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On morphisms of crossed polymodules

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Abstract. In this paper, we prove that the category of crossed polymodules (i.e. crossed modules of polygroups) and their morphisms is finitely complete. We, therefore, generalize the group theoretical case of this completeness property of crossed modules.

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1. Introduction

For a given set X, an ordinary algebraic structure on X equips with binary operations $X \times X \to X$. In general case, one can define a hyperstructure [9] in which the binary operation is replaced with a binary hyperoperation $X \times X \to P^*(X)$. As a particular case of hyperstructures, polygroups [5, 7] are introduced to generalize the group theory. Consequently, there exists an inclusion functor Grp $\to$ PGrp from the category of groups to the category of polygroups. Polygroups are applied in many areas such as geometry, lattice theory, combinatorics and color scheme.

A crossed module of groups [14] $\mathcal{G} = (\partial\colon H \to G, \triangleright)$ is defined by a group homomorphism $\partial\colon H \to G$ together with a (left) group action of G on H satisfying $\partial(g \triangleright h) = g\,\partial(h)\,g^{-1}$ and $\partial(h) \triangleright h' = h\,h'\,h^{-1}$ for all $h, h' \in H$ and $g \in G$. The complete philosophy of crossed modules from the topological and algebraic point of view

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can be found in [3, 4]. We also refer [11, 13] for some geometric and visual approaches to the crossed modules.

Crossed modules of polygroups (namely crossed polymodules) are defined in [1] to generalize the group crossed modules. Explicitly, any group crossed module is a crossed polymodule. This yields an inclusion functor XMod $\to$ PXMod from the category of crossed modules of groups to the category of crossed modules of polygroups. It is proven in [10, 15] that the category of crossed modules of groups (in two different general approaches both cover group crossed modules) and their morphisms is complete.

In this paper, we show that the category of crossed polymodules and their morphisms is complete; namely it has product, pullback and equalizer objects.

2. Preliminaries

We recall some notions from [1, 5, 6, 8] which will be used in sequel.

Definition 2.1 A polygroup is a multi-valued system $P = \left\langle P, \circ, e, ^{-1} \right\rangle$ consists of:

- a binary hyperoperation $\circ \colon P \times P \to \mathcal{P}^*(P)$ where $\mathcal{P}^*(P) = \mathcal{P}(P) \setminus \emptyset$,
- a unary operation $^{-1} \colon P \to P$,
- a fixed element $e \in P$,

such that satisfying:

P1) $(x \circ y) \circ z = x \circ (y \circ z)$,

P2) $e \circ x = x \circ e = x$,

P3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$

for all $x, y, z \in P$.

Remark 1 *If $x \in P$ and A, B are non-empty subsets of P, then*

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b\,,$$

where $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

Remark 2 *It follows directly from the polygroup conditions that*

$$e \in x \circ x^{-1} \cap x^{-1} \circ x, \quad e^{-1} = e, \quad \left(x^{-1}\right)^{-1} = x,$$

for all $x \in P$.

Example 2.2 Suppose that H is a subgroup of a group G. Define a system

$$G//H = \left\langle \{HgH \mid g \in G\}, *, H, ^{-1} \right\rangle,$$

where $(HgH)^{-1} = Hg^{-1}H$ and also

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \mid h \in H\}.$$

Then the algebra of double cosets $G//H$ is a polygroup.

Remark 3 *Every group is a polygroup. Therefore, we have the inclusion functor* Grp $\to$ PGrp *from the category of groups to the category of polygroups.*

Definition 2.3 Let $\langle P, \circ, e,^{-1} \rangle$ and $\langle P', \star, e,^{-1} \rangle$ be two polygroups and ϕ be a mapping from P into P' such that $\phi(e) = e$. Then ϕ is said to be

- an inclusion homomorphism, if $\phi(x \circ y) \subseteq \phi(x) \star \phi(y)$,
- a weak homomorphism, if $\phi(x \circ y) \cap \phi(x) \star \phi(y) \neq \emptyset$,
- a strong homomorphism, if $\phi(x \circ y) = \phi(x) \star \phi(y)$,

for all $x, y \in P$.

