On $\beta$--topological vector spaces

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Abstract. We introduce and study a new class of spaces, namely $\beta$--topological vector spaces via $\beta$--open sets. The relationships among these spaces with some existing spaces are investigated. In addition, some important and useful characterizations of $\beta$--topological vector spaces are provided.

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1. Introduction

It is well-known that the advent of topological vector spaces brought a revolution in the study of various branches of functional analysis. Because of nice properties and usefulness, these spaces remain a fundamental notion in fixed point theory, operator theory and various other advanced branches of mathematics. In 2015, Khan et al. [4] introduced and studied the s-topological vector spaces which are a generalization of topological vector spaces. In 2016, Khan and Iqbal [5] introduced the irresolute topological vector spaces which are a particular brand of s-topological vector spaces but they are independent of topological vector spaces. Ibrahim [3] initiated the study of $\alpha$--topological vector spaces which are contained in the class of s-topological vector spaces. In this paper, we introduce a new class of spaces, namely, $\beta$--topological vector spaces. Some general properties of $\beta$--topological vector spaces along with their relationships with certain other types.

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of spaces are investigated. Furthermore, a broad characterizations of these spaces are presented.

2. Preliminaries

Let $X$ be a topological space. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$ respectively. We represent the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. The notations $\epsilon$ and $\delta$ denote negligibly small positive real numbers.

**Definition 2.1** [1, 6, 7] A subset $A$ of a topological space $X$ is called
(a) semi-open if $A \subseteq \text{Cl}(\text{Int}(A))$,
(b) $\alpha$—open if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$,
(c) $\beta$—open if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$.

Clearly, every open set is $\alpha$—open and every $\alpha$—open set is semi-open and every semi-open set is $\beta$—open but, in general, the converse need not be true.

**Example 2.2** Let $X = \mathbb{R}$ with the usual topology. Consider $A = [1, 2], B = (1, 2) \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then $A$ is semi-open but it is neither open nor $\alpha$—open. Also, notice that $B$ is $\beta$—open which is not semi-open.

The complement of a $\beta$—open set is said to be $\beta$—closed set. The intersection of all $\beta$—closed sets containing a subset $A \subseteq X$ is called the $\beta$—closure of $A$ and is denoted by $\beta\text{Cl}(A)$. It is known that a subset $A$ of $X$ is $\beta$—closed if and only if $A = \beta\text{Cl}(A)$. A point $x \in \beta\text{Cl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $\beta$—open set $U$ in $X$ containing $x$. The $\beta$—interior of a subset $A \subseteq X$ is the union of all $\beta$—open sets in $X$ contained in $A$ and is denoted by $\beta\text{Int}(A)$. A point $x$ of $X$ is called $\beta$—interior point of a subset $A$ if there exists a $\beta$—open set $U$ in $X$ containing $x$ such that $x \in U \subseteq A$. The set of all $\beta$—interior points of $A$ is equal to $\beta\text{Int}(A)$. The family of all $\beta$—open (resp. $\beta$—closed) sets in $X$ will be denoted by $\beta\text{O}(X)$ (resp. $\beta\text{C}(X)$).

Also we recall some definitions that will be used in the sequel.

**Definition 2.3** Let $L$ be a vector space over the field $F$ ($\mathbb{R}$ or $\mathbb{C}$). Let $T$ be a topology on $L$ such that

(1) For each $x, y \in L$ and each open neighborhood $W$ of $x + y$ in $L$, there exist open neighborhoods $U$ and $V$ of $x$ and $y$ respectively, in $L$ such that $U + V \subseteq W$,

(2) For each $\lambda \in F, x \in L$ and each open neighborhood $W$ of $\lambda x$ in $L$, there exist open neighborhoods $U$ of $\lambda$ in $F$ and $V$ of $x$ in $L$ such that $U.V \subseteq W$.

Then the pair $(L, T)$ is called topological vector space.

