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Best proximity point theorems in $\frac{1}{2}$-modular metric spaces

H. Hosseini[a,*], M. Eshaghi Gordji[b]

[a] *Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.*
[b] *Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran.*

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Abstract. In this paper, first we introduce the notion of $\frac{1}{2}$-modular metric spaces and weak (α, Θ)-ω-contractions in this spaces and we establish some results of best proximity points. Finally, as consequences of these theorems, we derive best proximity point theorems in modular metric spaces endowed with a graph and in partially ordered metric spaces. We present an example to illustrate the usability of these theorems.

Keywords: (α, Θ)-ω-contractions, best proximity point, $\frac{1}{2}$-modular metric space.

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1. Introduction and preliminaries

The background literature on best proximity theory and associated fixed point theory in (ordered) metric spaces, Banach spaces and probabilistic and fuzzy metric spaces is very abundant in the literature, see, for instance, [7, 8, 11, 15] and references therein.

Let A and B be two nonempty subsets of a metric space (X, d). A best proximity point of a mapping $T : A \to B$ is a point $x \in A$ satisfying the equality $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Best proximity theory provide sufficient conditions that assure the existence of such points. For more details on this approach, again we refer the reader to the above cited papers and references therein.

*Corresponding author.
E-mail address: hoda.hosseini@kiau.ac.ir (H. Hosseini); madjid.eshaghi@gmail.com (M. Eshaghi Gordji).

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Modular metric spaces are a natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. Modular metric spaces were introduced in [2, 3]. The introduction of this new concept is justified by the physical interpretation of the modular. Roughly, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$, it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But, in this paper, we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [18] on vector spaces and modular function spaces introduced by Musielak [17] and Orlicz [19].

In recent years, many researchers studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (for instance lithium polymetachrylate). An interesting model for these fluids, is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case that p is a function [4]. We remark that the usual approach in dealing with the Dirichlet energy problem [5, 6] is to convert the energy functional, naturally defined by a modular, to a convoluted and complicated problem which involves the Luxemburg norm.

In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. Recently, there was a strong interest to study the existence of fixed points in the setting of modular function spaces after the first paper [13] was published in 1990. For more on metric fixed point theory and modular function spaces, see [12, 14].

In this paper, first we introduce the notion of $\frac{1}{2}$-modular metric spaces and weak $(\alpha, \Theta) - \omega$-contractions in this spaces and we establish some results of best proximity points. Finally, as consequences of these theorems, we derive best proximity point theorems in modular metric spaces endowed with a graph and in partially ordered metric spaces. We present an example to illustrate the usability of these theorems.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function. For simplicity, we will write $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 [2, 3] A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leqslant \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If in the Definition 1.1, we use the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, ω is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x, y) \leqslant \tfrac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \tfrac{\mu}{\lambda+\mu}\omega_\mu(z, y).$$

If we replace (iii) by

(iv) $\omega_{\frac{\lambda+\mu}{2}}(x,y) \leqslant \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$,

then the pair (X, ω) is called $\frac{1}{2}$-modular metric space. Note that, since (iv) implies (iii), each $\frac{1}{2}$-modular metric space is a modular metric space.

Remark 1 *If ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,+\infty)$ for all $x, y \in X$. Indeed, if $0 < \mu < \lambda$, then*

$$\omega_\lambda(x,y) \leqslant \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Definition 1.2 [2, 3] Let ω be a pseudomodular on X and $x_0 \in X$ fixed. Consider the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \;\text{ such that }\; \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X_ω^* are called modular spaces (around x_0).

It is clear that $X_\omega \subset X_\omega^*$ but this inclusion may be proper in general. Let ω be a modular on X. From [2, 3], we deduce that the modular space X_ω can be equipped with a (nontrivial) metric, induced by ω and defined by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant \lambda\} \;\text{ for all }\; x, y \in X_\omega.$$

If ω is a convex modular on X, according to [2, 3] the two modular spaces coincide, that is, $X_\omega^* = X_\omega$, and this common set can be endowed with the metric d_ω^* defined by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant 1\} \;\text{ for all }\; x, y \in X_\omega.$$

These distances will be called Luxemburg distances. Example 2.4 presented by Abdou and Khamsi [1] is an important motivation for developing the theory of modular metric spaces. Other examples may be found in [2, 3].

Definition 1.3 Let X_ω be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Then

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to +\infty$. x will be called the ω-limit of (x_n).
(2) $(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to +\infty$.
(3) M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.
(4) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.
(5) M is called ω-bounded if we have $\delta_\omega(M) = \sup\{\omega_1(x,y); x, y \in M\} < +\infty$.

