Some topological properties of fuzzy strong b-metric spaces

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Abstract. In this study, we investigate topological properties of fuzzy strong b-metric spaces defined in [13]. Firstly, we prove Baire’s theorem for these spaces. Then we define the product of two fuzzy strong b-metric spaces defined with same continuous t-norms and show that $X_1 \times X_2$ is a complete fuzzy strong b-metric space if and only if $X_1$ and $X_2$ are complete fuzzy strong b-metric spaces. Finally it is proven that a subspace of a separable fuzzy strong b-metric space is separable.

Keywords: Fuzzy strong b-metric space, strong b-metric space, complete, separable.

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1. Introduction and Preliminaries

The notion of strong b-metric space is obtained by modifying the “relaxed triangle inequality” in the definition of b-metric (or metric type) space [2, 3, 6, 8, 10].

Definition 1.1 [11] Let $X$ be a non-empty set, $K \geq 1$ and $D : X \times X \rightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$,

1) $D(x, y) = 0$ if and only if $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq D(x, y) + KD(y, z)$.

Then $D$ is called a strong b-metric on $X$ and $(X, D, K)$ is called a strong b-metric space.

In these spaces, the strong b-metric $D$ is continuous and an open ball is open set [11] where these are not true in general for b-metric spaces [1].
After introducing the theory of fuzzy sets by Zadeh [15], fuzzy analogy of metric spaces were applied by different authors from different points of view [4, 5, 7, 9, 12].

In [13], Öner introduced and studied the notion of fuzzy strong b-metric spaces which is the fuzzy analogy of strong b-metric spaces and a generalization of fuzzy metric space introduced by George and Veeramani [7].

**Definition 1.2** [14] A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous t-norm if \( * \) satisfies the following conditions:
1) \( * \) is associative and commutative,
2) \( * \) is continuous,
3) \( a * 1 = a \) for all \( a \in [0, 1] \),
4) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), \( a, b, c, d \in [0, 1] \).

**Definition 1.3** [13] Let \( X \) be a non-empty set, \( K \geq 1 \), \( * \) is a continuous t-norm and \( M \) be a fuzzy set on \( X \times X \times (0, \infty) \) such that for all \( x, y, z \in X \) and \( t, s > 0 \),
1) \( M(x, y, t) > 0 \),
2) \( M(x, y, t) = 1 \) if and only if \( x = y \),
3) \( M(x, y, t) = M(y, x, t) \),
4) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + Ks) \),
5) \( M(x, y, :) : (0, \infty) \rightarrow [0, 1] \) is continuous.

Then \( M \) is called a fuzzy strong b-metric on \( X \) and \((X, M, *, K)\) is called a fuzzy strong b-metric space.

For \( t > 0 \), open balls and closed balls with center \( x \) and radius \( r \in (0, 1) \) were defined in [13] as follows:

\[
B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}, \quad B[x, r, t] = \{ y \in X : M(x, y, t) \geq 1 - r \}
\]

and it was proven that every fuzzy strong b-metric spaces \((X, M, *, K)\) induces a Hausdorff and first countable topology \( \tau_M \) on \( X \) which open balls are open and closed balls are closed and the family of sets \( \{B(x, r, t) : x \in X, 0 < r < 1, t > 0\} \) form a base.

**Proposition 1.4** [13] \( M(x, y, :) : (0, \infty) \rightarrow [0, 1] \) is nondecreasing for all \( x, y \in X \).

**Definition 1.5** [13] Let \((X, M, *, K)\) be a fuzzy strong b-metric space, \( x \in X \) and \( \{x_n\} \) be a sequence in \( X \). Then
1) \( \{x_n\} \) is said to converge to \( x \) if for any \( t > 0 \) and any \( r \in (0, 1) \) there exists a natural number \( n_0 \) such that \( M(x_n, x, t) > 1 - r \) for all \( n \geq n_0 \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \rightarrow x \) as \( n \to \infty \).
2) \( \{x_n\} \) is said to be a Cauchy sequence if for any \( r \in (0, 1) \) and any \( t > 0 \) there exists a natural number \( n_0 \) such that \( M(x_n, x_m, t) > 1 - r \) for all \( n, m \geq n_0 \).
3) \((X, M, *, K)\) is said to be a complete fuzzy strong b-metric space if every Cauchy sequence is convergent.

