Suzuki-Berinde type fixed-point and fixed-circle results on $S$-metric spaces

N. Taş\textsuperscript{*}

\textit{Department of Mathematics, Bahcesir University, 10145 Bahcesir, Turkey.}

Received 26 March 2018; Revised 10 September 2018; Accepted 11 September 2018.

Communicated by Hamidreza Rahimi

Abstract. In this paper, the notions of a Suzuki-Berinde type $F$-contraction and a Suzuki-Berinde type $F_S$-contraction are introduced on a $S$-metric space. Using these new notions, a fixed-point theorem is proved on a complete $S$-metric space and a fixed-circle theorem is established on a $S$-metric space. Some examples are given to support the obtained results.

© 2018 IAUCTB. All rights reserved.

Keywords: $S$-metric, Suzuki-Berinde type contraction, fixed point, fixed circle.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

The fixed-point theory was started with the classical Banach contraction principle [2]. This principle has been generalized using different approaches. One of these approaches is to generalize the used contractive conditions (for example, see [3, 4, 8, 18, 20, 25, 26]). Another approach is to generalize the used metric spaces. For example, the concept of $S$-metric space was introduced for this purpose as a generalization of metric spaces [23]. Using this space, new fixed-point theorems were obtained with various approaches such as generalized Banach’s contractive conditions, Rhoades’ condition, Wardowski’s condition and etc (for more details, see [5–7, 9, 10, 14–16, 19, 21–24]).

Recently, the fixed-circle problem has been considered and studied a new direction of the extensions of the fixed-point results on metric and $S$-metric spaces. For example, in [13], some fixed-circle theorems were proved using Caristi’s inequality with existence and uniqueness conditions on metric spaces. In [12], using a family of some functions, a

\textsuperscript{*} E-mail address: nihaltas@balikesir.edu.tr (N. Taş).
A sequence was given with discontinuity application. Therefore, some fixed-circle results were studied using different approaches on $S$-metric spaces (see [11, 17]).

Motivated by the above studies, in this paper we prove a fixed-point theorem and a fixed-circle theorem using the Suzuki-Berinde type contractive conditions on $S$-metric spaces. In Section 2, we recall some definitions, results and examples related to $S$-metric spaces. In Section 3, we define two new notions of a Suzuki-Berinde type $F_S$-contraction and a Suzuki-Berinde type $F^3_S$-contraction. Using these contractive conditions, we present a fixed-point theorem and a fixed-circle theorem with some illustrative examples on $S$-metric spaces.

2. Preliminaries

In this section, we recall some necessary notions and results about $S$-metric spaces.

**Definition 2.1** [23] Let $X$ be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $u, v, w, a \in X$:

$(S1) \quad S(u, v, w) = 0$ if and only if $u = v = w$,

$(S2) \quad S(u, v, w) \leq S(u, u, a) + S(v, v, a) + S(w, w, a)$.

Then $S$ is called a $S$-metric on $X$ and the pair $(X, S)$ is called a $S$-metric space.

**Definition 2.2** [23] Let $(X, S)$ be a $S$-metric space and $\{u_n\}$ be a sequence in this space.

1. A sequence $\{u_n\} \subset X$ converges to $u \in X$ if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(u_n, u_n, u) < \varepsilon$.

2. A sequence $\{u_n\} \subset X$ is a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(u_n, u_n, u_m) < \varepsilon$.

3. The $S$-metric space $(X, S)$ is complete if every Cauchy sequence is a convergent sequence.

**Lemma 2.3** [23] Let $(X, S)$ be a $S$-metric space and $u, v \in X$. Then we have

$$S(u, u, v) = S(v, v, u).$$

The relationships between a metric and a $S$-metric were studied in different papers such as [6, 7, 16]. In [7], a formula of a $S$-metric space which is generated by a metric $d$ was given as follows:

Let $(X, d)$ be a metric space. Then the function $S_d : X \times X \times X \to [0, \infty)$ defined by $S_d(u, v, w) = d(u, w) + d(v, w)$ for all $u, v, w \in X$ is a $S$-metric on $X$. The $S$-metric $S_d$ is called the $S$-metric generated by $d$ [16]. We note that there exists a $S$-metric which is not generated by any metric $d$ as seen in the following example.