**Definition 2.4** [4] Let $L$ be a vector space over the field $F$ ($\mathbb{R}$ or $\mathbb{C}$) and let $T$ be a topology on $L$ such that

(1) For each $x, y \in L$ and each open set $W$ in $L$ containing $x + y$, there exist semi-open sets $U$ and $V$ in $L$ containing $x$ and $y$ respectively such that $U + V \subseteq W$,

(2) For each $\lambda \in F, x \in L$ and each open set $W$ in $L$ containing $\lambda x$, there exist semi-open sets $U$ in $F$ containing $\lambda$ and $V$ in $L$ containing $x$ such that $U.V \subseteq W$.

Then the pair $(L, T)$ is called s-topological vector space.

**Definition 2.5** [5] Let $L$ be a vector space over the field $F$ ($\mathbb{R}$ or $\mathbb{C}$) and $T$ be a topology on $L$ such that

(1) For each $x, y \in L$ and each semi-open set $W$ in $L$ containing $x + y$, there exist semi-open sets $U$ and $V$ in $L$ containing $x$ and $y$ respectively such that $U + V \subseteq W$, 

(2) For each $\lambda \in F$ and each semi-open set $W$ in $L$ containing $\lambda x$, there exist semi-open sets $U$ in $F$ containing $\lambda$ and $V$ in $L$ containing $x$ such that $U.V \subseteq W$.
(2) For each \( \lambda \in F \), \( x \in L \) and each semi-open set \( W \) in \( L \) containing \( \lambda x \), there exist semi-open sets \( U \) in \( F \) containing \( \lambda \) and \( V \) in \( L \) containing \( x \) such that \( U.V \subseteq W \).

Then the pair \((L(F), T)\) is called irresolute topological vector space.

Definition 2.6 [3] Let \( L \) be a vector space over the field \( F \) (\( \mathbb{R} \) or \( \mathbb{C} \)) and \( T \) be a topology on \( L \) such that

1. For each \( x, y \in L \) and each \( \alpha \)-open set \( W \) in \( L \) containing \( x + y \), there exist \( \alpha \)-open sets \( U \) and \( V \) in \( L \) containing \( x \) and \( y \) respectively such that \( U + V \subseteq W \).
2. For each \( \lambda \in F \), \( x \in L \) and each \( \alpha \)-open set \( W \) in \( L \) containing \( \lambda x \), there exist \( \alpha \)-open sets \( U \) in \( F \) containing \( \lambda \) and \( V \) in \( L \) containing \( x \) such that \( U.V \subseteq W \).

Then the pair \((L(F), T)\) is called \( \alpha \)-topological vector space.

3. \( \beta \)-topological vector spaces

The purpose of this section is to define and investigate some basic properties of \( \beta \)-topological vector spaces.

Definition 3.1 Let \( E \) be a vector space over the field \( K \), where \( K = \mathbb{R} \) or \( \mathbb{C} \) with standard topology. Let \( \tau \) be a topology on \( E \) such that the following conditions are satisfied:

1. For each \( x, y \in E \) and each open set \( W \subseteq E \) containing \( x + y \), there exist \( \beta \)-open sets \( U \) and \( V \) in \( E \) containing \( x \) and \( y \) respectively, such that \( U + V \subseteq W \).
2. For each \( \lambda \in K \), \( x \in E \) and each open set \( W \subseteq E \) containing \( \lambda x \), there exist \( \beta \)-open sets \( U \) in \( K \) containing \( \lambda \) and \( V \) in \( E \) containing \( x \) such that \( U.V \subseteq W \).

Then the pair \((E(K), \tau)\) is called \( \beta \)-topological vector space (written in short, \( \betaTVS \)).

First of all we present some examples of \( \beta \)-topological vector spaces and then these examples will be used in the sequel for investigating the relationships of \( \beta \)-topological vector spaces with certain other types of spaces.

Example 3.2 Consider the field \( K = \mathbb{R} \) with the standard topology. Let \( E = \mathbb{R} \) be the real vector space, is also endowed with the standard topology. Then \((E(K), \tau)\) is \( \beta \)-topological vector space.