**Theorem 1.6** [13] Let \((X, M, *, K)\) be a fuzzy strong b-metric space, \( x \in X \) and \( \{x_n\} \) be a sequence in \( X \). \( \{x_n\} \) converges to \( x \) if and only if \( M(x_n, x, t) \to 1 \) as \( n \to \infty \), for each \( t > 0 \).

In this study, we investigate the further topological properties of fuzzy strong b-metric spaces. Firstly, we prove Baire’s theorem for these spaces. Then we define the product of two fuzzy strong b-metric spaces defined with same continuous t-norms and show that \( X_1 \times X_2 \) is a complete fuzzy strong b-metric space if and only if \( X_1 \) and \( X_2 \) are complete.
fuzzy strong b-metric spaces. Finally, it is proven that a subspace of a separable fuzzy strong b-metric space is separable.

2. Main results

Theorem 2.1 (Baire’s theorem). Let \((X, M, *, K)\) be a complete fuzzy strong b-metric space. Then the intersection of a countable number of dense open sets is dense.

Proof. Let \(X\) be the given complete fuzzy strong b-metric space, \(B_0\) be a nonempty open set and \(D_1, D_2, D_3, \ldots \) be dense open sets in \(X\). Since \(D_1\) is dense in \(X, B_0 \cap D_1 \neq \emptyset\). Let \(x \in B_0 \cap D_1\). Since \(B_0 \cap D_1\) is open, there exist \(0 < r_1 < 1\) and \(t_1 > 0\) such that \(B(x, r_1, t_1) \subset B_0 \cap D_1\). Choose \(r'_1 < r_1\) and \(t'_1 = \min\{t_1, 1\}\) such that \(B(x, r'_1, t'_1) \subset B_0 \cap D_1\). Let \(B_1 = B(x, r'_1, t'_1)\). Since \(D_2\) is dense in \(X, B_1 \cap D_2 \neq \emptyset\). Let \(x \in B_1 \cap D_2\). Since \(B_1 \cap D_2\) is open, there exist \(0 < r_2 < 1/2\) and \(t_2 > 0\) such that \(B(x, r_2, t_2) \subset B_1 \cap D_2\). Choose \(r'_2 < r_2\) and \(t'_2 = \min\{t_2, 1/2\}\) such that \(B(x, r'_2, t'_2) \subset B_1 \cap D_2\). Let \(B_2 = B(x, r'_2, t'_2)\). Similarly, proceeding by induction, we can find \(x_n \in B_{n-1} \cap D_n\). Since \(B_{n-1} \cap D_n\) is open, there exist \(0 < r_n < 1/n\) and \(t_n > 0\) such that \(B(x, r_n, t_n) \subset B_{n-1} \cap D_n\). Choose \(r'_n < r_n\) and \(t'_n = \min\{t_n, 1/n\}\) such that \(B(x, r'_n, t'_n) \subset B_{n-1} \cap D_n\). Let \(B_n = B(x, r'_n, t'_n)\). Now we claim that \(\{x_n\}\) is a Cauchy sequence. For a given \(t > 0\) and \(0 < \varepsilon < 1\), choose \(n_0\) such that \(1/n_0 < t, 1/n_0 < \varepsilon\). Then for \(n, m \geq n_0\)

\[
M(x_n, x_m, t) \geq M\left(x_n, x_m, \frac{1}{n_0}\right) \geq 1 - \frac{1}{n_0} \geq 1 - \varepsilon.
\]

Therefore, \(\{x_n\}\) is Cauchy sequence. Since \(X\) is complete, there exists \(x \in X\) such that \(x_n \to x\). But \(x_k \in B(x, r'_k, t'_k)\) for all \(k \geq n\). Since \(B(x, r'_n, t'_n)\) is closed, \(x \in B(x, r'_n, t'_n) \subset B_{n-1} \cap D_n\) for all \(n\). Thus, \(B_0 \cap (\bigcap_{n=1}^{\infty} D_n) \neq \emptyset\). Hence, \(\bigcap_{n=1}^{\infty} D_n\) is dense in \(X\). ■