Definition 1.4 [22] Let T be a self-mapping on X and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geqslant 1.$$

Definition 1.5 [11] A non-self-mapping T is called α-proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \geqslant 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B), \end{cases} \Longrightarrow \alpha(u_1, u_2) \geqslant 1$$

for all $x_1, x_2, u_1, u_2 \in A$, where $\alpha : A \times A \to [0, \infty)$.

Let A and B be two subsets of modular metric (X, ω). We denote by $A_0(\lambda)$ and $B_0(\lambda)$ the following sets:

$$A_0(\lambda) = \{x \in A : \omega_\lambda(x, y) = \omega_\lambda(A, B) \text{ for some } y \in B\},$$
$$B_0(\lambda) = \{y \in B : \omega_\lambda(x, y) = \omega_\lambda(A, B) \text{ for some } x \in A\},$$

where $\omega_\lambda(A, B) = \inf\{\omega_\lambda(x, y) \ : x \in A, y \in B\}$.

Also we extend the notion of regular metric space by the following method which is suitable for best proximity point results. We say ω is proximal regular if ω is regular and, $\omega_\lambda(x, y) = \omega_\lambda(A, B)$ for some $\lambda > 0$ if and only if $\omega_\lambda(x, y) = \omega_\lambda(A, B)$ for all $\lambda > 0$.

Let A and B be two subsets of modular metric (X, ω) and $T : A \to B$ be a non-self mapping. We say, x^* is best proximity point of T, if $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$ for all $\lambda > 0$.

2. Main results

Consistent with Jleli and Samet [10], we denote by Δ_Θ the set of all functions $\Theta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

(Θ_1) Θ is increasing;
(Θ_2) for all sequences $\{\alpha_n\} \subseteq (0, +\infty)$, $\lim_{n\to+\infty} \alpha_n = 0$ iff $\lim_{n\to+\infty} \Theta(\alpha_n) = 1$;
(Θ_3) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that $\lim_{t\to 0^+} \dfrac{\Theta(t) - 1}{t^r} = \ell$.

Definition 2.1 Let (X, ω) be a modular metric space and $T : A \to B$ be a non-self-mapping. Also suppose that $\alpha : X \times X \to [\mathrm{O}, +\infty)$ is a function. We say that T is a weak $(\alpha, \Theta) - \omega$-contraction if for all $x, y, u, v \in A$ with

$$\begin{cases} \alpha(x, y) \geqslant 1, \\ \omega_\lambda(u, v) > 0 \\ \omega_\lambda(u, Tx) = \omega_\lambda(A, B) \\ \omega_\lambda(v, Ty) = \omega_\lambda(A, B) \end{cases}$$

we have

$$\Theta\big(\omega_\lambda(u, v)\big) \leqslant \Big[\max\Big\{\Theta\big(\omega_\lambda(x, y)\big), \Theta\big(\omega_\lambda(x, u)\big), \Theta\big(\omega_\lambda(y, v)\big)\Big\}\Big]^k$$

for all $\lambda > 0$ where $0 \leqslant k < 1$ and $\Theta \in \Delta_\Theta$.

Now we state and prove our main result of this section.

Theorem 2.2 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that there exist two functions $\alpha : A \times A \to [0, +\infty)$ and $\Theta \in \Delta_\Theta$ such that the following assertions hold:

(i) T is an α- proximal admissible mapping,
(ii) T is a weak (α, Θ)-ω-contraction and continuous mapping,
(iii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that,

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A, B) \text{ and } \alpha(x_0, x_1) \geqslant 1 \text{ for all } \lambda > 0.$$

Then T has a best proximity point.

Proof. By (iii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that

$$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A, B) \text{ and } \alpha(x_0, x_1) \geqslant 1 \text{ for all } \lambda > 0.$$

On the other hand, $T(A_0(\lambda)) \subseteq B_0(\lambda)$. So there exists $x_2 \in A_0(\lambda)$ such that $\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A, B)$. Since T is α-proximal admissible mapping, we have $\alpha(x_1, x_2) \geqslant 1$; that is, $\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A, B)$ and $\alpha(x_1, x_2) \geqslant 1$. Since $T(A_0(\lambda)) \subseteq B_0(\lambda)$, there exists $x_3 \in A_0(\lambda)$ such that $\omega_\lambda(x_3, Tx_2) = \omega_\lambda(A, B)$. Thus, we have

$$\omega_\lambda(x_2, Tx_1) = \omega_\lambda(A, B), \omega_\lambda(x_3, Tx_2) = \omega_\lambda(A, B), \alpha(x_1, x_2) \geqslant 1.$$