Definition 2.4 Let $\mathcal{P} = \langle P, \circ, e,^{-1} \rangle$ be a polygroup and Ω be a non-empty set. A map $\alpha\colon P \times \Omega \to \mathcal{P}^*(\Omega)$ is called a (left) polygroup action on Ω if the following axioms hold:

1) $\alpha(e, w) = \{w\} = w$, for all $w \in \Omega$,
2) $\alpha(h, \alpha(g, w)) = \bigcup\limits_{x \in h \circ g} \alpha(x, w)$, for all $g, h \in P$ and $w \in \Omega$,
3) $\bigcup\limits_{w \in \Omega} \alpha(g, w) = \Omega$, for all $g \in P$,
4) $x \in \alpha(g, y) \Rightarrow y \in \alpha(g^{-1}, x)$, for all $g \in P$.

From the second condition, we get

$$\bigcup_{w_0 \in \alpha(g,w)} \alpha(h, w_0) = \bigcup_{x \in h \circ g} \alpha(x, w).$$

For any $w \in \Omega$, we write $g \rhd w := \alpha(g, w)$. Therefore, we have

1) $e \rhd w = w$,
2) $h \rhd (g \rhd w) = (h \circ g) \rhd w$, where $g \rhd A = \bigcup\limits_{a \in A} g \rhd a$ and similarly $B \rhd w = \bigcup\limits_{b \in B} b \rhd w$, for all $A \subseteq \Omega$ and $B \subseteq P$,
3) $\bigcup\limits_{w \in \Omega} g \rhd w = \Omega$,
4) $x \in g \rhd y \Rightarrow y \in g^{-1} \rhd x$, for all $g \in P$.

***Example* 2.5** Suppose that $\langle P, \circ, e,^{-1} \rangle$ is a polygroup. Then, we have the following possible actions of P on itself:

- $g \rhd x := x \circ g^{-1}$,
- $g \rhd x := g \circ x$,
- via conjugation, i.e. $g \rhd x := g \circ x \circ g^{-1}$,

for all $x, g \in P$.

Now let us define crossed modules of polygroups and give some examples of them.

Definition 2.6 A crossed module of polygroups (i.e. a crossed polymodule) $\mathcal{X} = (C, P, \partial)$ consist of polygroups $\langle C, \star, e,^{-1} \rangle$ and $\langle P, \circ, e,^{-1} \rangle$ together with a strong homomorphism $\partial\colon C \to P$ and a (left) action $P \times C \to \mathcal{P}^*(C)$ satisfying:

PX1) $\partial(p \rhd c) = p \circ \partial(c) \circ p^{-1}$,

PX2) $\partial(c) \rhd c' = c \star c' \star c^{-1}$, for all $c, c' \in C$ and $p \in P$.

Let $\mathcal{X} = (C, P, \partial)$ and $\mathcal{X}' = (C', P', \partial')$ be two crossed polymodules. A crossed polymodule morphism $f = (f_1, f_0)\colon \mathcal{X} \to \mathcal{X}'$ is a tuple of strong homomorphism such that the diagram

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "\partial"'] & C' \arrow[d, "\partial'"] \\ P \arrow[r, "f_0"'] & P' \end{tikzcd}$$

commutes, and $f_1(p \triangleright c) = f_0(p) \triangleright f_1(c)$ for all $p \in P$ and $c \in C$.

Consequently, we have the category of crossed polymodules denoted by PXMod.

***Example* 2.7** Some well known crossed polymodule examples are given below.

1) Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. The identity map $\mathrm{id}_P\colon P \to P$ is a crossed polymodule with the action of P on itself by conjugation.

2) Let $N \subset R$ be a normal subpolygroup of P (i.e. $p^{-1} \circ N \circ p \subseteq N$ for all $p \in P$). The inclusion map $N \to R$ is a crossed polymodule where the action is defined by conjugation.

3) Every group crossed module is a crossed polymodule. Therefore we have the inclusion functor XMod $\to$ PXMod from the category of crossed modules of groups to the category of crossed modules of polygroups.

