

## \*-K-g-Frames in Hilbert $\mathcal{A}$ -modules

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**Abstract.** In this paper, we introduce the concepts of \*-K-g-Frames in Hilbert  $\mathcal{A}$ -modules and we establish some results.

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**Keywords:** Frame, \*-frame, \*-g-frame, \*-K-g-frame,  $C^*$ -algebra, Hilbert  $\mathcal{A}$ -modules.

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### 1. Introduction

Frames were first introduced in 1952 by Duffin and Schaefer [6] in the study of nonharmonic Fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert  $C^*$ -modules.

In this article, a new notion of frames is introduced: \*-K-g-Frames as generalization of \*-g-frames in Hilbert  $C^*$ -modules introduced by Alijani [2] and we establish some results. The paper is organized as follows: In section 2, we briefly recall the definitions and basic properties of  $C^*$ -algebra, Hilbert  $C^*$ -modules, frames, g-frames, \*-frames, \*-g-frames and K-frames in Hilbert  $C^*$ -modules. In section 3, we introduce the \*-K-g-Frame, the analysis operator, the synthesis operator and the frame operator. In section 4, we investigate tensor product of Hilbert  $C^*$ -modules, we show that tensor product of \*-K-g-Frame for Hilbert  $C^*$ -modules  $\mathcal{H}$  and  $\mathcal{K}$ , present \*-K-g-Frame for  $\mathcal{H} \otimes \mathcal{K}$ , and tensor product of their frame operators is the frame operator of the tensor product of \*-K-g-Frame.

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## 2. Preliminaries

Let  $I$  and  $J$  be countable index sets. In this section we briefly recall the definitions and basic properties of  $C^*$ -algebra, Hilbert  $C^*$ -modules, frame, g-frame,  $*$ -frame,  $*$ -g-frame and K-frame in Hilbert  $C^*$ -modules. For information about frames in Hilbert spaces we refer to [3]. Our reference for  $C^*$ -algebras is [4, 5]. For a  $C^*$ -algebra  $\mathcal{A}$ , an element  $a \in \mathcal{A}$  is positive ( $a \geq 0$ ) if  $a = a^*$  and  $sp(a) \subset \mathbf{R}^+$ .  $\mathcal{A}^+$  denotes the set of positive elements of  $\mathcal{A}$ .

**Definition 2.1** [9] Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\mathcal{H}$  be a left  $\mathcal{A}$ -module, such that the linear structures of  $\mathcal{A}$  and  $\mathcal{H}$  are compatible.  $\mathcal{H}$  is a pre-Hilbert  $\mathcal{A}$ -module if  $\mathcal{H}$  is equipped with an  $\mathcal{A}$ -valued inner product  $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$  such that is sesquilinear, positive definite and respects the module action. In the other words,

- (i)  $\langle x, x \rangle_{\mathcal{A}} \geq 0$  for all  $x \in \mathcal{H}$  and  $\langle x, x \rangle_{\mathcal{A}} = 0$  if and only if  $x = 0$ .
- (ii)  $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, y \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$  for all  $a \in \mathcal{A}$  and  $x, y, z \in \mathcal{H}$ .
- (iii)  $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$  for all  $x, y \in \mathcal{H}$ .

For  $x \in \mathcal{H}$ , we define  $\|x\| = \|\langle x, x \rangle_{\mathcal{A}}\|^{\frac{1}{2}}$ . If  $\mathcal{H}$  is complete with  $\|\cdot\|$ , it is called a Hilbert  $\mathcal{A}$ -module or a Hilbert  $C^*$ -module over  $\mathcal{A}$ . For every  $a$  in  $C^*$ -algebra  $\mathcal{A}$ , we have  $|a| = (a^*a)^{\frac{1}{2}}$  and the  $\mathcal{A}$ -valued norm on  $\mathcal{H}$  is defined by  $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$  for  $x \in \mathcal{H}$ .

Let  $\mathcal{H}$  and  $\mathcal{K}$  be two Hilbert  $\mathcal{A}$ -modules, A map  $T : \mathcal{H} \rightarrow \mathcal{K}$  is said to be adjointable if there exists a map  $T^* : \mathcal{K} \rightarrow \mathcal{H}$  such that  $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$  for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ .

From now on, we assume that  $\{V_i\}_{i \in I}$  and  $\{W_j\}_{j \in J}$  are two sequences of Hilbert  $\mathcal{A}$ -modules. We also reserve the notation  $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$  for the set of all adjointable operators from  $\mathcal{H}$  to  $\mathcal{K}$  and  $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$  is abbreviated to  $End_{\mathcal{A}}^*(\mathcal{H})$ .

**Definition 2.2** [7] Let  $\mathcal{H}$  be a Hilbert  $\mathcal{A}$ -module. A family  $\{x_i\}_{i \in I}$  of elements of  $\mathcal{H}$  is a frame for  $\mathcal{H}$ , if there exist two positive constants  $A, B$  such that for all  $x \in \mathcal{H}$ ,

$$A\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle x, x_i \rangle_{\mathcal{A}} \langle x_i, x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}. \quad (1)$$

The numbers  $A$  and  $B$  are called lower and upper bounds of the frame, respectively. If  $A = B = \lambda$ , the frame is  $\lambda$ -tight. If  $A = B = 1$ , it is called a normalized tight frame or a Parseval frame. If the sum in the middle of (1) is convergent in norm, the frame is called standard.

**Definition 2.3** [10] We call a sequence  $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, V_i) : i \in I\}$  a g-frame in Hilbert  $\mathcal{A}$ -module  $\mathcal{H}$  with respect to  $\{V_i : i \in I\}$  if there exist two positive constants  $C, D$  such that for all  $x \in \mathcal{H}$ ,

$$C\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq D\langle x, x \rangle_{\mathcal{A}}. \quad (2)$$

The numbers  $C$  and  $D$  are called lower and upper bounds of the g-frame, respectively. If  $C = D = \lambda$ , the g-frame is  $\lambda$ -tight. If  $C = D = 1$ , it is called a g-Parseval frame. If the sum in the middle of (2) is convergent in norm, the g-frame is called standard.

**Definition 2.4** [1] Let  $\mathcal{H}$  be a Hilbert  $\mathcal{A}$ -module over a unital  $C^*$ -algebra. A family  $\{x_i\}_{i \in I}$  of elements of  $\mathcal{H}$  is a  $*$ -frame for  $\mathcal{H}$ , if there exist strictly nonzero elements

$A, B \in \mathcal{A}$  such that for all  $x \in \mathcal{H}$ ,

$$A\langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle x, x_i \rangle_{\mathcal{A}} \langle x_i, x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}} B^*. \tag{3}$$

The elements  $A$  and  $B$  are called lower and upper bounds of the  $*$ -frame, respectively. If  $A = B = \lambda_1$ , the  $*$ -frame is  $\lambda_1$ -tight. If  $A = B = 1$ , it is called a normalized tight  $*$ -frame or a Parseval  $*$ -frame. If the sum in the middle of (3) is convergent in norm, the  $*$ -frame is called standard.

**Definition 2.5** [2] We call a sequence  $\{\Lambda_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}, V_i) : i \in I\}$  a  $*$ -g-frame in Hilbert  $\mathcal{A}$ -module  $\mathcal{H}$  over a unital  $C^*$ -algebra with respect to  $\{V_i : i \in I\}$  if there exist strictly nonzero elements  $A, B$  in  $\mathcal{A}$  such that for all  $x \in \mathcal{H}$ ,

$$A\langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}} B^*. \tag{4}$$

The elements  $A$  and  $B$  are called lower and upper bounds of the  $*$ -g-frame, respectively. If  $A = B = \lambda_1$ , the  $*$ -g-frame is  $\lambda_1$ -tight. If  $A = B = 1$ , it is called a  $*$ -g-Parseval frame. If the sum in the middle of (4) is convergent in norm, the  $*$ -g-frame is called standard. The  $*$ -g-frame operator  $S_{\Lambda}$  is defined by:

$$S_{\Lambda} x = \sum_{i \in I} \Lambda_i^* \Lambda_i x, \quad \forall x \in \mathcal{H}.$$

In [8], Gavruta introduced  $K$ -frames to study atomic systems for operators in Hilbert spaces.

