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2n-Weak module amenability of semigroup algebras

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Abstract. Let S be an inverse semigroup with the set of idempotents E. We prove that the semigroup algebra $\ell^1(S)$ is always $2n$-weakly module amenable as an $\ell^1(E)$-module, for any $n \in \mathbb{N}$, where E acts on S trivially from the left and by multiplication from the right. Our proof is based on a common fixed point property for semigroups.

Keywords: 2n-weak module amenability, inverse semigroup, semigroup algebra, Banach module, module derivation.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear map $D : \mathcal{A} \to \mathcal{X}$ is called a derivation if $D(ab) = aD(b) + D(a)b$ for all $a, b \in \mathcal{A}$. Each map of the form $a \to ax - xa$, where $x \in \mathcal{X}$, is a continuous derivation which will be called an inner derivation.

For any Banach $\mathcal{A}$-module $\mathcal{X}$, its dual space $\mathcal{X}^*$ is naturally equipped with a Banach $\mathcal{A}$-module structure via

$$\langle x, af\rangle = \langle xa, f\rangle \quad , \quad \langle x, fa\rangle = \langle ax, f\rangle \ \ (a \in \mathcal{A}, f \in \mathcal{X}^*, x \in \mathcal{X}).$$

Note that the Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule under the algebra multiplication. So $\mathcal{A}^{(n)}$, the n-th dual space of $\mathcal{A}$, is naturally a Banach $\mathcal{A}$-bimodule in the

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above sense for each $n \in \mathbb{N}$. The Banach algebra $\mathcal{A}$ is called n-weakly amenable if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. If $\mathcal{A}$ is n-weakly amenable for each $n \in \mathbb{N}$ then it is called permanently weakly amenable.

The concept of n-weakly amenability was introduced by Dales, Ghahramani and Grønbæk in [8]. Johnson showed in [13] that for any locally compact group G, the group algebra $L^1(G)$ is always 1-weakly amenable. It was shown further in [8] that $L^1(G)$ is in fact n-weakly amenable for all odd numbers n. Whether this is still true for even numbers n was left open in [8]. Later in [12] Johnson proved that $\ell^1(G)$ is $2n$-weakly amenable for each $n \in \mathbb{N}$ whenever G is a free group. The problem has been resolved affirmatively for general locally compact group G in [7] and in [14] independently, using a theory established in [15]. In [21], as an application of a common fixed point property for semigroups, a short proof to $2m$-weak amenability of $L^1(G)$ was presented. Mewomo in [16] investigate the n-weak amenability of semigroup algebras and showed that for a Rees matrix semigroup S, $\ell^1(S)$ is n-weakly amenable when n is odd. Also he obtained a similar result for a regular semigroup S with finitely many idempotents.

Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule with compatible actions; that is,

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha) \ \ (a, b \in \mathcal{A}, \alpha \in \mathcal{U}).$$

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathcal{U}$-bimodule with compatible actions; that is,

$$\alpha.(ax) = (\alpha.a)x, \quad a(\alpha.x) = (a.\alpha)x, \quad (\alpha.x)a = \alpha.(xa) \ \ (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in \mathcal{X}),$$

and similarly for the right or two-sided actions. Then $\mathcal{X}$ is called a Banach $\mathcal{A}$-$\mathcal{U}$-module, and is called a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module whenever $\alpha.x = x.\alpha$ for all $\alpha \in \mathcal{U}$ and $x \in \mathcal{X}$.

Let $\mathcal{A}$ and $\mathcal{U}$ be as above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathcal{U}$-module. A bounded map $D : \mathcal{A} \to \mathcal{X}$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = aD(b) + D(a)b \ \ (a, b \in \mathcal{A}),$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \ \ (a \in \mathcal{A}, \alpha \in \mathcal{U}).$$

Note that D is not necessarily linear and if there exists a constant $M > 0$ such that $\| D(a) \| \leqslant M \| a \|$, for each $a \in \mathcal{A}$, then D is bounded and its boundedness implies its norm continuity. When $\mathcal{X}$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module, each $x \in \mathcal{X}$ defines an $\mathcal{U}$-module derivation $D_x(a) = ax - xa \, (a \in \mathcal{A})$, these are called inner module derivations.