3. Limits in PXMod/X

We know from [12] that the category of polygroups is complete. Briefly, the cartesian product $P \times R$ is the product object with the projection morphisms (that are strong homomorphisms). Moreover, suppose that $\alpha : P \to S$ and $\beta : R \to S$ are two strong homomorphisms. Then the subobject of the cartesian product:

$$P \times_S R = \{(p, r) \mid \alpha(p) = \beta(r)\}, \tag{1}$$

gives the pullback of α, β, which is called fiber product. These two objects guarantee the existence of equalizer object, see [2] for details.

3.1 *Limits in the Category of Crossed X-Polymodules*

Definition 3.1 The category of crossed polymodules with a fixed codomain X forms a subcategory of PXMod denoted by PXMod/X. These kind of crossed polymodules will be called crossed X-polymodules. A morphism in PXMod/X is defined by a tuple (f_1, f_0) where $f_0 = \mathrm{id}$.

Notation 3.2 From now on, any polygroup operation will be denoted by "$\circ$", and any polygroup action will be denoted by "$\triangleright$" for the sake of simplicity.

Lemma 3.3 Suppose that we have two crossed polymodules (P, S, α) and (R, S, β). Then there exists a crossed polymodule $\partial : P \times_S R \to S$, where $\partial(p, r) = \alpha(p) = \beta(r)$ with the action of S on $P \times_S R$, $s \triangleright (p, r) = (s \triangleright p, s \triangleright r)$ for all $s \in S$ and $(p, r) \in P \times_S R$.

Proof. It is clear that the action is well-defined. Moreover ∂ is a strong homomorphism, since

$$\partial\left((p,r)\circ\left(p',r'\right)\right)=\partial\left(p\circ p',r\circ r'\right)=\alpha\left(p\circ p'\right)=\alpha\left(p\right)\circ\alpha\left(p'\right)=\partial\left(p,r\right)\circ\partial\left(p',r'\right)$$

for all $(p,r)\,,(p',r') \in P \times_S R$.

Moreover, ∂ satisfies the crossed polymodule conditions as follows.

PX1)

$$\begin{aligned}\partial\left(s\triangleright(p,r)\right)&=\partial\left(s\triangleright p,s\triangleright r\right)\\&=\alpha(s\triangleright p)\\&=s\circ\alpha\left(p\right)\circ s^{-1}\\&=s\circ\partial\left(p,r\right)\circ s^{-1},\end{aligned}$$

PX2)

$$\begin{aligned}\partial\left(p',r'\right)\triangleright(p,r)&=\alpha\left(p'\right)\triangleright(p,r)\\&=\left(\alpha\left(p'\right)\triangleright p,\alpha\left(p'\right)\triangleright r\right)\\&=\left(\alpha\left(p'\right)\triangleright p,\beta\left(r'\right)\triangleright r\right)\\&=\left(p'\circ p\circ {p'}^{-1},r'\circ r\circ {r'}^{-1}\right)\\&=\left(p',r'\right)\circ(p,r)\circ(p',r')^{-1}\end{aligned}$$

for all $(p,r)\,,(p',r') \in P \times_S R$ and $s \in S$. ■

Lemma 3.4 Let $(\alpha, \mathrm{id})\colon (P, X, \gamma) \to (S, X, \partial')$ be a crossed polymodule morphism. Then there exists a crossed polymodule (P, S, α) where the action of S on P are defined along ∂', namely, $s \triangleright p = \partial'(s) \triangleright p$.

Proof. Since (α, id) is a crossed polymodule morphism, the diagram

$$\begin{tikzcd} P \arrow[dr, "\alpha"] \arrow[dd, "\gamma"'] & \\ & S \arrow[dl, "\partial'"] \\ X & \end{tikzcd}$$

commutes; i.e. $\alpha(x \triangleright p) = x \triangleright \alpha(p)$, for all $x \in X$ and $p \in P$. Thus

PX1)

$$\alpha(s\triangleright p)=\alpha\big(\partial'(s)\triangleright p\big)=\partial'(s)\triangleright\alpha(p)=s\circ\alpha(p)\circ s^{-1},$$

PX2)

$$\alpha(p) \triangleright p' = \partial'(\alpha(p)) \triangleright p' = \gamma(p) \triangleright p' = p \circ p' \circ p^{-1}$$

for all $s \in S$ and $p, p' \in P$. ∎

Proposition 3.5 If (A, B, ∂) and (B, C, ∂') are crossed polymodules such that C acts on A in a compatible way with B, namely $\partial'(b) \triangleright a = b \triangleright a$. Then $(A, C, \partial'\partial)$ becomes a crossed polymodule.

