A note on spectral mapping theorem

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Abstract. This paper aims to present the well-known spectral mapping theorem for multi-variable functions.

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1. Introduction

Spectral mapping theorem is a basic theorem in functional analysis. This theorem says that, if $f$ is an analytic function, then

$$\sigma (f (x)) = f (\sigma (x))$$

for an element of a Banach Algebra $\Omega$ (see for example [5, Proposition 2.8] and [6, Theorem 2.1.14]). Similar results have been studied for other types of spectrum as the joint spectrum (see [2,3]). Here, we prove an extension of this theorem for multi-variable functions. We conclude...
this short paper by recalling some notations which will be used in the sequel. Let $\Omega$ be a commutative $C^{*}$-algebra. Then for any $x$ in $\Omega$ we have

$$\sigma(x) = \{ \varphi(x) : \varphi \in M_\Omega \},$$

where $\varphi$ is character and $M_\Omega$ denotes the maximal ideal space of $\Omega$. In the sequel an essential role is played by bi-analytic functions. A continuous function $F$ on $\mathbb{C}^2$ is said to be bi-analytic, if for all $x_0, y_0$ in $\mathbb{C}$,

$$F(x) = F(x, y_0), \quad F(y) = F(x_0, y)$$

are analytic on $\mathbb{C}$. For instance

$$F(x, y) = e^{(x+y)}, \quad F(x, y) = e^{(x)} + e^{(y)}$$

are bi-analytic functions.

Assume that $A$ and $B$ are subsets of $\mathbb{C}$ and $F$ is two variable function. We define

$$F(A, B) := \{ F(a, b) : a \in A, b \in B \}.$$ 

The next section includes an extension of (1).

2. The main result

In order to derive our main result, we need the following observation.

**Lemma 2.1** Every bi-analytic function $f$, has the representation

$$f(x, y) = \sum_{m, n=0}^{\infty} \alpha_{m,n} x^m y^n,$$

where $\alpha_{m,n}$ are complex numbers.

**Proof.** First of all, consider $f(x, y)$ as function in $y$. Since $f(y) = f(x_0, y)$ is analytic, therefore

$$f(x, y) = \sum_n \alpha_n(x) y^n. \quad (2)$$

It remains to prove, $\alpha_n(x)$ is analytic. Since $\alpha_0(x) = f(x, 0)$, so $\alpha_0$ is analytic. Also $\alpha_1(x) = f_y(x, 0)$, and therefore $\alpha_1$ is analytic. In a similar manner one can get $\alpha_i$ are analytic. Whence, it has power series of the form

$$\alpha_n(x) = \sum_m \alpha_{m,n} x^m. \quad (3)$$

By replacing (3) into (2) we get the desired result.}

Now we use Lemma to prove an analogous result but different technique to the main theorem of Harte [4] in an easy fashion as an offshoot of our work.
**Theorem 2.2** Let $\Lambda$ be a commutative $C^*$-algebra and $F$ be a bi-analytic function. Then for any $x, y$ in $\Lambda$ we have

$$\sigma(F(x, y)) \subseteq F(\sigma(x), \sigma(y)).$$

**Proof.** It follows from previous lemma that

$$F(x, y) = \sum_{m,n=0}^{\infty} \alpha_{m,n}x^m y^n.$$

On the other hand, bearing in mind that

$$\varphi(F(x, y)) = \varphi\left(\sum_{m,n=0}^{\infty} \alpha_{m,n}x^m y^n\right),$$

we have

$$\varphi(F(x, y)) = \varphi\left(\sum_{m,n=0}^{\infty} \alpha_{m,n}x^m y^n\right) = \sum_{m,n=0}^{\infty} \alpha_{m,n}\varphi(x)^m \varphi(y)^n = F(\varphi(x), \varphi(y)),$$

where $\varphi(x) \in \sigma(x)$ and $\varphi(y) \in \sigma(y)$. Therefore, $\varphi(F(x, y)) \in F(\varphi(x), \varphi(y))$ and since $\varphi$ is arbitrary we have $\sigma(F(x, y)) \subseteq F(\sigma(x), \sigma(y))$. This completes the proof. □

Related to the above theorem, the following remarks are worth mentioning.

(i) If we take $F(x, y) = x + y$ or $F(x, y) = xy$, then we have $\sigma(x + y) \subseteq \sigma(x) + \sigma(y)$ and $\sigma(xy) \subseteq \sigma(x)\sigma(y)$, which are well known results (see, e.g., [1, Corollary 3.2.10]).

(ii) In general, for some fixed $x, y$ it is enough $x, y$ commute together, and then we consider the $C^*$-algebra generated by $x, y$.

(iii) We guess the above theorem can be extended in the following way:

$$\sigma(F(x_1, x_2, \ldots, x_n)) \subseteq F(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)).$$

However, this does not appear to be easy to prove directly.

(iv) For completeness, we also state the extension of our result to spectral radius $r(x)$. More precisely, we have

$$r(F(x, y)) \subseteq F(r(x), r(y))$$

and in general

$$r(F(x_1, x_2, \ldots, x_n)) \subseteq F(r(x_1), r(x_2), \ldots, r(x_n)).$$

Notice that, the proof is based on the fact that

$$|F(x_1, x_2, \ldots, x_n)| \leq F(|x_1|, |x_2|, \ldots, |x_n|).$$

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