**Example 2.4** [16] Let $X = \mathbb{R}$. If we consider the function $S : X \times X \times X \to [0, \infty)$ defined by $S(u, v, w) = |u - w| + |u + w - 2v|$ for all $u, v, w \in X$, then $S$ is a $S$-metric on $X$ which is not generated by any metric $d$.

Also in [6], it was shown that every $S$-metric defines a metric $d_S(u, v) = S(u, u, v) + S(v, v, u)$ for all $u, v \in X$. But the function $d_S$ does not always define a metric since the triangle inequality does not satisfy for all elements of $X$.

**Example 2.5** [16] Let $X = \{1, 2, 3\}$. If we consider the function $S : X \times X \times X \to [0, \infty)$
defined by
\[ S(1, 1, 2) = S(2, 2, 1) = 5, \]
\[ S(2, 2, 3) = S(3, 3, 2) = S(1, 1, 3) = S(3, 3, 1) = 2, \]
\[ S(u, v, w) = 0 \text{ if } u = v = w, \]
\[ S(u, v, w) = 1 \text{ otherwise}. \]
for all \( u, v, w \in X \), then \( S \) is a \( S \)-metric on \( X \) which is not generated by any metric \( d \) and does not generate a metric \( d_S \).

Also the relationship between a \( b \)-metric defined in [1] and a \( S \)-metric was proved in the following theorem.

**Theorem 2.6** [21] Let \((X, S)\) be a \( S \)-metric space and \( d_S(u, v) = S(u, u, v) \) for all \( u, v \in X \). Then we have

1. \( d_S \) is a \( b \)-metric on \( X \),
2. \( u_n \to u \) in \((X, S)\) if and only if \( u_n \to u \) in \((X, d_S)\),
3. \( \{u_n\} \) is a Cauchy sequence in \((X, S)\) if and only if \( \{u_n\} \) is a Cauchy sequence in \((X, d_S)\).

The metric \( d_S \) is called the \( b \)-metric generated by \( S \). From the above relationships, it was important to study new fixed-point results on \( S \)-metric spaces.

### 3. New fixed-point and fixed-circle results on \( S \)-metric spaces

In this section, using the Suzuki-Berinde and Wardowski’s techniques, we give a fixed-point theorem and a fixed-circle theorem on \( S \)-metric spaces. Some illustrative examples are also presented for the validity of our results. For this purpose, we use the following known family of functions and a lemma.

Let \( \Delta_F \) be the set of all functions \( F : \mathbb{R}^+ \to \mathbb{R} \) satisfying the following conditions [26]:

\( (F_1) \) \( F \) is strictly increasing,
\( (F_2) \) For all sequence \( \{u_n\} \subseteq \mathbb{R}^+ \), \( \lim_{n \to \infty} u_n = 0 \) if and only if \( \lim_{n \to \infty} F(u_n) = -\infty \),
\( (F_3) \) There exists \( 0 < k < 1 \) such that \( \lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0 \).

**Lemma 3.1** [20] Let \( F : \mathbb{R}^+ \to \mathbb{R} \) be an increasing mapping and \( \{u_n\}_{n=1}^\infty \) be a sequence of positive real numbers. Then the followings hold:

1. If \( \lim_{n \to \infty} F(u_n) = -\infty \), then \( \lim_{n \to \infty} u_n = 0 \).
2. If \( \inf_{n \to \infty} F = -\infty \) and \( \lim_{n \to \infty} u_n = 0 \), then \( \lim_{n \to \infty} F(u_n) = -\infty \).

After that, Secelean replaced the condition \((F_2)\) by \((F'_2)\) as follows:

\( (F'_2) \) \( \inf_{n \to \infty} F = -\infty \) or \((F'_2)\) There exists a sequence \( \{u_n\}_{n=1}^\infty \) of positive real numbers such that \( \lim_{n \to \infty} F(u_n) = -\infty \).

Further, Piri et al. [18] used the following condition \((F'_3)\) instead of the condition \((F_3)\) to obtain some new fixed-point results.

\( (F'_3) \) \( F \) is continuous on \((0, \infty)\).

In the sequel, we consider \( F \) be the family of all functions \( F : \mathbb{R}^+ \to \mathbb{R} \) satisfying in conditions \((F_1)\), \((F'_2)\) and \((F'_3)\).

