be precontinuous functions. Then, \(g \circ f : X \to Z\) is precontinuous.

Proof. Suppose that \(f\) and \(g\) are precontinuous. For all \(A \subseteq Z\) we have \(p\text{Cl}(g \circ f)^{-1}(A) = p\text{Cl}(f^{-1}(g^{-1}(A))) \subset f^{-1}(p\text{Cl}(g^{-1}(A))) \subset f^{-1}(g^{-1}(p\text{Cl}(A))) = (g \circ f)^{-1}(p\text{Cl}(A))\). Hence, \(g \circ f : X \to Z\) is precontinuous. \(\blacksquare\)

Theorem 3.6 Let \((X, p\text{Cl})\) and \((Y, p\text{Cl})\) be generalized preclosure spaces with pre-grounded pre-isotonic \(p\text{Cl}\) and \(f : (X, p\text{Cl}) \to (Y, p\text{Cl})\) be a precontinuous function onto
Y. If X is preconnected, then Y is preconnected.

**Proof.** Suppose that \( \{0, 1\} \) is a generalized preclosure space with pre-grounded pre-isotonic \( pCl \) and \( g : Y \to \{0, 1\} \) is a precontinuous function. Since \( f \) is precontinuous, by Theorem 3.5, \( g \circ f : X \to \{0, 1\} \) is precontinuous. Since \( X \) is preconnected, \( g \circ f \) is constant and hence \( g \) is constant. By Theorem 3.4, \( Y \) is preconnected.

**Definition 3.7** Let \((Y, pCl)\) be a generalized preclosure space with pre-grounded pre-isotonic \( pCl \) and more than one element. A generalized preclosure space \((X, pCl)\) with pre-grounded pre-isotonic \( pCl \) is called \( Y \)-preconnected if any precontinuous function \( f : X \to Y \) is constant.

**Theorem 3.8** Let \((Y, pCl)\) be a generalized preclosure space with pre-grounded pre-isotonic \( pCl \) and more than one element. Then every \( Y \)-preconnected generalized preclosure space with pre-grounded pre-isotonic is preconnected.

**Proof.** Let \((X, pCl)\) be a \( Y \)-preconnected generalized preclosure space with pre-grounded pre-isotonic \( pCl \). Suppose that \( f : X \to \{0, 1\} \) is a precontinuous function, where \( \{0, 1\} \) is a \( T_1 \)-pre-grounded pre-isotonic space. Since \( Y \) is a generalized preclosure space with pre-grounded pre-isotonic pre-enlarging \( pCl \) and more than one element, then there exists a precontinuous injection \( g : \{0, 1\} \to Y \). By Theorem 3.5, \( g \circ f : X \to Y \) is precontinuous. Since \( X \) is \( Y \)-preconnected, then \( g \circ f \) is constant and hence, by Theorem 3.4, \( X \) is preconnected.

**Theorem 3.9** Let \((X, pCl)\) and \((Y, pCl)\) be generalized preclosure spaces with pre-grounded pre-isotonic \( pCl \) and \( f : (X, pCl) \to (Y, pCl) \) be a precontinuous function onto \( Y \). If \( X \) is \( Z \)-preconnected, then \( Y \) is \( Z \)-preconnected.

**Proof.** Suppose that \( g : Y \to Z \) is a precontinuous function. Then \( g \circ f : X \to Z \) is precontinuous. Since \( X \) is \( Z \)-preconnected, then \( g \circ f \) is constant. This implies that \( g \) is constant. Thus, \( Y \) is \( Z \)-preconnected.

**Definition 3.10** A generalized preclosure space \((X, pCl)\) is strongly preconnected if there is no countable collection of pairwise preclosure-separated sets \( \{A_n\} \) such that \( X = \bigcup A_n \).

**Theorem 3.11** Every strongly preconnected generalized preclosure space with pre-grounded pre-isotonic \( pCl \) is preconnected.

**Theorem 3.12** Let \((X, pCl)\) and \((Y, pCl)\) be generalized preclosure spaces with pre-grounded pre-isotonic \( pCl \) and \( f : (X, pCl) \to (Y, pCl) \) be a precontinuous function onto \( Y \). If \( X \) is strongly preconnected, then \( Y \) is strongly preconnected.