Again since T is α-proximal admissible mapping, $\alpha(x_2, x_3) \geqslant 1$. Hence, $\omega_\lambda(x_3, Tx_2) = \omega_\lambda(A, B)$ and $\alpha(x_2, x_3) \geqslant 1$. Continuing this process, we get $\omega_\lambda(x_{n+1}, Tx_n) = \omega_\lambda(A, B)$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then $\omega_\lambda(x_{n_0}, Tx_{n_0}) = \omega_\lambda(x_{n_0+1}, Tx_{n_0}) = \omega_\lambda(A, B)$; that is, x_{n_0} is best proximity point of T and we have no things for prove. Hence, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now, regularity of ω implies $\omega_\lambda(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. Since T is a weak (α, Θ)-ω-contraction, we derive

$$\begin{aligned}\Theta(\omega_\lambda(x_n, x_{n+1})) &\leqslant [\max\{\Theta(\omega_\lambda(x_{n-1}, x_n)), \Theta(\omega_\lambda(x_{n-1}, x_n)), \Theta(\omega_\lambda(x_n, x_{n+1}))\}]^k \\ &= [\max\{\Theta(\omega_\lambda(x_{n-1}, x_n)), \Theta(\omega_\lambda(x_n, x_{n+1}))\}]^k.\end{aligned}$$

Now, assume that $\max\{\Theta(\omega_\lambda(x_{n-1}, x_n)), \Theta(\omega_\lambda(x_n, x_{n+1}))\} = \Theta(\omega_\lambda(x_n, x_{n+1}))$. Then

$$\Theta(\omega_\lambda(x_n, x_{n+1})) \leqslant [\Theta(\omega_\lambda(x_n, x_{n+1}))]^k < \Theta(\omega_\lambda(x_n, x_{n+1})),$$

which is a contradiction. Hence, $\Theta(\omega_\lambda(x_n, x_{n+1})) \leqslant [\Theta(\omega_\lambda(x_{n-1}, x_n))]^k$ for all $n \in \mathbb{N}$ and $\lambda > 0$. Therefore,

$$\begin{aligned}1 < \Theta(\omega_\lambda(x_n, x_{n+1})) &\leqslant \Theta(\omega_\lambda(x_{n-1}, x_n))^k \\ &\leqslant \Theta(\omega_\lambda(x_{n-2}, x_{n-1}))^{k^2} \leqslant \cdots \leqslant \Theta(\omega_\lambda(x_0, x_1))^{k^n}\end{aligned} \tag{1}$$

Taking the limit as $n \to +\infty$ in (1), we get $\lim\limits_{n\to+\infty} \Theta(\omega_\lambda(x_n, x_{n+1})) = 1$ for all $\lambda > 0$ and since $\Theta \in \Delta_\Theta$, we obtain $\lim_{n\to+\infty} \omega_\lambda(x_n, x_{n+1}) = 0$ for all $\lambda > 0$. Thus, there exist $0 < r < 1$ and $0 < \ell \leqslant +\infty$ such that $\lim\limits_{n\to+\infty} \frac{\Theta(\omega_\lambda(x_n, x_{n+1}))-1}{[\omega_\lambda(x_n, x_{n+1})]^r} = \ell$. Now, let $C^{-1} \in (0, \ell)$. From the definition of limit, there exists $n_\lambda \in \mathbb{N}$ such that $\dfrac{\Theta(\omega_\lambda(x_n, x_{n+1})) - 1}{[\omega_\lambda(x_n, x_{n+1})]^r} \geqslant C^{-1}$ for all $n \geqslant n_\lambda$ and so, $n[\omega_\lambda(x_n, x_{n+1})]^r \leqslant nC[\Theta(\omega_\lambda(x_n, x_{n+1})) - 1]$. From (1), we deduce $n[\omega_\lambda(x_n, x_{n+1})]^r \leqslant nC[\Theta(\omega_\lambda(x_0, x_1))^{k^n} - 1]$ for all $n \geqslant n_\lambda$. Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} n[\omega_\lambda(x_n, x_{n+1})]^r = 0 \text{ for all } \lambda > 0. \tag{2}$$

From (2), there exists $N_\lambda \in \mathbb{N}$ such that $n[\omega_\lambda(x_n, x_{n+1})]^r \leqslant 1$ for all $\lambda > 0$ and $n \geqslant N_\lambda$. Thus,

$$\omega_\lambda(x_n, x_{n+1}) \leqslant \frac{1}{n^{1/r}} \quad \text{for all} \quad n \geqslant N_\lambda, \lambda > 0. \tag{3}$$