At first, we define the notion of Suzuki-Berinde type \( F_S \)-contraction on \( S \)-metric spaces.
**Definition 3.2** Let \((X, S)\) be a \(S\)-metric space and \(T : X \rightarrow X\) be a self-mapping. If there exist \(F \in \mathcal{F}\), \(\tau_1 > 0\) and \(\tau_2 \geq 0\) such that for each \(u, v \in X\) with \(Tu \neq Tv\), we have
\[
\frac{1}{3}S(Tu, Tu, u) < S(u, u, v)
\]
implies
\[
\tau_1 + F(S(Tu, Tu, Tu)) \leq F(S(u, u, v)) + \tau_2 \min \{S(Tu, Tu, u), S(Tv, Tv, u), S(Tu, Tu, v)\}.
\]
then \(T\) is called a Suzuki-Berinde type \(F_S\)-contraction on \(X\).

Using Definition 3.2, we prove the following fixed-point result.

**Theorem 3.3** Let \((X, S)\) be a complete \(S\)-metric space and \(T : X \rightarrow X\) be a self-mapping. If \(T\) is a Suzuki-Berinde type \(F_S\)-contraction on \(X\), then \(T\) has a unique fixed point \(u \in X\) and the sequence \(\{Tu_n\}\) converges to \(u\) for every \(u_0 \in X\).

**Proof.** Let \(u_0 \in X\) and the sequence \(\{u_n\}\) be defined by \(Tu_0 = u_1, T^2u_0 = u_2, \ldots, T^nu_0 = u_{n+1}\). If there exists \(n_0 \in \mathbb{N}\) such that \(u_{n_0+1} = u_{n_0}\), then \(u_{n_0}\) is a fixed point of \(T\). Therefore, assume that \(Tu_n = u_{n+1} \neq u_n\). Now, we have
\[
\frac{1}{3}S(Tu_n, Tu_n, u_n) = \frac{1}{3}S(u_n, u_{n+1}, u_{n+1})
\]
\[
< S(u_n, u_{n+1}, Tu_n)
\]
\[
= S(Tu_n, Tu_n, u_n),
\]
for all \(n \in \mathbb{N}\). Using the hypothesis, we get
\[
\tau_1 + F(S(Tu_n, Tu_n, u_n)) = \tau_1 + F(S(Tu_n, Tu_n, Tu_{n-1}))
\]
\[
\leq F(S(u_n, u_{n+1}, u_{n-1})) + \tau_2 \min \{S(Tu_{n-1}, Tu_{n-1}, u_n), S(Tu_n, Tu_n, u_{n-1})\}
\]
and so
\[
F(S(Tu_n, Tu_n, u_n)) \leq F(S(Tu_{n-1}, Tu_{n-2}, u_{n-1})) - \tau_1
\]
\[
\leq \cdots \leq F(S(Tu_0, Tu_0, u_0)) - n\tau_1,
\]
for all \(n \in \mathbb{N}\). Taking limit as \(n \to \infty\), we have \(\lim_{n \to \infty} F(S(Tu_n, Tu_n, u_n)) = -\infty\) and so
\[
\lim_{n \to \infty} S(Tu_n, Tu_n, u_n) = 0, \quad (1)
\]
since \(F \in \mathcal{F}\). Now, we show that the sequence \(\{u_n\}\) is Cauchy. On the contrary, \(\{u_n\}\) is not a Cauchy sequence. Suppose that there exists \(\varepsilon > 0\) and sequences \(\{x(n)\}\) and \(\{y(n)\}\) of natural numbers such that for \(x(n) > y(n) > n\), we have
\[
S(u_{x(n)}, u_{x(n)}, u_{y(n)}) \geq \varepsilon.
\]
Therefore, \(S(u_{x(n)-1}, u_{x(n)-1}, u_{y(n)}) < \varepsilon\) for all \(n \in \mathbb{N}\). Using the inequality (2), Lemma...
2.3 and the condition \((S2)\), we obtain
\[
\varepsilon \leq S(u_{x(n)}, u_{x(n)}, u_{y(n)}) \\
\leq 2S(u_{x(n)}, u_{x(n)}, u_{x(n)-1}) + S(u_{y(n)}, u_{y(n)}, u_{x(n)-1}) \\
< 2S(u_{x(n)}, u_{x(n)}, u_{x(n)-1}) + \varepsilon.
\]

Using the equality (1) and taking the limit, we get
\[
\lim_{n \to \infty} S(u_{x(n)}, u_{x(n)}, u_{y(n)}) = \varepsilon. \quad (3)
\]