Lemma 3.6 (Pullback). Suppose that we have two crossed polymodule morphisms

$$(\alpha, \mathrm{id})\colon (P, X, \gamma) \to (S, X, \partial') \text{ and } (\beta, \mathrm{id})\colon (R, X, \delta) \to (S, X, \partial').$$

There exists a crossed polymodule $P \times_S R \to X$, which leads to the categorical pullback object in PXMod/X.

Proof. By using crossed polymodule morphisms (α, id) and (β, id), we get the strong homomorphisms $\alpha\colon P \to S$ and $\beta\colon R \to S$. We already know that the pullback of these two morphisms is defined by the fiber product $P \times_S R$ that makes the diagram

$$\begin{tikzcd}
& P \times_S R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
P \arrow[dr, "\alpha"'] & & R \arrow[dl, "\beta"] \\
& S &
\end{tikzcd}$$

commutative and satisfies the universal property.

By using Lemma 3.4, α and β turn into crossed polymodules, thus we get a crossed polymodule $\partial\colon P \times_S R \to S$ in the sense of Lemma 3.3. Moreover, $\partial'\colon S \to X$ is already a crossed polymodule and X acts on $P \times_S R$ in a natural way. Therefore by using Remark 3.5, we get the crossed polymodule $\partial'\partial\colon P \times_S R \to X$, which leads to the pullback object in the category of crossed X-polymodules. All fitting into the diagram:

$$\begin{tikzcd}
& P \times_S R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\partial"] & \\
P \arrow[dr, "\alpha"'] \arrow[ddr, bend right, "\gamma"'] & & R \arrow[dl, "\beta"] \arrow[ddl, bend left, "\delta"] \\
& S \arrow[d, "\partial'"] & \\
& X &
\end{tikzcd}$$

∎

Proposition 3.7 The category of crossed X-polymodules has a terminal object $\mathrm{id}\colon X \to X$. Consequently, one can construct the product object as a pullback of the morphisms:

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & 1 & \end{tikzcd}$$

where $\mathcal{X}, \mathcal{X}'$ are two crossed X-polymodules and 1 is the terminal object.

This yields the following:

Proposition 3.8 (Product). Given two crossed polymodules $\alpha\colon P \to S$ and $\beta\colon R \to S$, their product is the crossed polymodule $\partial : P \times_S R \to S$.

Thus, we have proved the following.

Corollary 3.9 The category PXMod/X is finitely complete.

4. Limits in PXMod

In this section, we consider the general case of the previous section without any restriction on the codomains of crossed polymodules.

Remark 4 *Consider we have two crossed polymodules (C_1, P_1, ∂_1) and (C_2, P_2, ∂_2). There exists an action of $P_1 \times P_2$ on $C_1 \times C_2$ component-wise, namely*

$$(p_1, p_2) \triangleright (c_1, c_2) = (p_1 \triangleright c_1, p_2 \triangleright c_2).$$

Remark that, each $\triangleright$ denotes a different action above, corresponding on the structure.

Proposition 4.1 (Product). We have a crossed polymodule structure

$$(C_1 \times C_2, P_1 \times P_2, \partial)\,, \tag{2}$$

given by

$$\partial(c_1, c_2) = \big(\partial_1(c_1), \partial_2(c_2)\big)\,, \tag{3}$$

for all $(c_1, c_2) \in C_1 \times C_2$, since

PX1)

$$\begin{aligned} \partial\big((p_1, p_2) \triangleright (c_1, c_2)\big) &= \partial(p_1 \triangleright c_1, p_2 \triangleright c_2) \\ &= \big(\partial_1(p_1 \triangleright c_1), \partial_2(p_2 \triangleright c_2)\big) \\ &= \big(p_1 \circ \partial_1\,(c_1) \circ p_1^{-1}, p_2 \circ \partial_2\,(c_2) \circ p_2^{-1}\big) \\ &= (p_1, p_2) \circ \big(\partial_1(c_1), \partial_2(c_2)\big) \circ (p_1^{-1}, p_2^{-1}) \\ &= (p_1, p_2) \circ \partial(c_1, c_2) \circ (p_1, p_2)^{-1}\,, \end{aligned}$$