**Definition 2.6** [12] Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ . A family  $\{x_i\}_{i \in I}$  of elements in a Hilbert  $\mathcal{A}$ -module  $\mathcal{H}$  over a unital  $C^*$ -algebra is a  $K$ -frame for  $\mathcal{H}$ , if there exist two positive constants  $A, B$  such that for all  $x \in \mathcal{H}$ ,

$$A\langle K^* x, K^* x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle x, x_i \rangle_{\mathcal{A}} \langle x_i, x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}. \tag{5}$$

The numbers  $A$  and  $B$  are called lower and upper bounds of the  $K$ -frame, respectively.

### 3. $*$ -K-g-frames in Hilbert $\mathcal{A}$ -modules

**Definition 3.1** Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ . We call a sequence  $\{\Lambda_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$  is a  $*$ -K-g-frame in Hilbert  $\mathcal{A}$ -module  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  if there exist strictly nonzero elements  $A, B$  in  $\mathcal{A}$  such that

$$A\langle K^* x, K^* x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}} B^*, \forall x \in \mathcal{H}. \tag{6}$$

The numbers  $A$  and  $B$  are called lower and upper bounds of the  $*$ -K-g-frame, respectively. If

$$A\langle K^* x, K^* x \rangle_{\mathcal{A}} A^* = \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}}, \forall x \in \mathcal{H}. \tag{7}$$

The  $*$ -K-g-frame is  $A$ -tight.

**Remark 1**

- (1) Every  $*$ -g-frame for  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  is an  $*$ -K-g-frame, for any  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H}) : K \neq 0$ .
- (2) If  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  is a surjective operator, then every  $*$ -K-g-frame for  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  is a  $*$ -g-frame.

**Example 3.2** Let  $\mathcal{H}$  be a finitely or countably generated Hilbert  $\mathcal{A}$ -module. Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H}) : K \neq 0$ . Let  $\mathcal{A}$  be a Hilbert  $\mathcal{A}$ -module over itself with the inner product  $\langle a, b \rangle = ab^*$ . Let  $\{x_i\}_{i \in I}$  be an  $*$ -frame for  $\mathcal{H}$  with bounds  $A$  and  $B$ , respectively. For each  $i \in I$ , we define  $\Lambda_i : \mathcal{H} \rightarrow \mathcal{A}$  by  $\Lambda_i x = \langle x, x_i \rangle$  for all  $x \in \mathcal{H}$ .  $\Lambda_i$  is adjointable and  $\Lambda_i^* a = ax_i$  for each  $a \in \mathcal{A}$ . Also, we have

$$A\langle x, x \rangle A^* \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq B\langle x, x \rangle B^*, \forall x \in \mathcal{H}$$

Or

$$\langle K^* x, K^* x \rangle \leq \|K\|^2 \langle x, x \rangle, \forall x \in \mathcal{H}.$$

Then

$$\|K\|^{-1} A \langle K^* x, K^* x \rangle (\|K\|^{-1} A)^* \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \leq B \langle x, x \rangle B^*, \forall x \in \mathcal{H}.$$

So  $\{\Lambda_i\}_{i \in I}$  is an  $*$ -K-g-frame for  $\mathcal{H}$  with bounds  $\|K\|^{-1} A$  and  $B$ , respectively.

Let  $\{\Lambda_i\}_{i \in I}$  be an  $*$ -K-g-frame in  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$ . Define an operator  $T : \mathcal{H} \rightarrow \oplus_{i \in I} \mathcal{H}_i$  by  $Tx = \{\Lambda_i x\}_i$  for all  $x \in \mathcal{H}$ , then  $T$  is called the analysis operator. So its adjoint operator is  $T^* : \oplus_{i \in I} \mathcal{H}_i \rightarrow \mathcal{H}$  given by  $T^*(\{x_i\}_i) = \sum_{i \in I} \Lambda_i^* x_i$  for all  $\{x_i\}_i \in \oplus_{i \in I} \mathcal{H}_i$ . The operator  $T^*$  is called the synthesis operator. By composing  $T$  and  $T^*$ , the frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  is given by  $Sx = T^*Tx = \sum_{i \in I} \Lambda_i^* \Lambda_i x$ .