From (1) and (3), we can choose a natural number \(n_0 \in \mathbb{N}\) such that
\[
\frac{1}{3}S(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}) < \frac{\varepsilon}{3} < S(u_{x(n)}, u_{x(n)}, u_{y(n)}),
\]
for all \(n \geq n_0\). Using the hypothesis, we obtain
\[
\tau_1 + F(S(Tu_{x(n)}, Tu_{x(n)}, Tu_{y(n)})) \\
\leq F(S(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 \min \left\{ S(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}), S(Tu_{y(n)}, Tu_{y(n)}, u_{x(n)}), S(Tu_{x(n)}, Tu_{y(n)}, u_{x(n)}) \right\} \\
\leq F(S(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 \min \left\{ 2S(Tu_{y(n)}, Tu_{y(n)}, Tu_{x(n)}) + S(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}), \right. \\
\left. 2S(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}) + S(u_{y(n)}, u_{y(n)}, u_{x(n)}) \right\} \\
= F(S(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 S(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}).
\]

Using the condition \(\{F'_3\}\), and (1) and (3), we get \(\tau_1 + F(\varepsilon) \leq F(\varepsilon)\), which is a contradiction since \(\tau_1 > 0\). Therefore, \(\{u_n\}\) is a Cauchy sequence. From the completeness of \(X\), there exists \(u \in X\) such that \(u_n \to u\) as \(n \to \infty\). Thus we have \(\lim_{n \to \infty} S(u_n, u_n, u) = 0\).

Now, we claim that
\[
\frac{1}{3}S(Tu_n, Tu_n, u_n) < S(u_n, u_n, u) \quad \text{or} \quad \frac{1}{3}S(T^2u_n, T^2u_n, Tu_n) < S(Tu_n, Tu_n, u), \quad (4)
\]
for all \(n \in \mathbb{N}\). On the contrary, we assume that there exists \(m \in \mathbb{N}\) such that
\[
\frac{1}{3}S(Tu_m, Tu_m, u_m) \geq S(u_m, u_m, u) \quad \text{and} \quad \frac{1}{3}S(T^2u_m, T^2u_m, Tu_m) \geq S(Tu_m, Tu_m, u). \quad (5)
\]

Thus, using Lemma 2.3, we get
\[
3S(u_m, u_m, u) \leq S(Tu_m, Tu_m, u_m) \\
= S(u_m, u_m, Tu_m) \\
\leq 2S(u_m, u_m, u) + S(Tu_m, Tu_m, u),
\]
which implies
\[
S(u_m, u_m, u) \leq S(Tu_m, Tu_m, u). \quad (6)
\]
From the inequalities (5) and (6), we obtain
\[ S(u_m, u_m, u) \leq S(T u_m, T u_m, u) \leq \frac{1}{3} S(T^2 u_m, T^2 u_m, T u_m). \] (7)

Also using the hypothesis and Lemma 2.3, we have
\[ \frac{1}{3} S(T u_m, T u_m, u_m) = \frac{1}{3} S(u_m, u_m, u_{m+1}) \]
\[ < S(u_m, u_m, u_{m+1}) \]
\[ = S(T u_m, T u_m, u_m) \]
and
\[ \tau_1 + F(S(T^2 u_m, T^2 u_m, T u_m)) = \tau_1 + F(S(T u_m, T u_m, T^2 u_m)) \]
\[ \leq F(S(u_m, u_m, T u_m)) \]
\[ + \tau_2 \min \left\{ \frac{S(u_m, u_m, T u_m), S(u_m, u_m, T^2 u_m)}{S(T u_m, T u_m, u_m)}, \right\} \]
\[ = F(S(T u_m, T u_m, u_m)), \]
which implies
\[ \tau_1 + F(S(T^2 u_m, T^2 u_m, T u_m)) \leq F(S(T u_m, T u_m, u_m)). \] (8)

From the inequality (8), then we have
\[ F(S(T^2 u_m, T^2 u_m, T u_m)) < F(S(T u_m, T u_m, u_m)), \]
and therefore, using strictly increasing property of \( F \) we obtain that
\[ S(T^2 u_m, T^2 u_m, T u_m) < S(T u_m, T u_m, u_m) \] (9)