Now, for $\lambda = \dfrac{1}{m-n}$ and $m > n \geqslant N_\lambda$, and by (3), we get

$$\omega_1(x_n, x_m) = \omega_{(m-n)\frac{1}{m-n}}(x_n, x_m) \leqslant \sum_{i=n}^{m-1} \omega_{\frac{1}{m-n}}(x_i, x_{i+1}) \leqslant \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1$, then $\lim_{n\to+\infty} \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} = 0$, and hence $\omega_1(x_n, x_m) \to 0$ as $m, n \to +\infty$. Thus, we have proved that $\{x_n\}$ is a ω-Cauchy sequence. The hypothesis of ω-completeness of X ensures that there exists $x^* \in X$ such that $\omega_1(x_{n+1}, x^*) \to 0$ as $n \to +\infty$. Now, since T is a ω-continuous mapping, then $\omega_1(Tx_n, Tx^*) \to 0$ as $n \to +\infty$. Thus,

$$\begin{aligned}\omega_1(A, B) \leqslant \omega_1(x^*, Tx^*) = \omega_{\frac{1+1}{2}}(x^*, Tx^*) &\leqslant \omega_1(x^*, Tx_n) + \omega_1(Tx_n, Tx^*)\\ &= \omega_{\frac{1+1}{2}}(x^*, Tx_n) + \omega_1(Tx_n, Tx^*)\\ &\leqslant \omega_1(x^*, x_{n+1}) + \omega_1(x_{n+1}, Tx_n) + \omega_1(Tx_n, Tx^*)\\ &= \omega_1(x^*, x_{n+1}) + \omega_1(A, B) + \omega_1(Tx_n, Tx^*).\end{aligned}$$

Taking limit as $n \to +\infty$, we get $\omega_1(x^*, Tx^*) = \omega_1(A, B)$ and hence $\omega_\lambda(x^*, Tx^*) = \omega_\lambda(A, B)$ (because ω is proximal regular). Thus, T has a best proximity point. ■

Theorem 2.3 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that there exist two functions $\alpha : A \times A \to [0, +\infty)$ and $\Theta \in \Delta_\Theta$ such that the following assertions hold:

(i) T is an α-proximal admissible mapping,
(ii) T is a weak (α, Θ)-ω-contraction where Θ continuous,
(iii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A, B)$ and $\alpha(x_0, x_1) \geqslant 1$ for all $\lambda > 0$,
(iv) if $\{x_n\}$ is a sequence in X for all $n \in \mathbb{N} \cup \{0\}$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$ with $\omega_1(x_n, x)$ as $n \to +\infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$,
(v) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x, y_n) \to \omega_1(A, B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$.

Then T has a best proximity point.

Proof. By (iii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A, B)$ and $\alpha(x_0, x_1) \geqslant 1$ for all $\lambda > 0$. As in the proof of Theorem 2.2, we can deduce that a sequence $\{x_n\}$ starting at x_0 is ω-Cauchy and so converges to a point $x^* \in X$, where

$$\omega_\lambda(x_{n+1}, Tx_n) = \omega_\lambda(A, B), \alpha(x_n, x_{n+1}) \geqslant 1 \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } \lambda > 0. \quad (4)$$

Also, from (iv), we have $\alpha(x_n, x^*) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus,

$$\begin{aligned}\omega_1(A, B) = \omega_1(x_{n+1}, Tx_n) &= \omega_{\frac{1+1}{2}}(x_{n+1}, Tx_n)\\ &\leqslant \omega_1(x_{n+1}, x^*) + \omega_1(x^*, Tx_n)\\ &= \omega_1(x_{n+1}, x^*) + \omega_{\frac{1+1}{2}}(x^*, Tx_n)\\ &\leqslant \omega_1(x_{n+1}, x^*) + \omega_1(x^*, x_{n+1}) + \omega_1(x_{n+1}, Tx_n)\\ &= \omega_1(x_{n+1}, x^*) + \omega_1(x^*, x_{n+1}) + \omega_1(A, B);\end{aligned}$$

that is,

$$\omega_1(A,B) \leqslant \omega_1(x_{n+1},x^*) + \omega_1(x^*,Tx_n) \leqslant \omega_1(x_{n+1},x^*) + \omega_1(x^*,x_{n+1}) + \omega_1(A,B).$$

By letting limit as $n \to \infty$ in the above inequality, we derive $\lim_{n\to\infty} \omega_1(x^*,Tx_n) = \omega_1(A,B)$ and then, by condition (iv), $x^* \in A_0(1)$. Since $T(A_0(1)) \subset B_0(1)$, then there exists $z \in A_0(1)$ such that $\omega_1(z,Tx^*) = \omega_1(A,B)$.