Note that  $S$  need not be invertible in general. But under some condition  $S$  will be invertible.

**Theorem 3.3** Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  be a surjective operator. If  $\{\Lambda_i\}_{i \in I}$  is an  $*$ -K-g-frame in  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$ , then the frame operator  $S$  is positive, invertible and adjointable. Moreover we have the reconstruction formula,  $x = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} x, \forall x \in \mathcal{H}$ .

**Proof.** Result of (2) in Remark 1 and Theorem 3.8 in [2]. ■

Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ , in the following theorem using an  $*$ -g-frame we constructed an  $*$ -K-g-frame.

**Theorem 3.4** Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  and  $\{\Lambda_i\}_{i \in I}$  be an  $*$ -g-frame in  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  with bounds  $A, B$ . Then  $\{\Lambda_i K\}_{i \in I}$  is an  $*$ -K-g-frame in  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  with bounds  $A, \|K\|B$ . The frame operator of  $\{\Lambda_i K\}_{i \in I}$  is  $S' = K^*SK$ , where  $S$  is the frame operator of  $\{\Lambda_i\}_{i \in I}$ .

**Proof.** Form

$$A\langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}} B^*$$

for all  $x \in \mathcal{H}$ , we get

$$A\langle Kx, Kx \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle \Lambda_i Kx, \Lambda_i Kx \rangle_{\mathcal{A}} \leq B\langle Kx, Kx \rangle_{\mathcal{A}} B^* \leq \|K\| B\langle x, x \rangle_{\mathcal{A}} (\|K\| B)^*.$$

Then  $\{\Lambda_i K\}_{i \in I}$  is an  $*\text{-}K^*$ -g-frame in  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i : i \in I\}$  with bounds  $A, \|K\|B$ . By definition of  $S$ , we have  $SKx = \sum_{i \in I} \Lambda_i^* \Lambda_i Kx$ . Then

$$K^*SKx = K^* \sum_{i \in I} \Lambda_i^* \Lambda_i Kx = \sum_{i \in I} K^* \Lambda_i^* \Lambda_i Kx.$$

Hence  $S' = K^*SK$ . ■

**Corollary 3.5** Let  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  and  $\{\Lambda_i\}_{i \in I}$  be an  $*\text{-}g$ -frame. Then  $\{\Lambda_i S^{-1}K\}_{i \in I}$  is an  $*\text{-}K^*$ -g-frame, where  $S$  is the frame operator of  $\{\Lambda_i\}_{i \in I}$ .