After tasting this example, an immediate question that comes into mind is that is there any other topology on \( \mathbb{R} \) which turn it out a \( \beta \)-topological vector space. The answer is in affirmative. In fact, there are topologies on \( \mathbb{R} \) other than the standard topology which turn it out a \( \beta \)-topological vector space. Let us present some examples of them.

Example 3.3 Consider \( F = \mathbb{R} \) with the standard topology. Let \( E = \mathbb{R} \) be the vector space of real numbers over the field \( F \), is endowed with the topology \( \tau = \{ \emptyset, D, \mathbb{R} \} \), where \( D \) denotes the set of irrational numbers. Then

1. For each \( x, y \in E \), we have two cases:
   Case (I) If \( x + y \) is rational, then the only open neighborhood of \( x + y \) in \( E \) is \( \mathbb{R} \). So, there is nothing to prove.
   Case (II) If \( x + y \) is irrational, then for open neighborhood \( W = D \) of \( x + y \) in \( E \). We have following sub-cases:
     Sub-case (i) If both \( x \) and \( y \) are irrational, we can choose \( \beta \)-open sets \( U = \{ x \} \) and \( V = \{ y \} \) in \( E \) such that \( U + V \subseteq W \).
     Sub-case (ii) If one of \( x \) or \( y \) is rational, say \( y \). Then, for the selection of \( \beta \)-open sets \( U = \{ x \} \) and \( V = \{ p, y \} \) in \( E \), where \( p \in D \) such that \( p + x \in D \), we have \( U + V \subseteq W \).
   This verifies the first condition of \( \beta \)-topological vector spaces.
(2) Let $\lambda \in \mathbb{R}$ and $x \in E$. If $\lambda x$ is rational, then it is straightforward to prove. Suppose $\lambda x$ is irrational. Let $W = D$ be an open neighborhood of $\lambda x$. The following cases arise:

Case (I) If both $\lambda$ and $x$ are irrational, then, choose $\beta$–open sets $U = [(\lambda - \epsilon, \lambda + \epsilon) \cap \mathbb{Q}] \cup \{\lambda\}$ in $\mathbb{R}$ containing $\lambda$ and $V = \{x\}$ in $E$ containing $x$, we see that $U.V \subseteq W$.

Case (II) If $\lambda$ is rational and $x$ is irrational, then for the selection of $\beta$–open sets $U = (\lambda - \epsilon, \lambda + \epsilon) \cap \mathbb{Q}$ in $\mathbb{R}$ containing $\lambda$ and $V = \{x\}$ in $E$ containing $x$, we have $U.V \subseteq W$.

Case (III) Finally, suppose $\lambda$ is irrational and $x$ is rational. Choose $\beta$–open sets $U = (\lambda - \epsilon, \lambda + \epsilon) \cap D$ of $\mathbb{R}$ and $V = \{x, p\}$ of $E$ such that $p \in D$ with $pq$ is irrational for each $q \in U$, we find that $U.V \subseteq W$.

This proves that $(E(\mathbb{R}), \tau)$ is $\beta$–topological vector space.

**Example 3.4** Let $E = \mathbb{R}$ be the vector space of real numbers over the field $K$, where $K = \mathbb{R}$ with standard topology and the topology $\tau$ on $E$ be generated by the base $B = \{(a, b), [c, d] : a, b, c$ and $d$ are real numbers with $0 < c < d\}$. We show that $(E(\mathbb{R}), \tau)$ is $\beta$–topological vector space. For which we have to verify the following two conditions:

(1) Let $x, y \in L$. Then, for open neighborhood $W = [x + y, x + y + \epsilon)$ (resp. $(x + y - \epsilon, x + y + \epsilon)$) of $x + y$ in $E$, we can opt for $\beta$–open sets $U = [x, x + \delta)$ (resp. $(x - \delta, x + \delta)$) and $V = [y, y + \delta)$ (resp. $(y - \delta, y + \delta)$) of $E$ containing $x$ and $y$ respectively, such that $U + V \subseteq W$ for each $\delta < \frac{\epsilon}{2}$.