Proposition 2.2 Let \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) be fuzzy strong b-metric spaces. For \((x_1, x_2), (y_1, y_2) \in X_1 \times X_2\), consider

\[
M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).
\]

Then \((X_1 \times X_2, M, *, K)\) is a fuzzy strong b-metric space where \(K = \max\{K_1, K_2\}\

Proof. 1) Since \(M_1(x_1, y_1, t) > 0\) and \(M_2(x_2, y_2, t) > 0\) this implies that

\[
M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0.
\]

Therefore, \(M((x_1, x_2), (y_1, y_2), t) > 0\).

2) Suppose that \((x_1, x_2) = (y_1, y_2)\). This implies that \(x_1 = y_1\) and \(x_2 = y_2\). Hence, for all \(t > 0\), we have \(M_1(x_1, y_1, t) = 1\) and \(M_2(x_2, y_2, t) = 1\). It follows that

\[
M((x_1, x_2), (y_1, y_2), t) = 1.
\]

Conversely, suppose that \(M((x_1, x_2), (y_1, y_2), t) = 1\). This implies that

\[
M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1.
\]

Since \(0 < M_1(x_1, y_1, t) \leq 1\) and \(0 < M_2(x_2, y_2, t) \leq 1\), it follows that \(M_1(x_1, y_1, t) = 1\) and \(M_2(x_2, y_2, t) = 1\). Thus, \(x_1 = y_1\) and \(x_2 = y_2\). Therefore \((x_1, x_2) = (y_1, y_2)\).
3) To prove that \( M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t) \). We observe that

\[
\begin{align*}
M_1(x_1, y_1, t) &= M_1(y_1, x_1, t), \\
M_2(x_2, y_2, t) &= M_2(y_2, x_2, t).
\end{align*}
\]

It follows that for all \((x_1, x_2), (y_1, y_2) \in X_1 \times X_2 \) and \( t > 0 \),

\[
M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t).
\]

4) Since \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) are fuzzy strong b-metric spaces, we have

\[
\begin{align*}
M_1(x_1, z_1, t + K_1 s) &\geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s), \\
M_2(x_2, z_2, t + K_2 s) &\geq M_2(x_2, y_2, t) * M_2(y_2, z_2, s)
\end{align*}
\]

for all \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2 \) and \( s > 0 \). Since \( K = \max\{K_1, K_2\} \), we get

\[
M((x_1, x_2), (z_1, z_2), t + K s) = M_1(x_1, z_1, t + K s) * M_2(x_2, z_2, t + K s)
\]

\[
\geq M_1(x_1, z_1, t + K_1 s) * M_2(x_2, z_2, t + K_2 s)
\]

\[
\geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) * M_2(x_2, y_2, t) * M_2(y_2, z_2, s)
\]

\[
\geq M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * M_1(y_1, z_1, s) * M_2(y_2, z_2, s)
\]

\[
\geq M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s).
\]

5) Note that \( M_1(x_1, y_1, t) \) and \( M_2(x_2, y_2, t) \) are continuous with respect to \( t \) and * is continuous. It follows that

\[
M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)
\]

is also continuous.

\[\blacksquare\]

**Proposition 2.3** Let \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) be fuzzy strong b-metric spaces. Then \((X_1 \times X_2, M, *, K)\) is complete if and only if \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) are complete.