First assume that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $\omega_1(x_{k_n+1},z) = 0$ and $k_n > k_{n-1}$ where $k_0 = 1$,

$$\omega_1(x^*,z) = \omega_{\frac{1+1}{2}}(x^*,z) \leqslant \omega_1(x^*,x_{k_n+1}) + \omega_1(x_{k_n+1},z),$$

and so we get $\omega_1(x^*,z) = 0$. Now, regularity of ω ensure that $x^* = z$; that is, $x^* = z$ is best proximity point of T. Next we assume $\omega_1(x_{n+1},z) > 0$. Since T is weak (α,Θ)-ω-contraction, then we can write

$$\Theta(\omega_1(z,x_{n+1})) \leqslant \left[\max\left\{\Theta(\omega_1(x^*,x_n)),\Theta(\omega_1(x^*,z)),\Theta(\omega_1(x_n,x_{n+1}))\right\}\right]^k$$

If $\max\left\{\Theta(\omega_1(x^*,x_n)),\Theta(\omega_1(x^*,z)),\Theta(\omega_1(x_n,x_{n+1}))\right\} = \Theta(\omega_1(x^*,x_n))$, then

$$\Theta(\omega_1(z,x_{n+1})) \leqslant \left[\Theta(\omega_1(x^*,x_n))\right]^k \leqslant \Theta(\omega_1(x^*,x_n)),$$

which implies $\omega_1(z,x_{n+1}) \leqslant \omega_1(x^*,x_n)$; that is, $\lim_{n\to\infty} \omega_1(z,x_{n+1}) = 0$.

If $\max\left\{\Theta(\omega_1(x^*,x_n)),\Theta(\omega_1(x^*,z)),\Theta(\omega_1(x_n,x_{n+1}))\right\} = \Theta(\omega_1(x_n,x_{n+1}))$, then similarly we can deduce $\lim_{n\to\infty} \omega_1(z,x_{n+1}) = 0$. So

$$\omega_1(z,x^*) = \omega_{\frac{1+1}{2}}(z,x^*) \leqslant \omega_1(z,x_{n+1}) + \omega_1(z,x_{n+1}).$$

By taking limit as $n \to \infty$ in the above inequality we derive $\omega_1(z,x^*) = 0$. Then $z = x^*$. Now if $\max\left\{\Theta(\omega_1(x^*,x_n)),\Theta(\omega_1(x^*,z)),\Theta(\omega_1(x_n,x_{n+1}))\right\} = \Theta(\omega_1(x^*,z))$, Then

$$\Theta(\omega_1(z,x_{n+1})) \leqslant [\Theta(\omega_1(x^*,z))]^k. \tag{5}$$

Also,

$$\omega_1(z,x_{n+1}) = \omega_{\frac{1+1}{2}}(z,x_{n+1}) \leqslant \omega_1(z,x^*) + \omega_1(x^*,x_{n+1}),$$
$$\omega_1(z,x^*) = \omega_{\frac{1+1}{2}}(z,x^*) \leqslant \omega_1(z,x_{n+1}) + \omega_1(x_{n+1},x^*).$$

Letting limit as $n \to \infty$ in the above inequality we deduce $\lim_{n\to\infty} \omega_1(z,x_{n+1}) = \omega_1(z,x^*)$. and hence, from (5) and continuity of Θ, we get

$$\Theta(\omega_1(x^*,z)) \leqslant [\Theta(\omega_1(x^*,z))]^k < \Theta(\omega_1(x^*,z)),$$

which is a contradiction. Hence, $z = x^*$ and x^* is best proximity point of T. ■

***Example* 2.4** Let $X = \mathbb{R}^2$ endowed with the $\frac{1}{2}$-modular metric $\omega_\lambda : X \times X \times [0,+\infty)$ given by $\omega_\lambda(x,y) = \left(\frac{\lambda}{\lambda+1}\right) d(x,y)$, where $d : X \times X \to [0,+\infty)$ is the metric

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. Thus (X, ω) is a complete $\frac{1}{2}$-modular metric space. Define the sets $A = \Big\{(0, x) \in \mathbb{R}^2; x \in \mathbb{R}\Big\}$ and $B = \Big\{(1, x) \in \mathbb{R}^2; x \in \mathbb{R}\Big\}$, so that $d(A, B) = 1$ and $\omega_\lambda(A, B) = \dfrac{\lambda}{\lambda + 1}$ for all $\lambda > 0$. Clearly A and B are nonempty ω-closed subsets of X. Define $T : A \to B$ by

$$T(x_1, x_2) = \begin{cases} (1, 2\pi) \text{ if } (x_1, x_2) \in A \backslash V \\ (1, \frac{1}{2n}) \text{ if } (x_1, x_2) = (0, \frac{1}{n}), \ \ \forall n \in \mathbb{N}, \\ (1, 0) \quad \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

where $V = \big\{(0, \frac{1}{n}) : n \in \mathbb{N}\big\} \cup \{(0, 0)\}$. Notice that $A_0(\lambda) = A$, $B_0(\lambda) = B$. Also define $\alpha : A \times A \to [0, \infty)$ by

$$\alpha((0, x), (0, y), t) = \begin{cases} 2, \text{ if } \ (0, x), (0, y) \in V \\ \dfrac{1}{4}, \text{ otherwise.} \end{cases}$$