If $\mathcal{X}$ is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module, then so is $\mathcal{X}^*$, where the actions of $\mathcal{A}$ and $\mathcal{U}$ on $\mathcal{X}^*$ are naturally defined as above. So by letting $\mathcal{X}^{(0)} = \mathcal{X}$, if we define $\mathcal{X}^{(n)}$ $(n \in \mathbb{N})$ inductively by $\mathcal{X}^{(n)} = (\mathcal{X}^{(n-1)})^*$, then $\mathcal{X}^{(n)}$ is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathcal{U}$-module, as we have not assumed the compatibility condition $a(\alpha.b) = (a.\alpha)b \ \ (a, b \in \mathcal{A}, \alpha \in \mathcal{U})$. If we consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a.\alpha)b - a(\alpha.b)$ for $a, b \in \mathcal{A}, \alpha \in \mathcal{U}$, then J is an $\mathcal{U}$-submodule of $\mathcal{A}$. So the quotient Banach algebra $\mathcal{A}/J$ is a Banach $\mathcal{U}$-module with compatible actions and hence from definition of J, when $\mathcal{A}/J$ acts on itself by algebra multiplication, it is a Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. Therefore, $(\mathcal{A}/J)^{(n)}$ $(n \in \mathbb{N})$ is a Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. In general $\mathcal{A}/J$ is not a commutative $\mathcal{U}$-module. If $\mathcal{A}/J$ is a commutative $\mathcal{U}$-module, then $(\mathcal{A}/J)^{(n)}$ $(n \geqslant 0)$ is a commutative Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. Now it is clear when $\mathcal{A}$ is a commutative $\mathcal{U}$-module, then $J = \{0\}$ and hence by multiplication of $\mathcal{A}$ from both sides, $\mathcal{A}^{(n)}$ $(n \geqslant 0)$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module.

Let the Banach algebra $\mathcal{A}$ be a Banach $\mathcal{U}$-module with compatible actions. From the above observations, $(\mathcal{A}/J)^{(n)}$ $(n \geqslant 0)$ is a Banach $\mathcal{A}$-$\mathcal{U}$-module by the $\mathcal{A}$-module actions $a\Phi = (a+J)\Phi$ and $\Phi a = \Phi(a+J)$ for $a, b \in \mathcal{A}, \Phi \in (\mathcal{A}/J)^{(n)}$ (the $\mathcal{U}$-module actions are similar to actions on $(\mathcal{A}/J)^{(n)}$ as $\mathcal{U}$-module). Note that whenever $\mathcal{A}/J$ is a commutative $\mathcal{U}$-module, then $(\mathcal{A}/J)^{(n)}$ $(n \geqslant 0)$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module by the above actions. Now we are ready to define the notion of n-weak module amenability. We say that $\mathcal{A}$ is n-weakly module amenable $(n \in \mathbb{N})$ if $(\mathcal{A}/J)^{(n)}$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module, and each continuous module derivation $D : \mathcal{A} \to (\mathcal{A}/J)^{(n)}$ is inner; that is $D(a) = D_\Phi(a) = a\Phi - \Phi a$ for some $\Phi \in (\mathcal{A}/J)^{(n)}$ and all $a \in \mathcal{A}$. Also $\mathcal{A}$ is called permanently weakly module amenable if $\mathcal{A}$ is n-weakly module amenable for each $n \in \mathbb{N}$. This definition is quite natural since $(\mathcal{A}/J)^{(n)}$ $(n \geqslant 0)$ is always a Banach $\mathcal{A}$-$\mathcal{U}$-module.