(2) Let $x \in E$ and $\lambda \in K$. Consider open neighborhood $W = [\lambda x, \lambda x + \epsilon)$ (resp. $(\lambda x - \epsilon, \lambda x + \epsilon)$) of $\lambda x$ in $E$. We have following cases:

Case (I). If $\lambda > 0$ and $x > 0$, then clearly $\lambda x > 0$. We can choose $\beta$–open sets $U = [\lambda, \lambda + \delta)$ (resp. $(\lambda - \delta, \lambda + \delta)$) in $K$ containing $\lambda$ and $V = [x, x + \delta)$ (resp. $(x - \delta, x + \delta)$) in $E$ containing $x$ such that $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{\lambda + x + 1}$.

Case (II). If $\lambda < 0$ and $x < 0$, then $\lambda x > 0$. We can choose $\beta$–open sets $U = (\lambda - \delta, \lambda)$ (resp. $(\lambda - \delta, \lambda + \delta)$) in $K$ and $V = (x - \delta, x + \delta)$ (resp. $(x - \delta, x + \delta)$) in $E$ such that $U.V \subseteq W$ for sufficiently appropriate $\delta < \frac{\epsilon}{\lambda x + 1}$.

Case (III). If $\lambda = 0$ and $x > 0$ (resp. $\lambda > 0$ and $x = 0$). Then $\lambda x = 0$. Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of $0$ in $E$. We can opt for $\beta$–open sets $U = (\delta, \delta)$ (resp. $U = (\lambda - \delta, \lambda + \delta)$) in $\mathbb{R}$ containing $\lambda$ and $V = (x - \delta, x + \delta)$ (resp. $V = (\lambda - \delta, \lambda + \delta)$) in $E$ containing $x$ such that $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{\lambda x + 1}$ (resp. $\delta < \frac{\epsilon}{\lambda + 1}$).

Case (IV). If $\lambda = 0$ and $x < 0$ (resp. $\lambda < 0$ and $x = 0$). Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of $0$ in $E$. Then, for the selection of $\beta$–open sets $U = (\delta, \delta)$ (resp. $U = (-\delta, \delta)$) in $\mathbb{R}$ and $V = (x - \delta, x + \delta)$ (resp. $V = (-\delta, \delta)$) in $E$, we have $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{\lambda x + 1}$ (resp. $\delta < \frac{\epsilon}{\lambda + 1}$).

Case (V). If $\lambda = 0$ and $x = 0$. Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of $0$ in $E$, we can find $\beta$–open sets $U = (\delta, \delta)$ of $\mathbb{R}$ and $V = (\delta, \delta)$ of $E$, such that $U.V \subseteq W$ for each $\delta < \sqrt{\epsilon}$.

Case (VI). If $\lambda < 0$, $x > 0$ (resp. $\lambda > 0$, $x < 0$). In this case, there is only one type of open neighborhood $W = (\lambda x - \epsilon, \lambda x + \epsilon)$ of $\lambda x$ in $E$. Choose $\beta$–open sets $U = (\lambda - \delta, \lambda + \delta)$ and $V = (x - \delta, x + \delta)$ in $E$, we have $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{\lambda x + 1}$ (resp. $\delta < \frac{\epsilon}{\lambda + 1}$). Hence, $(E(\mathbb{R}), \tau)$ is $\beta$–topological vector space.

The definitions clarify that every $s$–topological vector space is $\beta$–topological vector space but the converse is not true because, in general, Example 3.3 is not $s$–topological vector space.

From here on, $E$ denotes a $\beta$–topological vector space $(E(\mathbb{K}), \tau)$ unless stated explicitly and by a scalar we mean an element of the associated field $\mathbb{K}$ of a $\beta$–topological vector space $(E(\mathbb{K}), \tau)$. Now, we discuss some basic properties of $\beta$–topological vector spaces.
**Theorem 3.5** Let $A$ be any open subset of a $\beta-$topological vector space $E$. Then the following are true:

(i) $x + A \in \beta O(E)$ for each $x \in E$,
(ii) $\lambda A \in \beta O(E)$ for each non-zero scalar $\lambda$.