**Proof.** Suppose that \( Y \) is not strongly preconnected. Then, there exists a countable collection of pairwise preclosure-separated sets \( \{A_n\} \) such that \( Y = \bigcup A_n \). Since \( f^{-1}(A_n) \cap pCl(f^{-1}(A_m)) \leq f^{-1}(A_n) \cap f^{-1}(pCl(A_m)) = \phi \) for all \( n \neq m \), then the collection \( \{f^{-1}(A_n)\} \) is pairwise preclosure separated. This is a contradiction. Hence, \( Y \) is strongly preconnected.

**Theorem 3.13** Let \((X, (pCl)_X)\) and \((Y, (pCl)_Y)\) be generalized preclosure spaces. Then the following are equivalent for a function \( f : X \to Y \).

1. \( f \) is precontinuous,
2. \( f^{-1}(pInt(B)) \subseteq pInt(f^{-1}(B)) \) for each \( B \subseteq Y \).

**Theorem 3.14** Let \((X, pCl)\) be a generalized preclosure space with pre-grounded pre-isotonic \( pCl \). Then \((X, pCl)\) is strongly preconnected if and only if \((X, pCl)\) is \( Y \)-
preconnected for any countable $T_1$-pre-grounded pre-isotonic space $(Y, pCl)$.

**Proof.** Let $(X, pCl)$ be strongly preconnected. Suppose that $(X, pCl)$ is not $Y$-preconnected for some countable $T_1$-pre-grounded pre-isotonic space $(Y, pCl)$. There exists a precontinuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$. Since $f$ is precontinuous and $Y$ is pre-grounded, then for each $n \neq m$, $U_n \cap pCl(U_m) = f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) 
subseteq f^{-1}(pCl(\{y_n\})) \cap f^{-1}(pCl(\{y_m\})) \subsetneq f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \emptyset$. This contradict with the strong preconnectedness of $X$. Thus, $X$ is $Y$-preconnected. Conversely, let $X$ be $Y$-preconnected for any countable $T_1$-pre-grounded pre-isotonic space $(Y, pCl)$. Suppose that $X$ is not strongly preconnected. There exists a countable collection of pairwise pre-continuous separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space $(Z, pCl)$, where $Z$ is the set of integers and $pCl : P(Z) \to P(Z)$ is defined by $pCl(K) = K$ for each $K \subset Z$. Clearly $(Z, pCl)$ is countable $T_1$-pre-grounded pre-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \setminus U_k) = \{y\}$ where $x, y \in Z$ and $x \neq y$. Since $pCl(U_k) \cap U_n = \emptyset$ for all $n \neq k$, then $pCl(U_k) \cap \cup_{n \neq k}U_k = \emptyset$ and hence $pCl(U_k) \subset U_k$. Let $\emptyset \neq K \subseteq Z$. If $x, y \in K$ and then $f^{-1}(K) = X$ and $pCl(f^{-1}(K)) = pCl(K) \subset X = f^{-1}(K) = f^{-1}(pCl(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $pCl(f^{-1}(K)) = pCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(pCl(K))$. If $y \in K$ and $x \notin K$, then $f^{-1}(K) = X \setminus U_k$. Since $pCl(K) = K$ for each $K \subset Z$, then $pInt(K) = K$ for each $K \subset Z$. Also, $X \setminus U_K \cup \cup_{n \neq k}U_n \subset X \setminus pCl(U_k) = pInt(X \setminus U_k)$. Therefore, $f^{-1}(pInt(K)) = X \setminus U_k = f^{-1}(K) \subset pInt(X \setminus U_k) = pInt(f^{-1}(K))$. Hence we obtain that $f$ is precontinuous. Since $f$ is not constant, this is a contradiction with the $Z$-preconnectedness of $X$. Hence, $X$ is strongly preconnected. \hfill \blacksquare

4. Conclusion

Closure spaces in point-set topology will give some new topological properties (for example: separation axioms, compactness, connectedness, continuity) which have been found to be very useful in the study of certain objects of digital topology [9]. Thus we may stress once more the importance of preclosure operators as a branch of them and the possible application in computer graphics [5] and quantum physics [4].

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References