The notion of weak module amenability of a Banach algebra $\mathcal{A}$ which is a Banach $\mathcal{U}$-module with compatible actions is defined in [2], and studied in [1]. The main result of [2] is that the semigroup Banach algebra $\ell^1(S)$ on an inverse semigroup S is weakly module amenable, as an $\ell^1(E)$-module, when S is commutative. The definition of weak module amenability is modified in [1] and the above result is proved for an arbitrary inverse semigroup (with trivial left action). Then the notion of n-weak module amenability is introduced in [5] and proved that $\ell^1(S)$ is $(2n+1)$-weakly module amenable as an $\ell^1(E)$-module, for each $n \in \mathbb{N}$, where S is an inverse semigroup with the set of idempotents E.

In this paper, we show that the inverse semigroup algebra $\ell^1(S)$ is $2n$-weakly module amenable as an $\ell^1(E)$-module, for every number $n \in \mathbb{N}$, where E is the set of idempotents of S and E acts on S trivially from the left and by multiplication from the right. Our proof is based on a common fixed point property for semigroups. In fact in this article we show that a module version of the main result of [21] holds for inverse semigroups.

2. Main result

A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E. There is a natural order on E, defined by

$$e \leqslant d \Leftrightarrow ed = e \quad (e, d \in E),$$

and E is a commutative subsemigroup of S, which is also a semilattice [11, Theorem V.1.2]. Elements of the form ss^* are idempotents of S and in fact all elements of E are in this form.

The algebra $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$. Hence $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions. In this article we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left; that is,

$$\delta_e.\delta_s = \delta_s, \quad \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal J (see section 1) is the closed linear span of $\{\delta_{set} - \delta_{st} \,|\, s, t \in S, e \in E\}$. With the notations of the previous section $(\ell^1(S)/J)^{(n)}$ $(n \geqslant 0)$ is a Banach $\ell^1(S)$-$\ell^1(E)$-module. Note that we show the $\ell^1(E)$-module actions of $f \in \ell^1(E)$ on $\Phi \in (\ell^1(S)/J)^{(n)}$ by $f.\Phi$ and $\Phi.f$, and also denote the $\ell^1(S)$ module actions of $f \in \ell^1(S)$ on $\Phi \in (\ell^1(S)/J)^{(n)}$ by $f\Phi$ and Φf. In the next remark, we give some properties of these module actions.

Remark 1 *With the above notation, for all $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(n)}$ $(n \geqslant 0)$ we have the followings*

(i) $\delta_e.\Phi = \Phi.\delta_e$;

(ii) $\delta_e\Phi = \Phi\delta_e = \Phi$.

Proof. For all $e, d \in E$, we have $\delta_e - \delta_d = \delta_{ee} - \delta_{ede} - \delta_{dd} + \delta_{ded} \in J$. So $\delta_e + J = \delta_d + J$. Now for any $s \in S$ and $e \in E$, we find

$$\delta_{es} + J = (\delta_e + J)(\delta_s + J) = (\delta_{ss^*} + J)(\delta_s + J) = \delta_s + J.$$

Similarly, we get $\delta_{se} + J = \delta_s + J$ for $e \in E$ and $s \in S$. Hence, we have

$$\delta_e.(\delta_s + J) = \delta_s + J = \delta_{se} + J = (\delta_s + J).\delta_e$$

and

$$\delta_e(\delta_s + J) = (\delta_e + J)(\delta_s + J) = \delta_{es} + J = \delta_s + J = \delta_{se} + J = (\delta_s + J)(\delta_e + J) = (\delta_s + J)\delta_e$$

for all $e \in E$ and $s \in S$. Since $lin\{\delta_s \,|\, s \in S\}$ is dense in $\ell^1(S)$ and J is closed in $\ell^1(S)$, it follows that $\delta_e.(f + J) = (f + J).\delta_e$ and $\delta_e(f + J) = f + J = (f + J)\delta_e$ for all $e \in E$ and $f \in \ell^1(S)$. So, by induction on n, we arrive at $\delta_e.\Phi = \Phi.\delta_e$ and $\delta_e\Phi = \Phi\delta_e = \Phi$ for all $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(n)}$ $(n \geqslant 0)$. ■

In view of this remark (i), we find that $(\ell^1(S)/J)^{(n)}$ $(n \geqslant 0)$ is a commutative $\ell^1(E)$-module.