**Proof.** (i) Let $y \in x + A$. Then there exist $\beta-$open sets $U, V \in \beta O(E)$ containing $-x$ and $y$, respectively, such that

$$U + V \subseteq A \Rightarrow -x + V \subseteq U + V \subseteq A \Rightarrow V \subseteq x + A \Rightarrow y \in Int_\beta(x + A).$$

Hence, $x + A = Int_\beta(x + A)$. This proves that $x + A$ is $\beta-$open set in $E$.

(ii) Let $x \in \lambda A$. By the definition of $\beta-$topological vector spaces, there exist $\beta-$open sets $U$ in $K$ containing $\frac{1}{\lambda}$ and $V$ in $E$ containing $x$ such that

$$U, V \subseteq A \Rightarrow x \in V \subseteq \lambda A \Rightarrow x \in Int_\beta(\lambda A) \Rightarrow \lambda A = Int_\beta(\lambda A).$$

Thus, $\lambda A \in \beta O(E)$. ■

**Corollary 3.6** For any open subset $A$ of a $\beta-$topological vector space $E$, the following are true:

(i) $x + A \subseteq Cl(Int(Cl(x + A)))$ for each $x \in E$,
(ii) $\lambda A \subseteq Cl(Int(Cl(\lambda A)))$ for each non-zero scalar $\lambda$.

**Theorem 3.7** Let $F$ be any closed subset of a $\beta-$topological vector space $E$. Then the following are true:

(i) $x + F \in \beta C(E)$ for each $x \in E$,
(ii) $\lambda F \in \beta C(E)$ for each non-zero scalar $\lambda$.

**Proof.** (i) Suppose that $y \in \beta Cl(x + F)$. Consider $z = -x + y$ and let $W$ be any open set in $E$ containing $z$. Then there exist $\beta-$open sets $U$ and $V$ in $E$ such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. Since $y \in \beta Cl(x + F)$, $(x + F) \cap V \neq \emptyset$. So, there is $a \in (x + F) \cap V$. Now

$$-x + a \in F \cap (U + V) \subseteq F \cap W \Rightarrow F \cap W \neq \emptyset \Rightarrow z \in Cl(F) = F \Rightarrow y \in x + F.$$  

Hence, $x + F = \beta Cl(x + F)$. This proves that $x + F$ is $\beta-$closed set in $E$.

(ii) Assume that $x \in \beta Cl(\lambda F)$ and let $W$ be any open neighborhood of $y = \frac{1}{\lambda} x$ in $E$. Since $E$ is $\beta$TVS, there exist $\beta-$open sets $U$ in $K$ containing $\frac{1}{\lambda}$ and $V$ in $E$ containing $x$ such that $U, V \subseteq W$. By hypothesis, $(\lambda F) \cap V \neq \emptyset$. Therefore, there is $a \in (\lambda F) \cap V$. Now

$$\frac{1}{\lambda} a \in F \cap (U, V) \subseteq F \cap W \Rightarrow F \cap W \neq \emptyset \Rightarrow y \in Cl(F) = F \Rightarrow x \in \lambda F$$

and thereby, $\lambda F = \beta Cl(\lambda F)$. Hence, $\lambda F \in \beta C(E)$. ■

**Corollary 3.8** For any closed subset $F$ of a $\beta-$topological vector space $E$, the following are true:

(i) $Int(Cl(Int(x + F))) \subseteq x + F$ for each $x \in E$,
(ii) $Int(Cl(\lambda F)) \subseteq \lambda F$ for each non-zero scalar $\lambda$.

**Theorem 3.9** Let $A$ and $B$ be any subsets of a $\beta-$topological vector space $E$. Then $\beta Cl(A) + \beta Cl(B) \subseteq Cl(A + B)$.

