Non-additive Lie centralizer of infinite strictly upper triangular matrices

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Abstract. Let $F$ be an field of zero characteristic and $N_{\infty}(F)$ be the algebra of infinite strictly upper triangular matrices with entries in $F$, and $f : N_{\infty}(F) \rightarrow N_{\infty}(F)$ be a non-additive Lie centralizer of $N_{\infty}(F)$; that is, a map satisfying that $f([X,Y]) = [f(X),Y]$ for all $X,Y \in N_{\infty}(F)$. We prove that $f(X) = \lambda X$, where $\lambda \in F$.

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1. Introduction and preliminaries

Consider a ring $R$. An additive mapping $T : R \rightarrow R$ is called a left (respectively right) centralizer if $T(ab) = T(a)b$ (respectively $T(ab) = aT(b)$) for all $a, b \in R$. The mapping $T$ is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [11], Zalar proved the following interesting result: if $R$ is a 2-torsion free semiprime ring and $T$ is an additive mapping such that $T(a^2) = T(a)a$ (or $T(a^2) = aT(a)$), then $T$ is a centralizer. Vukman [10] considered additive maps satisfying similar condition, namely $2T(a^2) = T(a)a + aT(a)$ for any $a \in R$, and showed that if $R$ is a 2-torsion free semiprime ring, then $T$ is also a centralizer. Since then centralizers have been intensively

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investigated by many mathematicians, let us name only [2–5, 7] and references included in these works.

An additive map $f : R \to R$, where $R$ is a ring, is called a Lie centralizer of $R$ if $f([x, y]) = [f(x), y]$ for all $x, y \in R$, where $[x, y]$ is the Lie product of $x$ and $y$.


Comes from the inspiration of this paper articles [1, 4, 6] in which the authors deal with triangular algebras and rings and various maps connected to commutativity. In this note we will consider non-additive Lie centralizers on strictly infinite upper triangular matrices over an field of zero characteristic.

Let us recall one basic fact. Let $F$ be an field of zero characteristic. Also, let $N_\infty(F)$, $D_\infty(F)$ and $T_\infty(F)$ denote the algebra of strictly infinite $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over $F$, the algebra of all infinite $\mathbb{N} \times \mathbb{N}$ diagonal matrices over $F$ and the algebra of all infinite $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over $F$, respectively.

Throughout this article, $J$ will represent the matrix $J = \sum_{i=1}^{\infty} E_{i,i+1}$ and $I_\infty = \sum_{i=1}^{\infty} E_{i,i}$.

By $C_{N_\infty(F)}(X)$, we will denote the centralizer of the element $X$ in the ring $N_\infty(F)$ and $f : N_\infty(F) \to N_\infty(F)$ will denote a non-additive map satisfying: $f([X,Y]) = [f(X), Y]$ for all $X, Y \in N_\infty(F)$. We will say that $f$ is a non-additive Lie centralizer of $N_\infty(F)$.

Notice that it is easy to check that the $N_\infty(F)$ has a trivial center $Z(N_\infty(F))$.

The main result in this paper is the following:

**Theorem 1.1** Let $F$ be an field of zero characteristic. If $f : N_\infty(F) \to N_\infty(F)$ is a non-additive Lie centralizer then there exists $\lambda \in F$ such that $f(X) = \lambda X$ for all $X \in N_\infty(F)$.

Notice that the converse is trivially true: every map $f(X) = \lambda X$ is a (non-additive) Lie centralizer.

## 2. Proof of the main result

Let’s start with some properties of Lie centralizers.

**Lemma 2.1** [6] Let $f$ be a non-additive Lie centralizer of $N_\infty(F)$. Then

1. $f(0) = 0$,
2. For every $X, Y \in N_\infty(F)$ we have $f([X, Y]) = [X, f(Y)]$,
3. $f$ is a commuting map, i.e. $f(X)X = XF(X)$ for all $X \in N_\infty(F)$.

**Proof.** (1) It suffices to notice that $f(0) = f([0, 0]) = [f(0), 0] = 0$.

(2) Observe that if $f([X, Y]) = [f(X), Y]$, then we have $f(XY - YX) = f(X)f(Y) - f(Y)f(X)$. Interchanging $X$ and $Y$ in the above identity, we have $f(YX - XY) = f(Y)X - f(X)f(Y)$. Replacing $X$ with $-X$, we arrive at $f(XY - YX) = Xf(Y) - f(Y)X$ which can be written as $f([X, Y]) = [X, f(Y)]$.