**Proof.** Result of the Theorem 3.4 for the  $*\text{-}g$ -frame  $\{\Lambda_i S^{-1}\}_{i \in I}$ . ■

### 4. Tensor Product

Suppose that  $\mathcal{A}, \mathcal{B}$  are unital  $C^*$ -algebras and we take  $\mathcal{A} \otimes \mathcal{B}$  as the completion of  $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$  with the spatial norm.  $\mathcal{A} \otimes \mathcal{B}$  is the spatial tensor product of  $\mathcal{A}$  and  $\mathcal{B}$ , also suppose that  $\mathcal{H}$  is a Hilbert  $\mathcal{A}$ -module and  $\mathcal{K}$  is a Hilbert  $\mathcal{B}$ -module. We want to define  $\mathcal{H} \otimes \mathcal{K}$  as a Hilbert  $(\mathcal{A} \otimes \mathcal{B})$ -module. Start by forming the algebraic tensor product  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  of the vector spaces  $\mathcal{H}, \mathcal{K}$  (over  $\mathbb{C}$ ). This is a left module over  $(\mathcal{A} \otimes_{\text{alg}} \mathcal{B})$  (the module action being given by  $(a \otimes b)(x \otimes y) = ax \otimes by$  ( $a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{H}, y \in \mathcal{K}$ )). For  $(x_1, x_2 \in \mathcal{H}, y_1, y_2 \in \mathcal{K})$  we define  $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{A} \otimes \mathcal{B}} = \langle x_1, x_2 \rangle_{\mathcal{A}} \langle y_1, y_2 \rangle_{\mathcal{B}}$ . We also know that for  $z = \sum_{i=1}^n x_i \otimes y_i$  in  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  we have  $\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}} = \sum_{i,j} \langle x_i, x_j \rangle_{\mathcal{A}} \langle y_i, y_j \rangle_{\mathcal{B}} \geq 0$  and  $\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}} = 0$  iff  $z = 0$ . This extends by linearity to an  $(\mathcal{A} \otimes_{\text{alg}} \mathcal{B})$ -valued sesquilinear form on  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$ , which makes  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  into a semi-inner-product module over the pre- $C^*$ -algebra  $(\mathcal{A} \otimes_{\text{alg}} \mathcal{B})$ . The semi-inner-product on  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  is actually an inner product, see [11]. Then  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  is an inner-product module over the pre- $C^*$ -algebra  $(\mathcal{A} \otimes_{\text{alg}} \mathcal{B})$ , and we can perform the double completion discussed in chapter 1 of [11] to conclude that the completion  $\mathcal{H} \otimes \mathcal{K}$  of  $\mathcal{H} \otimes_{\text{alg}} \mathcal{K}$  is a Hilbert  $(\mathcal{A} \otimes \mathcal{B})$ -module. We call  $\mathcal{H} \otimes \mathcal{K}$  the exterior tensor product of  $\mathcal{H}$  and  $\mathcal{K}$ . With  $\mathcal{H}, \mathcal{K}$  as above, we wish to investigate the adjointable operators on  $\mathcal{H} \otimes \mathcal{K}$ . Suppose that  $S \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  and  $T \in \text{End}_{\mathcal{B}}^*(\mathcal{K})$ . We define a linear operator  $S \otimes T$  on  $\mathcal{H} \otimes \mathcal{K}$  by  $S \otimes T(x \otimes y) = Sx \otimes Ty$  ( $x \in \mathcal{H}, y \in \mathcal{K}$ ). It is a routine verification that  $S^* \otimes T^*$  is the adjoint of  $S \otimes T$ , so in fact  $S \otimes T \in \text{End}_{\mathcal{A} \otimes \mathcal{B}}^*(\mathcal{H} \otimes \mathcal{K})$ . For more details, see [5, 11]. We note that if  $a \in \mathcal{A}^+$  and  $b \in \mathcal{B}^+$ , then  $a \otimes b \in (\mathcal{A} \otimes \mathcal{B})^+$ . Plainly if  $a, b$  are Hermitian elements of  $\mathcal{A}$  and  $a \geq b$ , then for every positive element  $x$  of  $\mathcal{B}$ , we have  $a \otimes x \geq b \otimes x$ .

Let  $I$  and  $J$  be countable index sets. Our next theorem is a generalization of theorem 2.2 in [1]

**Theorem 4.1** Let  $\mathcal{H}$  and  $\mathcal{K}$  be two Hilbert  $C^*$ -modules over unital  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Let  $\{\Lambda_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H}, V_i)$  be an  $*\text{-}K$ -g-frame for  $\mathcal{H}$  with bounds  $A$  and  $B$  and frame operators  $S_{\Lambda}$  and  $\{\Gamma_j\}_{j \in J} \subset \text{End}_{\mathcal{B}}^*(\mathcal{K}, W_j)$  be an  $*\text{-}L$ -g-frame for  $\mathcal{K}$  with bounds  $C$  and  $D$  and frame operators  $S_{\Gamma}$ . Then  $\{\Lambda_i \otimes \Gamma_j\}_{i \in I, j \in J}$  is an  $*\text{-}K \otimes L$ -g-frame for Hilbert  $\mathcal{A} \otimes \mathcal{B}$ -module  $\mathcal{H} \otimes \mathcal{K}$  with frame operator  $S_{\Lambda} \otimes S_{\Gamma}$  and bounds  $A \otimes C$  and  $B \otimes D$ .