Let $\begin{cases} \alpha(x, y) \geqslant 1 \\ \omega_\lambda(u, Tx) = \omega_\lambda(A, B) \\ \omega_\lambda(v, Ty) = \omega_\lambda(A, B) \end{cases}$, So $\begin{cases} (x, y) \in V \\ \omega_\lambda(u, Tx) = \omega_\lambda(A, B) \\ \omega_\lambda(v, Ty) = \omega_\lambda(A, B) \end{cases}$. Then

$$(u, x), (v, y) \in \left\{((0, 0), (0, 0)), \left(\left(0, \frac{1}{2n}\right), \left(0, \frac{1}{n}\right)\right) : n \in \mathbb{N}\right\}.$$

So, $\alpha(u, v) \geqslant 1$ or T is a α-proximal admissible mapping. Next we distinguish the following cases:

(i) if $(u, x) = ((0, \frac{1}{2n}), (0, \frac{1}{n}))$ and $(v, y) = ((0, \frac{1}{2m}), (0, \frac{1}{m}))$ for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
[\omega_\lambda(u, v)]\sqrt{\omega_\lambda(u, v)} &= \left[\left(\frac{\lambda}{t+1}\right) d(u, v)\right] \sqrt{\left(\frac{\lambda}{t+1}\right) d(u, v)} \\
&= \left[\left(\frac{\lambda}{\lambda+1}\right) \left|\frac{1}{2n} - \frac{1}{2m}\right|\right] \sqrt{\left(\frac{\lambda}{\lambda+1}\right) \left|\frac{1}{2n} - \frac{1}{2m}\right|} \\
&= \frac{1}{2\sqrt{2}} \left[\left(\frac{\lambda}{\lambda+1}\right) \left|\frac{1}{n} - \frac{1}{m}\right|\right] \sqrt{\left(\frac{\lambda}{\lambda+1}\right) \left|\frac{1}{n} - \frac{1}{m}\right|} \\
&= \left[\frac{1}{2\sqrt{2}} \omega_\lambda(x, y)\right] \sqrt{\omega_\lambda(x, y)} \\
&\leqslant \left[\frac{1}{2} \omega_\lambda(x, y)\right] \sqrt{\omega_\lambda(x, y)}.
\end{aligned}$$

(ii) if $(u,x) = ((0,0),(0,0))$ and $(v,y) = ((0,\frac{1}{2m}),(0,\frac{1}{m}))$ for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\omega_\lambda(u,v)\sqrt{\omega_\lambda(u,v)} &= \left[\left(\frac{\lambda}{\lambda+1}\right)d(u,v)\right]\sqrt{\left(\frac{\lambda}{\lambda+1}\right)d(u,v)} \\
&= \left(\frac{\lambda}{\lambda+1}\right)[\frac{1}{2m}]\sqrt{\left(\frac{t}{S+1}\right)[\frac{1}{2m}]} \\
&= \frac{1}{2\sqrt{2}}\left(\frac{\lambda}{\lambda+1}\right)[\frac{1}{m}]\sqrt{\left(\frac{t}{t+1}[\frac{1}{m}]\right)} \\
&= [\frac{1}{2\sqrt{2}}\omega_\lambda(x,y)]\sqrt{\omega_\lambda(x,y)} \\
&\leqslant [\frac{1}{2}\omega_\lambda(x,y)]\sqrt{\omega_\lambda(x,y)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Theta(\omega_\lambda(u,v)) &= e^{\omega_\lambda(u,v)\sqrt{\omega_\lambda(u,v)}} \\
&\leqslant e^{[\frac{1}{2}\omega_\lambda(x,y)]\sqrt{\omega_\lambda(x,y)}} \\
&= [e^{[\omega_\lambda(x,y)]\sqrt{\omega_\lambda(x,y)}}]^{\frac{1}{2}} \\
&= [\Theta(\omega_\lambda(x,y))]^{\frac{1}{2}} \\
&\leqslant [\Theta(\max\{\omega_\lambda(x,y)),\omega_\lambda(x,Tx)),\omega_\lambda(y,Ty)\}]^{\frac{1}{2}}.
\end{aligned}$$

So we omit details. We conclude that all the hypotheses of Theorem 2.2 are satisfied, with $\Theta : (0,\infty) \to [1,+\infty)$ given by $\Theta(t) = e^{t\sqrt{t}}$, for all $t \in (0,\infty)$ and so there exists a $x^* \in A$ such that $\omega_\lambda(x^*,Tx^*) = \omega_\lambda(A,B)$, for all $\lambda > 0$. Here $x^* = (0,0)$ is best proximity point of T.

By taking $\alpha(x,y) = 1$ for all $x,y \in X$ in Theorem 2.3 we deduce the following corollary.