Using the inequalities (5), (7) and (9), we obtain
\[ S(T^2 u_m, T^2 u_m, T u_m) < S(T u_m, T u_m, u_m) \]
\[ \leq 2S(T u_m, T u_m, u) + S(u_m, u_m, u) \]
\[ < \frac{2}{3} S(T^2 u_m, T^2 u_m, T u_m) + \frac{1}{3} S(T^2 u_m, T^2 u_m, T u_m) \]
\[ = S(T^2 u_m, T^2 u_m, T u_m), \]
which is a contradiction. Therefore, the inequalities given in (4) are satisfied. So using Lemma 2.3, for each \( n \in \mathbb{N} \), we get
\[ \tau_1 + F(S(T u_n, T u_n, T u)) \leq F(S(u_n, u_n, u)) + \tau_2 \min \left\{ \frac{S(u_n, u_n, T u_n), S(u_n, u_n, T u)}{S(u, u, T u)}, \right\}, \]
which implies
\[ \tau_1 + F(S(Tu_n, Tu_n, Tu)) \leq F(S(u_n, u_n, u)) + \tau_2 \min \left\{ \frac{S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tu)}{S(u, u, u_{n+1})} \right\} . \]

(10)

Using (10), the condition \( (F'_{2}) \) and Lemma 3.1, we obtain \( \lim_{n \to \infty} F(S(Tu_n, Tu_n, Tu)) = -\infty \) and \( \lim_{n \to \infty} S(Tu_n, Tu_n, Tu) = 0 \). Hence, we have

\[ S(u, u, Tu) = \lim_{n \to \infty} S(u_{n+1}, u_{n+1}, Tu) = \lim_{n \to \infty} S(Tu_n, Tu_n, Tu) = 0 \]

and so \( u \) is a fixed point of \( T \). Finally, we show that \( u \) is a unique fixed point of \( T \). On the contrary, \( v \) is another fixed point of \( T \) such that \( u \neq v \). Then we have \( S(Tu, Tu, Tv) = S(u, u, v) > 0 \) and

\[ \frac{1}{3} S(Tu, Tu, u) = 0 < S(u, u, v). \]

Using the hypothesis, we obtain

\[ F(S(u, u, v)) = F(S(Tu, Tu, Tv)) < \tau_1 + F(S(Tu, Tu, Tv)) \leq F(S(u, u, Tu)) + \tau_2 \min \left\{ \frac{F(S(u, u, Tu)), F(S(u, u, Tv))}{F(S(v, v, Tu))} \right\} , \]

which implies \( F(S(u, u, v)) < F(S(u, u, v)) \). Now, using the strictly increasing property of \( F \), we get \( S(u, u, v) < S(u, u, v) \), which is a contradiction. Therefore, \( u \) is a unique fixed point of \( T \).

\[ \Box\]

Now we give the following illustrative example.

**Example 3.4** Let us consider the sequence \( \{A_n\} \) defined as \( A_n = 2 + 4 + \cdots + 2n = n(n + 1) \). Let \( X = \{A_n : n \in \mathbb{N}\} \) and the function \( S : X \times X \times X \to [0, \infty) \) be defined as in Example 2.4. Then \( (X, S) \) is a complete \( S \)-metric space and the \( S \)-metric is not generated by any metric. Let us consider the self-mapping \( T : X \to X \) defined by

\[ Tu = \begin{cases} 
A_1 & u = A_1 \\
A_{n-1} & u = A_n \text{ for } (n > 1)
\end{cases} \]

for all \( u \in X \). If we take the mapping \( F(t) = -\frac{1}{t} + t, \tau_1 = 4 \) and \( \tau_2 = 0 \), then \( T \) is a Suzuki-Berinde type \( F_S \)-contraction on \( X \). Indeed, we show this under the following cases:

**Case 1.** Let \( 1 = n < m \). Then we have

\[ S(TA_m, TA_m, TA_1) = 2 |TA_m - TA_1| = 2 |A_{m-1} - A_1| = 2 [4 + 6 + \cdots + 2(m - 1)] \]

and

\[ S(A_m, A_m, A_1) = 2 |A_m - A_1| = 2 [4 + 6 + \cdots + 2m] . \]
Since \( m > 1 \), so we get
\[
4 - \frac{1}{2[4 + 6 + \cdots + 2(m - 1)]} + 2\left[4 + 6 + \cdots + 2(m - 1)\right] \\
< -\frac{1}{2[4 + 6 + \cdots + 2m]} + 2\left[4 + 6 + \cdots + 2(m - 1) + 2m\right].
\]

Hence, we have
\[
4 - \frac{1}{2|TA_m - TA_1|} + 2|TA_m - TA_1| < -\frac{1}{2|A_m - A_1|} + 2|A_m - A_1|.
\]

Case 2. By the similar arguments used in above, we get
\[
4 - \frac{1}{2|TA_m - TA_1|} + 2|TA_m - TA_1| < -\frac{1}{2|A_m - A_1|} + 2|A_m - A_1|
\]
for \( 1 \leq m < n \).