**Proof.** Let $x \in \beta Cl(A)$, $y \in \beta Cl(B)$ and let $W$ be any open neighborhood of $x + y$ in $E$. Then, by the definition of $\beta-$topological vector spaces, there exist $\beta-$open sets $U, V \in \beta O(E)$ such that $x \in U$, $y \in V$ and $U + V \subseteq W$. By assumption, there are $a \in A \cap U$ and $b \in B \cap V$. Consequently, $a + b \in (A + B) \cap (U + V) \subseteq (A + B) \cap W \Rightarrow (A + B) \cap W \neq \emptyset$.
4. Characterizations

In this section, we obtain some useful characterizations of \( \beta \)-topological vector spaces.

Theorem 4.1 For a subset \( A \) of a \( \beta \)-topological vector space \( E \), the following are valid:

(a) \( \beta Cl(x + A) \subseteq x + Cl(A) \) for each \( x \in E \),
(b) \( x + \beta Cl(A) \subseteq Cl(x + A) \) for each \( x \in E \),
(c) \( x + Int(A) \subseteq \beta Int(x + A) \) for each \( x \in E \),
(d) \( Int(x + A) \subseteq x + \beta Int(A) \) for each \( x \in E \).

Proof. (a) Let \( y \in \beta Cl(x + A) \) and consider \( z = -x + y \) in \( E \). Let \( W \) be any open neighborhood of \( z \). Then we get \( \beta \)-open sets \( U \) containing \( -x \) and \( V \) containing \( y \) in \( E \) such that \( U + V \subseteq W \). Whence we find that \( (x + A) \cap V \neq \emptyset \) \( \Rightarrow \) there is \( a \in E \) such that \( a \in (x + A) \cap V \). Now, \( -x + a \in A \cap (U + V) \subseteq A \cap W \Rightarrow A \cap W \neq \emptyset \) and hence, \( z \in Cl(A) \); that is, \( y \in x + Cl(A) \). Therefore, \( \beta Cl(x + A) \subseteq x + Cl(A) \).
(b) Let \( z = x + \beta Cl(A) \). Then \( z = x + y \) for some \( y \in \beta Cl(A) \). Notice that for any open neighborhood \( W \) of \( z \), there exist \( \beta \)-open sets \( U, V \in \beta O(E) \) such that \( x \in U \), \( y \in V \) and \( U + V \subseteq W \). Since \( y \in \beta Cl(A) \), \( A \cap V \neq \emptyset \Rightarrow \) there is \( a \in A \cap V \). Now, \( x + a \in (x + A) \cap (U + V) \subseteq (x + A) \cap W \Rightarrow (x + A) \cap W \neq \emptyset \Rightarrow z \in Cl(x + A) \).

Hence, the assertion follows.
(c) Let \( y \in x + Int(A) \). Then \( U + V \subseteq Int(A) \) where \( U, V \in \beta O(E) \) such that \( -x \in U \) and \( y \in V \). Whence we have \( -x + V \subseteq U + V \subseteq A \Rightarrow V \subseteq x + A \). Since \( V \) is \( \beta \)-open, \( y \in \beta Int(x + A) \) and consequently, \( x + Int(A) \subseteq \beta Int(x + A) \).
(d) Let \( y \in Int(x + A) \). Then \( y = x + a \) for some \( a \in A \). Since \( E \) is \( \beta TVS \), there exist \( U, V \in \beta O(E) \) such that \( x \in U \), \( a \in V \) and \( U + V \subseteq Int(x + A) \). Now \( x + V \subseteq U + V \subseteq Int(x + A) \subseteq x + A \) implies that \( y \in x + \beta Int(A) \). Therefore, the assertion follows.

The following is the analog of Theorem 4.1.