Corollary 2.5 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that there exist continuous function $\Theta \in \Delta_\Theta$ such that the following assertions hold:

(i) if, for all $x,y,u,v \in A$ with $\begin{cases} \omega_\lambda(u,v) > 0 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B), \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases}$ then we have

$$\Theta(\omega_\lambda(u,v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x,y)),\Theta(\omega_\lambda(x,u)),\Theta(\omega_\lambda(y,v))\right\}\right]^k$$

for all $\lambda > 0$ where $0 \leqslant k < 1$,

(ii) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x,y_n) \to \omega_1(A,B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$.

Then T has a best proximity point.

From the above corollary we can deduce the following results.

Corollary 2.6 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that there exist continuous function $\Theta \in \Delta_\Theta$ such that the following assertions hold:

(i) if for all $x, y, u, v \in A$ with $\begin{cases} \omega_\lambda(u,v) > 0 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases}$ we have

$$\Theta(\omega_\lambda(u,v)) \leqslant [\omega_\lambda(x,y))]^k$$

for all $\lambda > 0$ where $0 \leqslant k < 1$,

(ii) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x, y_n) \to \omega_1(A, B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$.

Then T has a best proximity point.

Corollary 2.7 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that there exist continuous function $\Theta \in \Delta_\Theta$ such that the following assertions hold:

(i) if for all $x, y, u, v \in A$ with $\begin{cases} \omega_\lambda(u,v) > 0 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases}$ we have

$$\Theta(\omega_\lambda(u,v)) \leqslant \left[\frac{\Theta(\omega_\lambda(x,u) + \Theta(\omega_\lambda(y,v))}{2}\right]^k$$

for all $\lambda > 0$ where $0 \leqslant k < 1$,

(ii) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x, y_n) \to \omega_1(A, B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$.

Then T has a best proximity point.

3. Some Best Proximity point Results in Modular Metric spaces endowed with a graph

As in [9], let (X_ω, ω) be a modular metric space and Δ denotes the diagonal of the cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover we may treat G as a weighted graph (see [8], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, N$.

Definition 3.1 [9] Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G; that is, for all $x, y \in X$ $(x, y) \in E(G)$ implies that

$(Tx, Ty) \in E(G)$ and T decreases the weights of the edges of G in the following way:

$$\exists \alpha \in (0,1) \text{ such that for all } x, y \in X, (x,y) \in E(G) \Longrightarrow d(Tx,Ty) \leqslant \alpha d(x,y).$$

Definition 3.2 [9] A mapping T: $X \to X$ is called G-continuous if for given $x \in X$ and sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ imply $Tx_n \to Tx$.

Definition 3.3 Let A, B be two nonempty closed subsets of a metric space (X, d) endowed with a graph G. We say that a nonself-mapping $T : A \to B$ is a weak (G,Θ)-ω-contraction if, for all $u, v, x, y \in A$,

$$\begin{cases} (x,y) \in E(G) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Longrightarrow \Theta(\omega_\lambda(u,v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x,y)), \Theta(\omega_\lambda(x,u)), \Theta(\omega_\lambda(y,v))\right\}\right]^k$$

for all $\lambda > 0$ where $0 \leqslant k < 1$ and $\Theta \in \Delta_\Theta$, and $\begin{cases} (x,y) \in E(G) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Longrightarrow (u,v) \in E(G).$

Theorem 3.4 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular and endowed with a graph G. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that the following assertions hold:

(i) T is a weak (G,Θ)-ω-contraction and continuous,
(ii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that,
$\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B)$ and $(x_0, x_1) \in E(G)$ for all $\lambda > 0$,

then T has a best proximity point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$ Firstly, we prove that T is a triangular α-proximal admissible mapping. To this aim, assume that $\begin{cases} \alpha(x,y) \geqslant 1 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B). \end{cases}$ Therefore, we have $\begin{cases} (x,y) \in E(G) \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B). \end{cases}$ Since T is a weak (G,Θ)-ω-contraction, we get $(u,v) \in E(G)$, that is $\alpha(u,v) \geqslant 1$ and

$$\Theta(\omega_\lambda(u,v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x,y)), \Theta(\omega_\lambda(x,u)), \Theta(\omega_\lambda(y,v))\right\}\right]^k,$$

where $T(A_0) \subseteq B_0$; that is, T is a continuous (G,Θ)-ω-contraction. From (ii) there exist $x_0, x_1 \in A_0$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B)$ and $(x_0, x_1) \in E(G)$, that is, $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B)$ and $\alpha(x_0, x_1) \geqslant 1$. Hence all the conditions of Theorem 2.2 are satisfied and T has a best proximity point. ∎

Similarly, by applying Theorem 2.3, we can deduce the following theorem.

Theorem 3.5 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space X_ω with ω proximal regular and endowed with a graph G. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that the following assertions hold:

(i) T is a weak (G,Θ)-ω-contraction where Θ is continuous,
(ii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B)$ and $(x_0, x_1) \in E(G)$ for all $\lambda > 0$,

(iii) if $\{x_n\}$ is a sequence in X_ω for all $n \in \mathbb{N} \cup \{0\}$ such that $(x_n, x_{n+1}) \in E(G)$ with $\omega_1(x_n, x)$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$,

(iv) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x, y_n) \to \omega_1(A, B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$.