Case 3. Let \( 1 < n < m \). Then we get
\[
\mathcal{S}(TA_m, TA_m, TA_n) = 2|TA_m - TA_n| = 2|A_m - A_n|
\]
\[
= 2[2n + 2(n + 1) + \cdots + 2(m - 1)]
\]
and
\[
\mathcal{S}(A_m, A_m, A_n) = 2|A_m - A_n| = 2[2(n + 1) + 2(n + 2) + \cdots + 2m].
\]

Since \( 1 < n < m \) and \( 4n + 4 \leq 4m \), we have
\[
4 - \frac{1}{2[2n + 2(n + 1) + \cdots + 2(m - 1)]} + 2[2n + 2(n + 1) + \cdots + 2(m - 1)] \\
< -\frac{1}{2[2(n + 1) + 2(n + 2) + \cdots + 2m]} + 2[2(n + 1) + 2(n + 2) + \cdots + 2(m - 1) + 2m].
\]
So we get
\[
4 - \frac{1}{2|TA_m - TA_n|} + 2|TA_m - TA_n| < -\frac{1}{2|A_m - A_n|} + 2|A_m - A_n|.
\]

Therefore, \( T \) is a Suzuki-Berinde type \( F_S \)-contraction and \( TA_1 = A_1 \); that is, \( A_1 \) is a unique fixed point of \( T \).

If we take \( \tau_2 = 0 \) then we get the following corollaries.

**Corollary 3.5** Let \((X, \mathcal{S})\) be a complete \( \mathcal{S} \)-metric space and \( T : X \to X \) be a self-mapping. If there exist \( \tau_1 > 0 \) and \( F \in \mathcal{F} \) such that for each \( u, v \in X \) with \( Tu \neq Tv \), we have \( \frac{1}{3}\mathcal{S}(Tu, Tu, u) < \mathcal{S}(u, u, v) \) implies \( \tau_1 + F(\mathcal{S}(Tu, Tu, Tv)) < F(\mathcal{S}(u, u, v)) \), then \( T \) has a unique fixed point \( u \in X \) and the sequence \( \{T^nu_0\} \) converges to \( u \) for every \( u_0 \in X \).
Corollary 3.6 Let $(X, S)$ be a complete $S$-metric space and $T : X \to X$ be a self-mapping. If there exist $\tau_1 > 0$ and $F \in \mathcal{F}$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have $\tau_1 + F(S(Tu, Tu, Tv)) \leq F(S(u, u, v))$, then $T$ has a unique fixed point $u \in X$ and the sequence $\{T^nu_0\}$ converges to $u$ for every $u_0 \in X$.

If we consider Theorem 2.6, then we get the Suzuki-Berinde type fixed-point theorem on $b$-metric spaces.

Theorem 3.7 Let $(X, d^S)$ be a complete $b$-metric space and $T : X \to X$ be a self-mapping. If there exist $F \in \mathcal{F}$, $\tau_1 > 0$ and $\tau_2 \geq 0$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have $\frac{1}{3}d^S(Tu, u) < d^S(u, v)$ implies

$$\tau_1 + F(d^S(Tu, Tv)) \leq F(d^S(u, v)) + \tau_2 \min\{d^S(Tu, u), d^S(Tv, u), d^S(Tu, v)\},$$

then $T$ has a unique fixed point $u \in X$ and the sequence $\{T^nu_0\}$ converges to $u$ for every $u_0 \in X$.