(3) From (1), one also gets $[f(X), X] = f([X, X]) = f(0) = 0$. □

**Remark 1** Let $f$ be a non-additive Lie centralizer of $N_\infty(F)$ and $X \in C_{N_\infty(F)}(Y)$. Then $f(X) \in C_{N_\infty(F)}(Y)$. Indeed, if $X \in C_{N_\infty(F)}(Y)$, then $[X,Y] = 0$ and

$$0 = f(0) = f([X,Y]) = [f(X),Y].$$
Lemma 2.2 Let $f$ be a non-additive Lie centralizer of $N_\infty(F)$. Then

1. if $A \in T_\infty(F)$, then $[D_0, A] = A$ if and only if $A = \sum_{i=1}^\infty a_i E_{i,i+1}$;

2. $f(\sum_{i=1}^\infty a_i E_{i,i+1}) = \sum_{i=1}^\infty b_i E_{i,i+1}$;

3. if $A = \sum_{i=1}^\infty a_i E_{i,i+1}$ for some $a_i \in A$, then $[J_\infty, A] = 0$ if and only if $A = aJ_\infty$ for some $a \in F$;

4. there exists $\lambda \in F$ such that $f(J) = \lambda J$.

Proof. Let $D_0 = \sum_{k=1}^\infty (-k) E_{k,k}$.

1. Consider $A = \sum_{i\leq j} a_{ij} E_{ij} \in T_\infty(F)$. Then $[D_0, A] = A$ if and only if $(p-n) a_{np} = a_{np}$ for all $1 \leq n \leq p \in \mathbb{N}$, and consequently $A = \sum_{i=1}^\infty a_{i,i+1} E_{i,i+1}$.

2. Hence, if $A = \sum_{i=1}^\infty a_i E_{i,i+1}$, $[D_0, A] = A$. Thus $f([D_0, A]) = [D_0, f(A)] = f(A)$.

Thus, $f(A) = \sum_{i=1}^\infty b_i E_{i,i+1}$.

3. As in (1), consider $A = \sum_{i=1}^\infty a_i E_{i,i+1}$ for some $a_i \in F$. Then $[J, A] = 0$ if and only if $A = aJ$ for some $a \in F$.

Indeed, $f(J) = \sum_{i=1}^\infty a_i E_{i,i+1}$ by (1). Thus, $0 = f(0) = f([J, J]) = [J, f(J)]$. Hence, there exists $\lambda \in F$ such that $f(J) = \lambda J$.

Lemma 2.3 [9] Suppose that $F$ is an arbitrary field. If $G, H \in UT_\infty(F)$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i \leq n - 1$, then $G$ and $H$ are conjugated in $UT_\infty(F)$.

Here $UT_\infty(F)$ is the multiplicative group of infinite $\mathbb{N} \times \mathbb{N}$ upper triangular matrices with only 1’s in the main diagonal. From the lemma above we obtain the following corollary.

Corollary 2.4 Let $F$ be a field. For every $A = \sum_{i \leq j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $B \in T_\infty(F)$ such that $B^{-1}AB = J$.

Proof. Let $A$ be a matrix in $N_\infty(F)$ of the mentioned form. Then $I_\infty + A$ is a unitriangular matrix, let’s notice first that there exists $B_1 \in D_\infty(F)$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. We can construct $B_1 \in D_\infty(F)$ recursively by:

$$(B_1)_{11} = 1, \quad (B_1)_{i,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \text{ for } i \geq 1.$$ 

Consider matrix $B_1^{-1}AB$ and $I_\infty + B_1^{-1}AB \in UT_\infty(F)$. The unitriangular matrices $I_\infty + J$ and $I_\infty + B_1^{-1}AB$ fulfill the condition in Lemma 2.3. Hence, there exists $B_2 \in UT_\infty(F)$ such that $I_\infty + J = B_2^{-1}((I_\infty + B_1^{-1}AB)B_2)$. Then, $J = B_2^{-1}(B_1^{-1}AB_1)B_2$. Taking $B = B_1B_2 \in T_\infty(F)$, we obtain $J = B^{-1}AB$ as wanted.

Lemma 2.5 Let $A = \sum_{i \leq j} a_{ij} E_{ij}$, be a matrix in $N_\infty(F)$ with $a_{i,i+1} \neq 0$ for every $i \geq 1$. Then there exists $\lambda_A \in F$ such that $f(A) = \lambda_A A$.

Proof. If $A = \sum_{i \leq j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $T \in T_\infty(F)$ such that $TAT^{-1} = J$, by the previous corollary. Define $h : N_\infty(F) \rightarrow N_\infty(F)$ by $h(X) = \lambda_A X$. 