Theorem 4.2 For a subset \( A \) of a \( \beta \)-topological vector space \( E \), the following are valid:

(a) \( \beta Cl(\lambda A) \subseteq \lambda Cl(A) \) for each non-zero scalar \( \lambda \),
(b) \( \lambda \beta Cl(A) \subseteq Cl(\lambda A) \) for each non-zero scalar \( \lambda \),
(c) \( \lambda Int(A) \subseteq \beta Int(\lambda A) \) for each non-zero scalar \( \lambda \),
(d) \( Int(\lambda A) \subseteq \lambda \beta Int(A) \) for each non-zero scalar \( \lambda \).

Theorem 4.3 Let \( A \) be any subset of a \( \beta \)-topological vector space \( E \). Then

(a) \( Int(Cl(\{x + A\})) \subseteq x + Cl(A) \) for each \( x \in E \),
(b) \( x + Int(Cl(Int(A))) \subseteq Cl(x + A) \) for each \( x \in E \),
(c) \( x + Int(A) \subseteq Cl(Int(Cl(x + A))) \) for each \( x \in E \),
(d) \( Int(x + A) \subseteq x + Cl(Cl(Int(A))) \) for each \( x \in E \).

Proof. (a) Since \( Cl(A) \) is closed, by Theorem 3.7, \( x + Cl(A) \) is \( \beta \)-closed. Consequently, \( Int(Cl(\{x + A\})) \subseteq x + Cl(A) \).
(b) In view of Theorem 3.7, \( -x + Cl(x + A) \) is \( \beta \)-closed and hence \( Int(Cl(Int(A))) \subseteq -x + Cl(x + A) \). Thereby the assertion follows.
(c) In consequence of Theorem 3.5, \( x + Int(A) \) is \( \beta \)-open. Therefore, \( x + Int(A) \subseteq Cl(Int(Cl(x + A))) \). Hence the assertion follows.
(d) Obvious.

The analog of Theorem 4.3 is the following:
Theorem 4.4 Let $A$ be any subset of a $\beta$–topological vector space $E$. Then
(a) $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq \lambda \text{Cl}(A)$ for each non-zero scalar $\lambda$,
(b) $\lambda \text{Int}(\text{Cl}(\text{Int}(A))) \subseteq \text{Cl}(\lambda A)$ for each non-zero scalar $\lambda$,
(c) $\lambda \text{Int}(A) \subseteq \text{Cl}(\text{Int}(\lambda A))$ for each non-zero scalar $\lambda$,
(d) $\text{Int}(\lambda A) \subseteq \lambda \text{Cl}(\text{Int}(A))$ for each non-zero scalar $\lambda$.

Theorem 4.5 For any open set $U$ in a $\beta$–topological vector space $E$, $x + \text{Int}(\text{Cl}(U)) \subseteq \text{Cl}(x + U)$ for each $x \in E$.

Proof. On account of Theorem 4.3(b), $x + \text{Int}(\text{Cl}(U)) \subseteq \text{Cl}(x + U)$. Since $U$ is open, we have $x + \text{Int}(\text{Cl}(U)) \subseteq \text{Cl}(x + U)$. This completes the proof. ■

Theorem 4.6 For any closed set $F$ in a $\beta$–topological vector space $E$, $\text{Int}(x + F) \subseteq x + \text{Cl}(\text{Int}(F))$ for each $x \in E$.

Proof. In view of Theorem 4.3(d), $\text{Int}(x + F) \subseteq x + \text{Cl}(\text{Int}(F)) = x + \text{Cl}(\text{Int}(F))$ because $F$ is closed. Hence the proof is finished. ■

Definition 4.7 [1] A mapping $f : X \to Y$ from a topological space $X$ to a topological space $Y$ is called $\beta$–continuous if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists a $\beta$–open set $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.

Theorem 4.8 For a $\beta$–topological vector space $E$, the following are true:
(a) the translation mapping $f_x : E \to E$ defined by $f_x(y) = x + y$ for all $y \in E$ is $\beta$–continuous,
(b) the mapping $f_\lambda : E \to E$ defined by $f_\lambda(x) = \lambda x$ for all $x \in E$ is $\beta$–continuous, where $\lambda$ is a fixed scalar.