For an inverse semigroup S, the quotient $S/\approx$ is a discrete group, where $\approx$ is an equivalence relation on S as follows:

$$s \approx t \Leftrightarrow \delta_s - \delta_t \in J \quad (s, t \in S).$$

Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S [17] of S (see [3, 18, 19]). As in [20, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(G_S)$. Also see [10]. In [4, Remark 1] it is shown that all congruences on inverse semigroup S is equivalent and the similar properties holds for another class of semigroups such as E-inversive semigroup ,E-inversive E-semigroups and eventually inverse semigroups.

Since for proof of the main result we use a common fixed point property for semigroups, now we recall some notions related to common fixed point theory. Let S be a (discrete) semigroup. The space of all bounded complex valued functions on S is denoted by $\ell^\infty(S)$. It is a Banach space with the uniform supremum norm. In fact $\ell^\infty(S) = (\ell^1(S))^*$. For each $s \in S$ and each $f \in \ell^\infty(S)$ let $\ell_s f$ be the left translate of f by s, that is $\ell_s f(t) = f(st)$ $(t \in S)$ (the right translate $r_s f$ is defined similarly). We recall that $f \in \ell^\infty(S)$ is weakly almost periodic if its left orbit $\mathcal{LO}(f) = \{\ell_s f \,|\, s \in S\}$ is relatively compact in the weak topology of $\ell^\infty(S)$. We denote by $WAP(S)$ the space of all weakly almost periodic functions on S, which is a closed subspace of $\ell^\infty(S)$ containing the constant function and invariant under the left and right translations. A linear functional $m \in WAP(S)^*$ is a mean on $WAP(S)$ if $\| m \| = m(1) = 1$. A mean m on $WAP(S)$ is a left invariant mean (abbreviated LIM) if $m(\ell_s f) = m(f)$ for all $s \in S$, and all $f \in WAP(S)$. If S is an inverse semigroup, it is well known that $WAP(S)$ always has a LIM [9, Proposition 2]. Let C be a subset of a Banach space $\mathcal{X}$. We say that $\Gamma = \{T_s \,|\, s \in S\}$ is a representation of S on C if for each $s \in S$, T_s is a mapping from C into C and $T_{st}(x) = T_s(T_t(x))$ $(s, t \in S, x \in C)$. We say that $x \in C$ is a common fixed point for (the representation of) S if $T_s(x) = x$ for all $s \in S$.

Let $\mathcal{X}$ be a Banach space and C a nonempty subset of $\mathcal{X}$. A mapping $T : C \to C$ is called nonexpansive if $\| T(x) - T(y) \| \leqslant \| x - y \|$ for all $x, y \in C$. The mapping T is called affine if C is convex and $T(\gamma x + \eta y) = \gamma T(x) + \eta T(y)$ for all constants $\gamma, \eta \geqslant 0$ with

$\gamma + \eta = 1$ and $x, y \in C$. A representation Γ of a semigroup S on C acts as nonexpansive affine mappings, if each T_s $(s \in S)$ is nonexpansive and affine.

A Banach space $\mathcal{X}$ is called L-embedded if there is a closed subspace $\mathcal{X}_0 \subseteq \mathcal{X}^{**}$ such that $\mathcal{X}^{**} = \mathcal{X} \oplus_{\ell^1} \mathcal{X}_0$. The class of L-embedded Banach spaces includes all $L^1(\Sigma, \mu)$ (the space of of all absolutely integrable functions on a measure space (Σ, μ)), preduals of von Neumann algebras, dual spaces of M-embedded Banach spaces and the Hardy space H_1. In particular, given a locally compact group G, the space $L^1(G)$ is L-embedded. So are its even duals $L^1(G)^{(2n)}$ $(n \geqslant 0)$. For more details, we refer the reader to [21] and the references therein. The next lemma is the common fixed point theorem for semigroups, which will be used in our proof to the main result.