**Proof.** By the definition of  $*$ -K-g-frame  $\{\Lambda_i\}_{i \in I}$  and  $*$ -L-g-frame  $\{\Gamma_j\}_{j \in J}$  we have

$$A\langle K^*x, K^*x \rangle_{\mathcal{A}}A^* \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}B^*, \forall x \in \mathcal{H},$$

$$C\langle L^*y, L^*y \rangle_{\mathcal{B}}C^* \leq \sum_{j \in J} \langle \Gamma_j y, \Gamma_j y \rangle_{\mathcal{B}} \leq D\langle y, y \rangle_{\mathcal{B}}D^*, \forall y \in \mathcal{K}.$$

Therefore,

$$(A\langle K^*x, K^*x \rangle_{\mathcal{A}}A^*) \otimes (C\langle L^*y, L^*y \rangle_{\mathcal{B}}C^*) \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \otimes \sum_{j \in J} \langle \Gamma_j y, \Gamma_j y \rangle_{\mathcal{B}}$$

$$\leq (B\langle x, x \rangle_{\mathcal{A}}B^*) \otimes (D\langle y, y \rangle_{\mathcal{B}}D^*)$$

for all  $x \in \mathcal{H}$  and all  $y \in \mathcal{K}$ . Then

$$(A \otimes C)(\langle K^*x, K^*x \rangle_{\mathcal{A}} \otimes \langle L^*y, L^*y \rangle_{\mathcal{B}})(A^* \otimes C^*) \leq \sum_{i \in I, j \in J} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \otimes \langle \Gamma_j y, \Gamma_j y \rangle_{\mathcal{B}}$$

$$\leq (B \otimes D)(\langle x, x \rangle_{\mathcal{A}} \otimes \langle y, y \rangle_{\mathcal{B}})(B^* \otimes D^*)$$

for all  $x \in \mathcal{H}$  and all  $y \in \mathcal{K}$ . Consequently, we have

$$(A \otimes C)\langle K^*x \otimes L^*y, K^*x \otimes L^*y \rangle_{\mathcal{A} \otimes \mathcal{B}}(A \otimes C)^* \leq \sum_{i \in I, j \in J} \langle \Lambda_i x \otimes \Gamma_j y, \Lambda_i x \otimes \Gamma_j y \rangle_{\mathcal{A} \otimes \mathcal{B}}$$

$$\leq (B \otimes D)\langle x \otimes y, x \otimes y \rangle_{\mathcal{A} \otimes \mathcal{B}}(B \otimes D)^*$$

for all  $x \in \mathcal{H}$  and all  $y \in \mathcal{K}$ . Then, for all  $x \otimes y$  in  $\mathcal{H} \otimes \mathcal{K}$ , we have

$$(A \otimes C)\langle (K \otimes L)^*(x \otimes y), (K \otimes L)^*(x \otimes y) \rangle_{\mathcal{A} \otimes \mathcal{B}}(A \otimes C)^*$$

$$\leq \sum_{i \in I, j \in J} \langle (\Lambda_i \otimes \Gamma_j)(x \otimes y), (\Lambda_i \otimes \Gamma_j)(x \otimes y) \rangle_{\mathcal{A} \otimes \mathcal{B}}$$

$$\leq (B \otimes D)\langle x \otimes y, x \otimes y \rangle_{\mathcal{A} \otimes \mathcal{B}}(B \otimes D)^*.$$

The last inequality is satisfied for every finite sum of elements in  $\mathcal{H} \otimes_{alg} \mathcal{K}$  and then it's satisfied for all  $z \in \mathcal{H} \otimes \mathcal{K}$ . It shows that  $\{\Lambda_i \otimes \Gamma_j\}_{i \in I, j \in J}$  is an  $*$ -K  $\otimes$  L-g-frame for Hilbert  $\mathcal{A} \otimes \mathcal{B}$ -module  $\mathcal{H} \otimes \mathcal{K}$  with lower and upper bounds  $A \otimes C$  and  $B \otimes D$ , respectively. By the definition of frame operator  $S_{\Lambda}$  and  $S_{\Gamma}$  we have

$$S_{\Lambda}x = \sum_{i \in I} \Lambda_i^* \Lambda_i x, \forall x \in \mathcal{H},$$

$$S_{\Gamma}y = \sum_{j \in J} \Gamma_j^* \Gamma_j y, \forall y \in \mathcal{K}.$$