Proof. (a) Let $y \in E$ and $V$ be an open set in $E$ containing $f_x(y) = x + y$. Then by the definition of $\beta$–topological vector spaces, we get $U, U' \in \beta O(E)$ such that $x \in U, y \in U'$ and $U + U' \subseteq V$ and consequently, $f_x(U') \subseteq V$. This proves that $f_x$ is $\beta$–continuous.
(b) Let $x \in E$ be an arbitrary. Let $W$ be any open set in $E$ containing $\lambda x$. Then there exist $\beta$–open sets $U$ in $K$ containing $\lambda$ and $V$ in $E$ containing $x$ such that $U.V \subseteq W$. Now $\lambda V \subseteq U.V \subseteq W \Rightarrow f_\lambda(x) \subseteq W$ and hence $f_\lambda$ is $\beta$–continuous. ■

Theorem 4.9 Let $E_1$ be a $\beta$–topological vector space, $E_2$ be a topological vector space over the same field $K$. Let $f : E_1 \to E_2$ be a linear map such that $f$ is continuous at 0. Then $f$ is $\beta$–continuous everywhere.

Proof. Let $x$ be any non-zero element of $E_1$ and $V$ be an open set in $E_2$ containing $f(x)$. Since translation of an open set in topological vector spaces is open, $V - f(x)$ is open set in $E_2$ containing 0. Since $f$ is continuous at 0, there exists an open set $U$ in $E_1$ containing 0 such that $f(U) \subseteq V - f(x)$. Furthermore, linearity of $f$ implies that $f(x + U) \subseteq V$. By Theorem 3.5, $x + U$ is $\beta$–open and hence $f$ is $\beta$–continuous at $x$. By hypothesis, $f$ is $\beta$–continuous at 0. This reflects that $f$ is $\beta$–continuous. ■

Corollary 4.10 Let $E$ be a $\beta$–topological vector space over the field $K$. Let $f : E \to K$ be a linear functional which is continuous at 0. Then the set $F = \{x \in E : f(x) = 0\}$ is $\beta$–closed.

Definition 4.11 [2] A topological space $X$ is called $\beta$–compact if every cover of $X$ by $\beta$–open sets of $X$ has a finite subcover. A subset $A$ of $X$ is said to be $\beta$–compact relative to $X$ if every cover of $A$ by $\beta$–open sets of $X$ has a finite subcover.

Theorem 4.12 Let $A$ be any $\beta$–compact set in a $\beta$–topological vector space $E$. Then $x + A$ is compact for each $x \in E$. 

Proof. Let $U = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $x + A$. Then $A \subseteq \cup_{\alpha \in \Lambda_0} (-x + U_\alpha)$. By hypothesis and Theorem 3.5, $A \subseteq \cup_{\alpha \in \Lambda_0} (-x + U_\alpha)$ for some finite $\Lambda_0 \subseteq \Lambda$. Whence we find that $x + A \subseteq \cup_{\alpha \in \Lambda_0} U_\alpha$. This shows that $x + A$ is compact. Hence, the proof is complete.

Theorem 4.13 Let $A$ be any $\beta-$compact set in a $\beta-$topological vector space $E$. Then $\lambda A$ is compact for each scalar $\lambda$.

Proof. If $\lambda = 0$ we are nothing to prove. Assume that $\lambda$ is non-zero. Let $U = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $\lambda A$. Then $A \subseteq \cup_{\alpha \in \Lambda_0} (\frac{1}{\lambda} U_\alpha)$. In view of Theorem 3.1, $\frac{1}{\lambda} U_\alpha$ is $\beta-$open and consequently, by hypothesis, $A \subseteq \cup_{\alpha \in \Lambda_0} (\frac{1}{\lambda} U_\alpha)$ for some finite $\Lambda_0 \subseteq \Lambda$. Whence we find that $\lambda A \subseteq \cup_{\alpha \in \Lambda_0} U_\alpha$. This proves that $\lambda A$ is compact.

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