Lemma 2.1 ([21, Theorem 2]) Let S be a discrete semigroup and Γ a representation of S on an L-embedded Banach space $\mathcal{X}$ as nonexpansive affine mappings. Suppose that $WAP(S)$ has a LIM and suppose that there is a nonempty bounded set $B \subset \mathcal{X}$ such that $B \subseteq \overline{T_s(B)}$ for all $s \in S$, then $\mathcal{X}$ contains a common fixed point for S.

We now can prove the main result of the paper.

Theorem 2.2 Let S be an inverse semigroup with the set of idempotents E. Consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with the trivial left action and natural right action. Then the semigroup algebra $\ell^1(S)$ is $2n$-weakly module amenable as an $\ell^1(E)$-module for each $n \in \mathbb{N}$.

Proof. Let $D : \ell^1(S) \to (\ell^1(S)/J)^{(2n)}$ be a continuous module derivation. Since $ss^* \in E$ for all $s \in S$, from Remark *1*(ii), we have

$$D(\delta_{ss^*}) = D(\delta_{ss^*ss^*}) = D(\delta_{ss^*} * \delta_{ss^*}) = \delta_{ss^*} D(\delta_{ss^*}) + D(\delta_{ss^*})\delta_{ss^*} = 2D(\delta_{ss^*}).$$

Hence, $D(\delta_{ss^*}) = 0$ for all $s \in S$. Define $\phi : S \to (\ell^1(S)/J)^{(2n)}$ by $\phi(s) = D(\delta_s)\delta_{s^*}$ for $s \in S$. We see that

$$\begin{aligned} \phi(st) = D(\delta_s * \delta_t)\delta_{(st)^*} &= (\delta_s D(\delta_t))\delta_{t^*} * \delta_{s^*} + (D(\delta_s)\delta_t)\delta_{t^*} * \delta_{s^*} \\ &= \delta_s(D(\delta_t)\delta_{t^*})\delta_{s^*} + (D(\delta_s)\delta_{tt^*})\delta_{s^*} \\ &= \delta_s(D(\delta_t)\delta_{t^*})\delta_{s^*} + D(\delta_s)\delta_{s^*} \\ &= \delta_s\phi(t)\delta_{s^*} + \phi(s), \end{aligned} \tag{1}$$

for all $s, t \in S$. Let $B = \phi(S)$. Then B is a nonempty bounded subset of $(\ell^1(S)/J)^{(2n)}$. For any $s \in S$ define the mapping $T_s : (\ell^1(S)/J)^{(2n)} \to (\ell^1(S)/J)^{(2n)}$ by

$$T_s(\Phi) = \delta_s \Phi \delta_{s^*} + \phi(s) \quad (\Phi \in (\ell^1(S)/J)^{(2n)}).$$

Clearly, each T_s $(s \in S)$ is an affine mapping and for every $\Phi, \Psi \in (\ell^1(S)/J)^{(2n)}$ and $s \in S$, we have

$$\| T_s(\Phi) - T_s(\Psi) \| = \| \delta_s \Phi \delta_{s^*} + \phi(s) - \delta_s \Psi \delta_{s^*} + \phi(s) \| \leqslant \| \Phi - \Psi \| .$$

So each T_s $(s \in S)$ is nonexpansive. Now by using (1) for any $s, t \in S$ and $\Phi, \Psi \in (\ell^1(S)/J)^{(2n)}$, we find

$$\begin{aligned} T_{st}(\Phi) = \delta_{st}\Phi\delta_{(st)^*} + \phi(st) &= \delta_s(\delta_t \Phi \delta_{t^*})\delta_{s^*} + \delta_s\phi(t)\delta_{s^*} + \phi(s) \\ &= \delta_s T_t(\Phi)\delta_{s^*} + \phi(s) \\ &= T_s(T_t(\Phi)). \end{aligned}$$