Therefore,

$$\begin{aligned}
 (S_\Lambda \otimes S_\Gamma)(x \otimes y) &= S_\Lambda x \otimes S_\Gamma y \\
 &= \sum_{i \in I} \Lambda_i^* \Lambda_i x \otimes \sum_{j \in J} \Gamma_j^* \Gamma_j y \\
 &= \sum_{i \in I, j \in J} \Lambda_i^* \Lambda_i x \otimes \Gamma_j^* \Gamma_j y \\
 &= \sum_{i \in I, j \in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i x \otimes \Gamma_j y) \\
 &= \sum_{i \in I, j \in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j)(x \otimes y) \\
 &= \sum_{i \in I, j \in J} (\Lambda_i \otimes \Gamma_j)^*(\Lambda_i \otimes \Gamma_j)(x \otimes y).
 \end{aligned}$$

Now, by the uniqueness of frame operator, the last expression is equal to  $S_{\Lambda \otimes \Gamma}(x \otimes y)$ . Consequently, we have  $(S_\Lambda \otimes S_\Gamma)(x \otimes y) = S_{\Lambda \otimes \Gamma}(x \otimes y)$ . The last equality is satisfied for every finite sum of elements in  $\mathcal{H} \otimes_{alg} \mathcal{K}$  and then it's satisfied for all  $z \in \mathcal{H} \otimes \mathcal{K}$ . It shows that  $(S_\Lambda \otimes S_\Gamma)(z) = S_{\Lambda \otimes \Gamma}(z)$ . So  $S_{\Lambda \otimes \Gamma} = S_\Lambda \otimes S_\Gamma$ . ■

The two following theorems are a generalization of Theorem 3.5 in [9].

**Theorem 4.2** If  $Q \in \text{End}_A^*(\mathcal{H})$  is invertible and  $\{\Lambda_i\}_{i \in I} \subset \text{End}_{A \otimes B}^*(\mathcal{H} \otimes \mathcal{K}, V_i)$  is an \*-K-g-frame for  $\mathcal{H} \otimes \mathcal{K}$  with lower and upper bounds  $A$  and  $B$  respectively and frame operator  $S$ . If  $K$  commute with  $Q \otimes I$ , then  $\{\Lambda_i(Q^* \otimes I)\}_{i \in I}$  is an \*-K-g-frame for  $\mathcal{H} \otimes \mathcal{K}$  with lower and upper bounds  $\|Q^{*-1}\|^{-1}A$  and  $\|Q\|B$  respectively and frame operator  $(Q \otimes I)S(Q^* \otimes I)$ .

**Proof.** Since  $Q \in \text{End}_A^*(\mathcal{H})$ ,  $Q \otimes I \in \text{End}_{A \otimes B}^*(\mathcal{H} \otimes \mathcal{K})$  with inverse  $Q^{-1} \otimes I$ . It is obvious that the adjoint of  $Q \otimes I$  is  $Q^* \otimes I$ . An easy calculation shows that for every elementary tensor  $x \otimes y$ ,

$$\begin{aligned}
 \|(Q \otimes I)(x \otimes y)\|^2 &= \|Q(x) \otimes y\|^2 \\
 &= \|Q(x)\|^2 \|y\|^2 \\
 &\leq \|Q\|^2 \|x\|^2 \|y\|^2 \\
 &= \|Q\|^2 \|x \otimes y\|^2.
 \end{aligned}$$