**Proof.** By the similar arguments used in the proof of Theorem 3.3, it is clear. \[\blacksquare\]

In [11] and [23], a circle and a disc are defined on a $S$-metric space as follows, respectively:

$$C_{u_0, r}^S = \{u \in X : S(u, u, u_0) = r\} \quad \text{and} \quad D_{u_0, r}^S = \{x \in X : S(u, u, u_0) \leq r\}.$$

Definition 3.8 [11] Let $(X, S)$ be a $S$-metric space, $C_{u_0, r}^S$ be a circle and $T : X \to X$ be a self-mapping. If $Tu = u$ for every $u \in C_{u_0, r}^S$, then the circle $C_{u_0, r}^S$ is called as the fixed circle of $T$.

We introduce the notion of Suzuki-Berinde type $F_C^S$-contraction on $S$-metric spaces.

Definition 3.9 Let $(X, S)$ be a $S$-metric space and $T : X \to X$ be a self-mapping. $T$ is called a Suzuki-Berinde type $F_C^S$-contraction on $X$ if there exist $F \in \mathcal{F}$, $\tau_1 > 0$, $\tau_2 \geq 0$ and $u_0 \in X$ such that for each $u \in X$ with $Tu \neq u$, we have $\frac{1}{3}S(u, u, u_0) < S(Tu, Tu, u)$ implies

$$\tau_1 + F(S(Tu, Tu, u)) \leq F(S(u, u, u_0)) + \tau_2 \min\{S(Tu_0, Tu_0, u_0), S(Tu_0, Tu_0, u_0), S(Tu_0, Tu_0, u_0)\}.$$ 

Using Definition 3.9, we obtain the following proposition.

Proposition 3.10 Let $(X, S)$ be a $S$-metric space and $T : X \to X$ be a self-mapping. If $T$ is a Suzuki-Berinde type $F_C^S$-contraction on $X$ with $u_0 \in X$ then we have $Tu_0 = u_0$.

**Proof.** Assume that $Tu_0 \neq u_0$. From the definition of the Suzuki-Berinde type $F_C^S$-contraction, we get $\frac{1}{3}S(u_0, u_0, u_0) < S(Tu_0, Tu_0, u_0)$ and so

$$\tau_1 + F(S(Tu_0, Tu_0, u_0)) \leq F(S(u_0, u_0, u_0)) + \tau_2 \min\{S(Tu_0, Tu_0, u_0), S(Tu_0, Tu_0, u_0)\} = F(0) + \tau_2 S(Tu_0, Tu_0, u_0),$$

which is a contradiction with the definition of $F$. Therefore, we obtain $Tu_0 = u_0$. \[\blacksquare\]

Now we prove the fixed-circle theorem.
Theorem 3.11 Let \((X, S)\) be a \(S\)-metric space, \(T\) be a self-mapping on \(X\) satisfying the Suzuki-Berinde type \(F_C^S\)-contractive condition with \(u_0 \in X\) and \(r = \min \{S(Tu, Tu, u) : Tu \neq u\}\). If \(S(Tu, Tu, u_0) = r\) for all \(u \in C_{u_0, r}^S\), then \(C_{u_0, r}^S\) is a fixed circle of \(T\). Especially, \(T\) fixes every circle \(C_{u_0, r}^S\) with \(\rho < r\).

Proof. Let \(u \in C_{u_0, r}^S\) and \(Tu \neq u\). By the definition of \(r\), we have \(\frac{1}{3}S(u, u_0) = \frac{r}{3} < S(Tu, Tu, u)\). Now, using the Suzuki-Berinde type \(F_C^S\)-contractive property, Proposition 3.10, Lemma 2.3 and the strictly increasing property of \(S\), we obtain

\[
F(S(Tu, Tu, u)) \leq F(S(u, u, u_0)) - \tau_1 + \tau_2 \min \left\{ \frac{S(Tu_0, Tu_0, u_0), S(u, u, u_0)}{S(Tu, Tu, u_0)} \right\} = F(r) - \tau_1
\]

which is a contradiction. Therefore, we find \(Tu = u\) and so \(C_{u_0, r}^S\) is a fixed circle of \(T\).

Finally, we show that \(T\) also fixes any circle \(C_{u_0, \rho}^S\) with \(\rho < r\). Let \(u \in C_{u_0, \rho}^S\) and suppose that \(Tu \neq u\). By the Suzuki-Berinde type \(F_C^S\)-contractive property, we have

\[
F(S(Tu, Tu, u)) \leq F(S(u, u, u_0)) - \tau_1 < F(\rho) \leq F(S(Tu, Tu, u)),
\]

which is a contradiction. Hence we get \(Tu = u\). Thus, \(C_{u_0, \rho}^S\) is a fixed circle of \(T\). \(\blacksquare\)

As an immediate result of Theorem 3.11, we obtain the following corollary.