PX2)

$$\begin{aligned}
\partial(c_1,c_2) \triangleright (c_1',c_2') &= \big(\partial_1(c_1),\partial_2(c_2)\big) \triangleright (c_1',c_2') \\
&= \big(\partial_1(c_1) \triangleright c_1', \partial_2(c_2) \triangleright c_2'\big) \\
&= \big(c_1 \circ c_1' \circ c_1^{-1}, c_2 \circ c_2' \circ c_2^{-1}\big) \\
&= (c_1,c_2) \circ (c_1',c_2') \circ (c_1^{-1},c_2^{-1}) \\
&= (c_1,c_2) \circ (c_1',c_2') \circ (c_1,c_2)^{-1},
\end{aligned}$$

for all $(c_1,c_2),(c_1',c_2') \in C_1 \times C_2$ and $(p_1,p_2) \in (P_1,P_2)$. Hence (2) is the product object in the category of crossed polymodules, namely PXMod.

Proposition 4.2 (Pullback). Consider two crossed polymodule morphisms

$$\begin{aligned}
(f_1,g_1)\colon (C_1,P_1,\partial_1) &\to (C_3,P_3,\partial_3), \\
(f_2,g_2)\colon (C_2,P_2,\partial_2) &\to (C_3,P_3,\partial_3).
\end{aligned}$$

Recalling (1), define the fiber products $A = C_1 \times_{C_3} C_2$ and $B = P_1 \times_{P_3} P_2$ which are the pullbacks of (f_1,g_1) and (f_2,g_2) in the category of polygroups, respectively. Then we obtain a crossed polymodule (A,B,∂') where ∂' is the restriction of ∂ (3) to A, which is the pullback object in the category PXMod.

Remark 5 *Moreover, it is clear that the crossed polymodule* $(0,0,\mathrm{id})$ *is the zero object in this category. Therefore one can also obtain also equalizer object in the category* PXMod.

Consequently, we obtain:

Corollary 4.3 The category PXMod is finitely complete.

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References

[1] M. Alp, B. Davvaz, Crossed polymodules and fundamental relations, U.P.B. Sci. Bull. (Series A). 77 (1) (2015), 129–140.
[2] S. Awodey, Category Theory, Oxford: Oxford University Press, 2006.
[3] R. Brown, From groups to groupoids: A brief survey, Bull. Lond. Math. Soc. 19 (1987), 113–134.
[4] R. Brown, Modelling and computing homotopy types: I, Indag. Math. 29 (1) (2018) 459–482.
[5] S. D. Comer, Polygroups derived from cogroups, Journal of Algebra. 89 (2) (1984), 397 – 405.
[6] B. Davvaz, On polygroups and permutation polygroups, Math. Balkanica. 14 (1-2) (2000), 41–58.
[7] B. Davvaz, Polygroup Theory and Related Systems, World Scientific, 2012.
[8] B. Davvaz, M. Alp, Derivation and actor of crossed polymodules, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 25 (3) (2018), 203-218.
[9] B. Davvaz, T. Vougiouklis, A walk through weak hyperstructures. H_v-structures, Hackensack, NJ: World Scientific, 2019.
[10] K. Emir, S. Çetin, Limits in modified categories of interest, Bull. Iran. Math. Soc. 43 (7) (2017), 2617-2634.
[11] J. Faria Martins, R. Picken, On two-dimensional holonomy, Trans. Am. Math. Soc. 362 (11) (2010), 5657-5695.

[12] S. N. Hosseini, S. S. Mousavi, M. M. Zahedi, Category of polygroup objects, Bull. Iran. Math. Soc. 28 (1) (2002), 67-86.
[13] J. C. Morton, R. Picken, Transformation double categories associated to 2-group actions, Theory Appl. Categ. 30 (2015), 1429-1468.
[14] J. Whitehead, On adding relations to homotopy groups, Ann. Math. 42 (2) (1941), 409-428.
[15] M. Yavari, A. Salemkar, The category of generalized crossed modules, Categ. Gen. Algebr. Struct. Appl. 10 (1) (2019), 157-171.