So,  $Q \otimes I$  is bounded and it can be extended to  $\mathcal{H} \otimes \mathcal{K}$ . Similarly for  $Q^* \otimes I$ ,  $Q \otimes I$  is  $A \otimes B$ -linear, adjointable with adjoint  $Q^* \otimes I$ . Hence, for every  $z \in \mathcal{H} \otimes \mathcal{K}$ , we have

$$\|Q^{*-1}\|^{-1} \cdot |z| \leq |(Q^* \otimes I)z| \leq \|Q\| \cdot |z|.$$

By the definition of \*-K-g-frame, we have

$$A\langle K^* z, K^* z \rangle_{A \otimes B} A^* \leq \sum_{i \in I} \langle \Lambda_i z, \Lambda_i z \rangle_{A \otimes B} \leq B\langle z, z \rangle_{A \otimes B} B^*.$$

Then

$$\begin{aligned} A\langle K^*(Q^* \otimes I)z, K^*(Q^* \otimes I)z \rangle_{\mathcal{A} \otimes \mathcal{B} A^*} &\leq \sum_{i \in I} \langle \Lambda_i(Q^* \otimes I)z, \Lambda_i(Q^* \otimes I)z \rangle_{\mathcal{A} \otimes \mathcal{B}} \\ &\leq B\langle (Q^* \otimes I)z, (Q^* \otimes I)z \rangle_{\mathcal{A} \otimes \mathcal{B}} \\ &\leq \|Q\|B\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}}(\|Q\|B)^* \end{aligned}$$

or

$$\begin{aligned} A\langle K^*(Q^* \otimes I)z, K^*(Q^* \otimes I)z \rangle_{\mathcal{A} \otimes \mathcal{B} A^*} &= A\langle (Q^* \otimes I)K^*z, (Q^* \otimes I)K^*z \rangle_{\mathcal{A} \otimes \mathcal{B} A^*} \\ &\geq \|Q^{*-1}\|^{-1}A\langle K^*z, K^*z \rangle_{\mathcal{A} \otimes \mathcal{B}}(\|Q^{*-1}\|^{-1}A)^*. \end{aligned}$$

So, we have

$$\begin{aligned} &\|Q^{*-1}\|^{-1}A\langle K^*z, K^*z \rangle_{\mathcal{A} \otimes \mathcal{B}}(\|Q^{*-1}\|^{-1}A)^* \\ &\leq \sum_{i \in I} \langle \Lambda_i(Q^* \otimes I)z, \Lambda_i(Q^* \otimes I)z \rangle_{\mathcal{A} \otimes \mathcal{B}} \\ &\leq \|Q\|B\langle z, z \rangle_{\mathcal{A} \otimes \mathcal{B}}(\|Q\|B)^*. \end{aligned}$$

Now,

$$\begin{aligned} (Q \otimes I)S(Q^* \otimes I) &= (Q \otimes I)\left(\sum_{i \in I} \Lambda_i^* \Lambda_i\right)(Q^* \otimes I) \\ &= \sum_{i \in I} (Q \otimes I)\Lambda_i^* \Lambda_i(Q^* \otimes I) \\ &= \sum_{i \in I} (\Lambda_i(Q^* \otimes I))^* \Lambda_i(Q^* \otimes I), \end{aligned}$$

which completes the proof. ■

**Theorem 4.3** If  $Q \in \text{End}_{\mathcal{B}}^*(\mathcal{K})$  is invertible and  $\{\Lambda_i\}_{i \in I} \subset \text{End}_{\mathcal{A} \otimes \mathcal{B}}^*(\mathcal{H} \otimes \mathcal{K}, V_i)$  is an  $*$ -K-g-frame for  $\mathcal{H} \otimes \mathcal{K}$  with lower and upper bounds  $A$  and  $B$  respectively and frame operator  $S$ . If  $K$  commute with  $I \otimes Q$ , then  $\{\Lambda_i(I \otimes Q^*)\}_{i \in I}$  is an  $*$ -K-g-frame for  $\mathcal{H} \otimes \mathcal{K}$  with lower and upper bounds  $\|Q^{*-1}\|^{-1}A$  and  $\|Q\|B$  respectively and frame operator  $(I \otimes Q)S(I \otimes Q^*)$ .

**Proof.** Similar to the proof of the theorem 4.2. ■

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