Multiplying the left and right sides by $T$, introduce the set that we will denote by $S$ for all $A, B$. Let Lemma 2.6.

This set has an important property that is established below. Proof. If most two elements of $S$, then we can define $B$ and $B$ as follows:

\[
\begin{align*}
(B_1)_{ij} &= \begin{cases} a_{i,i+1} - b_i & \text{if } j = i + 1 \\ a_{ij} & \text{if } j > i + 1 \end{cases}, \\
(B_2)_{ij} &= \begin{cases} b_i & \text{if } j = i + 1 \\ 0 & \text{otherwise}, \end{cases}
\end{align*}
\]

where $b_i$ is a nonzero different elements of $F$ from $a_{i,i+1}$. It is easy to see that $B_1$ and $B_2$ are in $S$, and $A = B_1 + B_2$, so we wanted. \( \blacksquare \)

Now, we wish to extend Lemma 2.5 to all elements of $N_\infty(F)$. In order to do it, let’s introduce the set that we will denote by $S = \{ B = (b_{ij}) \in N_\infty(F) : b_{i,i+1} \neq 0 \forall i \geq 1 \}$. This set has an important property that is established below.

**Lemma 2.6** Let $F$ be a field. Every element of $N_\infty(F)$ can be written as a sum of at most two elements of $S$.

**Proof.** If $a_{i,i+1} \neq 0$ for all $i \geq 1$, then $A$ belongs in $S$, so there is nothing to prove. If $A$ is not in $S$, then we can define $B_1$ and $B_2$ as follows:

\[
\begin{align*}
(B_1)_{ij} &= \begin{cases} a_{i,i+1} - b_i & \text{if } j = i + 1 \\ a_{ij} & \text{if } j > i + 1 \end{cases}, \\
(B_2)_{ij} &= \begin{cases} b_i & \text{if } j = i + 1 \\ 0 & \text{otherwise}, \end{cases}
\end{align*}
\]

for all $A, B \in N_\infty(F)$. Hence, $h(J) = \lambda_A J$ by Lemma 2.2. Then

\[
Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}.
\]

Multiplying the left and right sides by $T^{-1}$ and $T$ respectively yields $f(A) = \lambda_A A$. \( \blacksquare \)

**Lemma 2.7** Let $F$ be a field. Then $f(A + B) = f(A) + f(B)$ for arbitrary elements $A, B \in N_\infty(F)$.

**Proof.** For any $A, B, X$ of $N_\infty(F)$, we have

\[
[f(A + B), X] = f([A + B, X])
\]

\[
= [A + B, f(X)]
\]

\[
= [A, f(X)] + [B, f(X)]
\]

\[
= [f(A), X] + [f(B), X]
\]

\[
= [f(A) + f(B), X],
\]

respectively yields $f(A) = \lambda_A A$.
which implies that \( f(A+B) - f(A) - f(B) \in \mathcal{Z}(N_\infty(F)) \). Thus, \( f(A+B) = f(A)+f(B) \).

**Proof of Theorem 1.1:** Let \( A, B \in \mathcal{S} \) be two non-commuting elements. By lemma 2.5, \( f(A) = \lambda_A A, f(B) = \lambda_B B, \lambda_A, \lambda_B \in F \). Since \( f \) is non-additive Lie centralizer, we have

\[
\begin{align*}
  f([A, B]) &= [f(A), B] = \lambda_A [A, B] \\
  &= [A, f(B)] = \lambda_B [A, B]
\end{align*}
\]

Then, \([A, B] \neq 0\) implies that \( \lambda_A = \lambda_B \). If \( A, B \in \mathcal{S} \) commute, then we take \( C \in \mathcal{S} \) that does not commute neither with \( A \) nor with \( B \). As we have just seen, \( \lambda_A = \lambda_C \) and \( \lambda_B = \lambda_B \). Given \( X \in N_\infty(F) \). We know by lemma 2.6 that there exist \( A, B \in \mathcal{S} \) such that \( X = A + B \) (we can assume that \( X \notin \mathcal{S} \)). Then \( f(X) = f(A) + f(B) \) by lemma 2.7. Thus, \( f(X) = \lambda_A A + \lambda_B B = f(X) - \lambda X \) for \( \lambda \in F \) such that \( f(A) = \lambda A \) for all \( A \in \mathcal{S} \); that is, \( f(X) = \lambda X \), and Theorem 1.1 is proved.

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**References**