Then T has a best proximity point.

4. Best Proximity Results in Partially Ordered Modular metric spaces

Recently Ran and Reurings [20] initiated the study of weaker contraction conditions by considering self-mappings in the setting of partially ordered metric space. Also such results were generalized by many authors. Here we consider some recent results of Mongkolkeha et al. [16] and Sadiq Basha et al. [21].

Definition 4.1 [21] Let $(X, d, \preceq)$ be a partially ordered metric space. We say that a non-self-mapping $T : A \to B$ is a proximally ordered-preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Longrightarrow u_1 \preceq u_2.$$

We extend the above definition to modular metric spaces by the following way.

Definition 4.2 Let X_ω be a partially ordered modular metric space. We say that a nonself- mapping $T : A \to B$ is a proximally ordered-preserving if and only if for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2 \\ \omega_\lambda(u_1, Tx_1) = \omega_\lambda(A, B) \\ \omega_\lambda(u_2, Tx_2) = \omega_\lambda(A, B) \end{cases} \Longrightarrow u_1 \preceq u_2.$$

Theorem 4.3 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space with ω proximal regular (X, ω) and endowed with a partial *order* $\preceq$. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that the following assertions hold:

(i) T is a continuous,

(ii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A, B)$ and $x_0 \preceq x_1$ for all $\lambda > 0$,

(iii) for all $\lambda > 0$ and some $0 \leqslant k < 1$ and $\Theta \in \Delta_\Theta$,

$$\begin{cases} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Longrightarrow \Theta(\omega_\lambda(u, v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x, y)), \Theta(\omega_\lambda(x, u)), \Theta(\omega_\lambda(y, v))\right\}\right]^k.$$

Then T has a best proximity point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$ Firstly, we prove that T is an α-proximal admissible mapping. To this aim, let

$\begin{cases} \alpha(x,y) \geqslant 1 \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases}$. Hence, we have $\begin{cases} x \preceq y \\ \omega_\lambda(u,Tx) = \omega_\lambda(A,B) \\ \omega_\lambda(v,Ty) = \omega_\lambda(A,B) \end{cases}$. Since T is a weak (G,Θ)-ω-contraction, we get $(u,v) \in E(G)$, that is, $\alpha(u,v) \geqslant 1$ and

$$\Theta(\omega_\lambda(u,v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x,y)), \Theta(\omega_\lambda(x,u)), \Theta(\omega_\lambda(y,v))\right\}\right]^k,$$

where $T(A_0) \subseteq B_0$. That is, T is a continuous (G,Θ)-ω-contraction. From (ii), there exist $x_0, x_1 \in A_0$ such that $\omega_\lambda(x_1, Tx_0) = \omega_\lambda(A,B)$ and $x_0 \preceq x_1$; that is, $\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B)$ and $\alpha(x_0,x_1) \geqslant 1$. Hence, all the conditions of Theorem 2.2 are satisfied and T has a best proximity point. ■

Similarly by applying Theorem 2.3, we can deduce the following theorem.

Theorem 4.4 Let A and B are two ω-closed subsets of complete $\frac{1}{2}$-modular metric space with ω proximal regular (X,ω) and endowed with a partial *order* $\preceq$. Let $T : A \to B$ be a non-self-mapping such that $T(A_0(\lambda)) \subseteq B_0(\lambda)$ and $A_0(\lambda) \neq \emptyset$. Assume that the following assertions hold:

(i) T is a weak (G,Θ)-ω-contraction where Θ is continuous,
(ii) there exist elements x_0 and x_1 in $A_0(\lambda)$ such that,
$\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B)$ and $x_0 \preceq x_1$ for all $\lambda > 0$,
(iii) if $\{x_n\}$ is a sequence in X_ω for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \preceq x_{n+1}$ with $\omega_1(x_n,x)$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,
(iv) for any sequence $\{y_n\}$ in $B_0(\lambda)$ and $x \in A$ satisfying $\omega_1(x,y_n) \to \omega_1(A,B)$ as $n \to +\infty$, then $x \in A_0(\lambda)$,
(v) for all $\lambda > 0$ and some $0 \leqslant k < 1$ and $\Theta \in \Delta_\Theta$,

$$\begin{cases} x \preceq y \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Longrightarrow \Theta(\omega_\lambda(u,v)) \leqslant \left[\max\left\{\Theta(\omega_\lambda(x,y)), \Theta(\omega_\lambda(x,u)), \Theta(\omega_\lambda(y,v))\right\}\right]^k.$$

Then T has a best proximity point.

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