So, $\Gamma = \{T_s \mid s \in S\}$ defines a representation of S on $(\ell^1(S)/J)^{(2n)}$ which is nonexpansive and affine. From definition of T_s and (1), for any $s, t \in S$ it follows that $T_s(\phi(t)) = \delta_s\phi(t)\delta_{s^*} + \phi(s) = \phi(st)$. Therefore $T_s(B) \subseteq B$ $(s \in S)$. Let $\Phi \in B$. Now by Remark *1*(ii) and the fact that $D(\delta_{ss^*}) = 0$ $(s \in S)$, we have

$$T_s(T_{s^*}(\Phi)) = T_{ss^*}(\Phi) = \delta_{ss^*}\Phi\delta_{ss^*} + \phi(ss^*) = \Phi \quad (s \in S).$$

Since $T_{s^*}(\Phi) \in B$, it follows that $T_s(B) = B$ for each $s \in S$. Here S is regarded as a discrete semigroup.

Since $\ell^1(S)/J \cong \ell^1(G_S)$, where G_S is the maximal group homomorphic image, it follows that $(\ell^1(S)/J)^{(2n)}$ is L-embedded. Also $WAP(S)$ has a LIM. So by Lemma 2.1, there is $\Upsilon \in (\ell^1(S)/J)^{(2n)}$ such that $T_s(\Upsilon) = \Upsilon$ for all $s \in S$ or $\delta_s\Upsilon\delta_{s^*} + \phi(s) = \Upsilon$ for all $s \in S$. So $\delta_s\Upsilon\delta_{s^*} + D(\delta_s)\delta_{s^*} = \Upsilon$ $(s \in S)$. Hence, we have $D(\delta_s) = \Upsilon\delta_s - \delta_s\Upsilon$ for all $s \in S$. By definition of left module action of $\ell^1(E)$ on $\ell^1(S)$, we have $\delta_e.\delta_s = \delta_s$ $(e \in E, s \in S)$. Since $lin\{\delta_s \mid s \in S\}$ is dense in $\ell^1(S)$, we find $\delta_e.f = f$ for all $e \in E$ and $f \in \ell^1(S)$. Hence $\delta_e.(f + J) = f + J$ $(e \in E, f \in \ell^1(S))$. Furthermore a routine inductive argument shows that for each $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(2n)}$ $(n \geqslant 0)$, we have $\delta_e.\Phi = \Phi$. From this result and the fact that D is a module mapping, for any $s \in S$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
D(\lambda\delta_s) &= D(\lambda\delta_{ss^*}.\delta_s) \\
&= \lambda\delta_{ss^*}.D(\delta_s) \\
&= \lambda\delta_{ss^*}.(\Upsilon\delta_s - \delta_s\Upsilon) \\
&= \lambda\delta_{ss^*}.(\Upsilon\delta_s) - \lambda(\delta_{ss^*}.\delta_s)\Upsilon \\
&= \lambda(\Upsilon\delta_s - \delta_s\Upsilon).
\end{aligned}$$

Since D is additive, we get $D(f) = \Upsilon f - f\Upsilon$ for any $f \in \ell^1(S)$ of finite support. But D is continuous and functions of finite support are dense in $\ell^1(S)$. Hence, we have

$$D(f) = \Upsilon f - f\Upsilon = D_{(-\Upsilon)}(f) \quad (f \in \ell^1(S)).$$

Therefore, D is inner. The proof is complete. ■

In [5], it has been proved that $\ell^1(S)$ is $(2n+1)$-weakly module amenable as an $\ell^1(E)$-module, for each $n \in \mathbb{N}$, where S is an inverse semigroup with the set of idempotents E. From this result and above theorem we get the next corollary.

Corollary 2.3 Let S be an inverse semigroup with the set of idempotents E. Consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with the trivial left action and natural right action. Then the semigroup algebra $\ell^1(S)$ is permanently weakly module amenable as an $\ell^1(E)$-module.

It should be noted that a similar result with the Corollary 2.4 of this paper has been obtained in [6] by a different proof.

With the notations in previous corollary, we have the next result.

Corollary 2.4 Each continuous module derivation $D : \ell^1(S) \to (\ell^1(G_S))^{(n)}$ $(n \in \mathbb{N})$ is inner.

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