**Proof.** Suppose that \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) are complete fuzzy strong b-metric spaces. Let \(\{a_n\}\) be a Cauchy sequence in \(X_1 \times X_2\). Note that \(a_n = (x^n_1, x^n_2)\) and \(a_m = (x^m_1, x^m_2)\). Also, \(M(a_n, a_m, t)\) converges to 1. Hence, \(M((x^n_1, x^n_2), (x^m_1, x^m_2), t)\) converges to 1 for each \(t > 0\). It follows that \(M_1(x^n_1, x^n_2, t) * M_2(x^m_1, x^m_2, t)\) converges to 1 for each \(t > 0\). Thus, \(M_1(x^n_1, x^n_2, t)\) converges to 1 and also, \(M_2(x^m_1, x^m_2, t)\) converges to 1. Therefore, \(\{x^n_1\}\) is a Cauchy sequence in \((X_1, M_1, *, K_1)\) and \(\{x^n_2\}\) is a Cauchy sequence in \((X_2, M_2, *, K_2)\). Since \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) are complete fuzzy strong b-metric spaces, there exists \(x_1 \in X_1\) and \(x_2 \in X_2\) such that \(M_1(x^n_1, x_1, t)\) converges to 1 and \(M_2(x^n_2, x_2, t)\) converges to 1 for each \(t > 0\). Let \(a = (x_1, x_2)\). Then \(a \in X_1 \times X_2\). It follows that \(M(a_n, a, t)\) converges to 1 for each \(t > 0\). This shows that \((X_1 \times X_2, M, *, K)\) is complete.

Conversely, suppose that \((X_1 \times X_2, M, *, K)\) is complete. We shall show that \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\) are complete. Let \(\{x^n_1\}\) and \(\{x^n_2\}\) be Cauchy sequences in \((X_1, M_1, *, K_1)\) and \((X_2, M_2, *, K_2)\), respectively. Thus, \(M_1(x^n_1, x_1, t)\) converges to 1 and \(M_2(x^n_2, x_2, t)\) converges to 1 for each \(t > 0\). It follows that

\[
M((x^n_1, x^n_2), (x^m_1, x^m_2), t) = M_1(x^n_1, x^m_1, t) * M_2(x^n_2, x^m_2, t)
\]
converges to 1. Then \((x^n_1, x^n_2)\) is a Cauchy sequence in \(X_1 \times X_2\). Since \((X_1 \times X_2, M, *, K)\) is complete, there exists \((x_1, x_2) \in X_1 \times X_2\) such that \(M(x^n_1, x^n_2), (x_1, x_2), t)\) converges to 1. Clearly, \(M_1(x^n_1, x_1, t)\) converges to 1 and \(M_2(x^n_2, x_2, t)\) converges to 1. Hence, \((X_1, M_1, *)\) and \((X_2, M_2, *)\) are complete. This completes the proof. ■

Proposition 2.4 A subspace of a separable fuzzy strong b-metric space \((X, M, *, K)\) is separable.

Proof. Let \(X\) be the given separable fuzzy strong b-metric space and \(Y\) be a subspace of \(X\). Let \(A = \{x_n : n \in \mathbb{N}\}\) be a countable dense subset of \(X\). For arbitrary but fixed \(n, k \in \mathbb{N}\), if there are points \(x \in X\) such that \(M(x_n, x, 1/k) > 1 - 1/k\), choose one of them and denote it by \(x_{nk}\). Let \(B = \{x_{nk} : n, k \in \mathbb{N}\}\). Then \(B\) is countable. Now, we claim that \(Y \subseteq B\). Let \(y \in Y\). Given \(r\) with \(0 < r < 1\) and \(t \geq 0\) we can find a \(k \in \mathbb{N}\) such that \((1 - 1/k) * (1 - 1/k) > 1 - r\) and \(1/k < t/2K\). Since \(A\) is dense in \(X\), there exists an \(m \in \mathbb{N}\) such that \(M(x_m, y, 1/k) > 1 - 1/k\). But, by definition of \(B\), there exists \(x_{mk}\) such that \(M(x_{mk}, x_m, 1/k) > 1 - 1/k\). Now, we have

\[
M(x_{mk}, y, t) \geq M(x_{mk}, x_m, \frac{t}{2}) * M(x_m, y, \frac{t}{2K}) \\
\geq M(x_{mk}, x_m, \frac{1}{k}) * M(x_m, y, \frac{1}{k}) \\
\geq (1 - \frac{1}{k}) * (1 - \frac{1}{k}) \\
> 1 - r.
\]

Thus, \(y \in B\). Hence, \(Y\) is separable. ■

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References