Corollary 3.12 Let \((X, S)\) be a \(S\)-metric space, \(T\) be a self-mapping on \(X\) satisfying the Suzuki-Berinde type \(F_C^S\)-contractive condition with \(u_0 \in X\) and \(r = \min \{S(Tu, Tu, u) : Tu \neq u\}\). If \(S(Tu, Tu, u_0) = r\) for all \(u \in C_{u_0, r}^S\), then \(T\) fixes the disc \(D_{u_0, r}^S\).

Example 3.13 Let \(X = \{1, 2, 5, \frac{5}{2}, e - \frac{1}{2}, e, e + \frac{1}{2}\}\) and the \(S\)-metric be defined as in Example 2.4. Then \((X, S)\) is a \(S\)-metric space. Let us define the self-mapping \(T : X \to X\) as

\[
Tu = \begin{cases} 
\frac{5}{2} & u = 2 \\
\text{otherwise} & 
\end{cases}
\]

for all \(u \in X\). Then the self-mapping \(T\) is a Suzuki-Berinde type \(F_C^S\)-contraction with \(F = \ln u, u_0 = e, \tau_1 = 0.5\) and \(\tau_2 > 0\). Indeed, for \(u = 2\), we get

\[
\frac{1}{3}S(u, u, u_0) = \frac{2e - 4}{3} < S(Tu, Tu, u) = 1 \\
\Rightarrow S(Tu, Tu, u) = 1 < S(u, u, u_0) = 2e - 4 \\
\Rightarrow \ln(1) < \ln(2e - 4) = \ln(2(e - 4)) \\
\Rightarrow 0.5 < \ln 2 + \ln(e - 4) \\
\Rightarrow \tau_1 + F(S(Tu, Tu, u)) \leq F(S(u, u, u_0)) \\
\quad + \tau_2 \min \{S(Tu_0, Tu_0, u_0), S(Tu_0, Tu_0, u), S(Tu, Tu, u_0)\}.
\]

Using Theorem 3.11, we get \(r = \min \{S(Tu, Tu, u) : Tu \neq u\} = 1\). It is clear that \(T\) fixes the circle \(C_{e, 1}^S = \{e - \frac{1}{2}, e + \frac{1}{2}\}\) and the disc \(D_{e, 1}^S = \{\frac{5}{2}, e - \frac{1}{2}, e, e + \frac{1}{2}\}\).
If we consider Theorem 2.6, then we get the Suzuki-Berinde type fixed-circle theorem on \( b \)-metric spaces.

**Theorem 3.14** Let \((X, d^S)\) be a \( b \)-metric space, \( T : X \to X \) be a self-mapping and \( r = \min \{ d^S(Tu, u) : Tu \neq u \} \). If there exist \( F \in \mathcal{F}, \tau_1 > 0, \tau_2 \geq 0 \) and \( u_0 \in X \) such that for each \( u \in X \) with \( Tu \neq u \), we have \( \frac{1}{2} d^S(u, u_0) < d^S(Tu, u) \) implies

\[
\tau_1 + F\left( d^S(Tu, u) \right) \leq F\left( d^S(u, u_0) \right) + d^S(Tu_0, u), d^S(Tu_0, u), d^S(Tu, u) \}
\]

then \( C^d_{u_0, r} = \{ u \in X : d^S(u, u_0) = r \} \) is a fixed circle of \( T \) with the condition \( d^S(Tu, u_0) = r \). Especially, \( T \) fixes every circle \( C^d_{u_0, r} \) with \( \rho < r \), that is, \( T \) fixes the disc \( D^d_{u_0, r} = \{ u \in X : d^S(u, u_0) \leq r \} \).

4. Conclusion

In this section, we prove a Suzuki-Berinde fixed-point theorem and a Suzuki-Berinde fixed-circle theorem using the Suzuki-Berinde and Wardowski’s techniques on \( S \)-metric spaces. Similarly, new fixed-point or fixed-circle results can be obtained using or modified the known techniques used in some fixed-point theorems on metric and some generalized metric spaces.

Acknowledgments

The author gratefully thanks to the Referees for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

References


S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in $S$-metric spaces, Mat. Vesnik. 64 (3) (2012), 258-266.

